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SPACELIKE CURVES OF CONSTANT BREADTH IN SEMI-RIEMANNIAN SPACE E_2^4

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Abstract In this paper, we investigate curves of constant breadth in E_2^4. Also, we obtain some characterizations according to the state of the spacelike curve in semi-Riemannian space E_2^4.

1 Introduction

The curves of constant breadth were first defined in 1778 by Euler. Then, Solow [12] and Blascke [1] investigated the curves of constant breadth. In Euclidean spaces E^3 and E^4, plane curves of constant breadth were studied by Kose [7], Magden and Yilmaz [8]. In [14], some geometric properties of plane curves of constant breadth in Minkowski 3-space were given. Also, these curves in Minkowski 4-space were obtained by Kazaz, Onder and Kocayigit [5]. A number of authors have, recently, studied the curves of constant breadth under different conditions (see [4, 6]).

In this study, we investigate the spacelike curves of constant breadth with timelike normal and first binormal and with timelike binormal and second binormal in E_2^4. Then we give some differential equations for these curves in semi-Riemannian space.

2 Preliminaries

In this section, we provide a brief view of the theory of curves in the semi-Riemannian space E_2^4. This space is an Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is rectangular coordinate system in E_2^4, [14]. An any vector $\overrightarrow{v} \in E_2^4$ can have one of the three causal characters; it is spacelike if $g(\overrightarrow{v}, \overrightarrow{v}) > 0$ or $\overrightarrow{v} = 0$, timelike if $g(\overrightarrow{v}, \overrightarrow{v}) < 0$ and null or lightlike if $g(\overrightarrow{v}, \overrightarrow{v}) = 0$ and $\overrightarrow{v} \neq 0$. Similarly, an any curve $\overrightarrow{\alpha} = \overrightarrow{\alpha}(s)$ in E_2^4 can locally be spacelike, timelike or null if its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively. Furthermore, the norm of a vector $\overrightarrow{v}$ is given by $\| \overrightarrow{v} \| = \sqrt{|g(\overrightarrow{v}, \overrightarrow{v})|}$. Thus, $\overrightarrow{v}$ is a unit vector if $g(\overrightarrow{v}, \overrightarrow{v}) = \pm 1$. The velocity of the curve $\overrightarrow{\alpha}$ is given by $\|\overrightarrow{\alpha}'\|$. Thus, a spacelike or a timelike $\overrightarrow{\alpha}$ is said to be parametrized by arclength function s, if $g(\overrightarrow{\alpha}', \overrightarrow{\alpha}') = \pm 1$. Let $\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B_1}, \overrightarrow{B_2}\}$ be the moving Frenet frame along the curve α in E_2^4. Here $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B_1}, \overrightarrow{B_2}$ are the tangent, the principal normal, the first binormal and the second binormal vector fields, respectively. Recall that a spacelike curve $\overrightarrow{\alpha}$ with timelike principal normal $\overrightarrow{N}$ and second binormal B_2. Then the following Frenet equations for the curve α are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equation $g(T,T) = 1,\ g(N,N) = -1,\ g(B_1,B_1) = 1,\ g(B_2,B_2) = -1$ and $g(T,N) = 0,\ g(T,B_1) = 0,\ g(T,B_2) = 0,\ g(N,B_1) = 0,\ g(N,B_2) = 0, g(B_1,B_2) = 0$.

If α is a spacelike curve with a timelike first binormal B_1 and second binormal B_2, then we write

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where $g(T,T) = 1,\ g(N,N) = 1,\ g(B_1,B_1) = -1,\ g(B_2,B_2) = -1$ and $g(T,N) = 0,\ g(T,B_1) = 0,\ g(T,B_2) = 0,\ g(N,B_1) = 0,\ g(N,B_2) = 0,\ g(B_1,B_2) = 0$. Also, here, k_1, k_2 and k_3 are first, second and third curvature of the curve α, respectively.

3 Some characterizations of spacelike curves of constant breadth in E_2^4

Let (C) be a unit speed regular spacelike curve in E_2^4, and $\overrightarrow{X(s)}$ position vector of the curve (C). The normal plane at every point $X(s)$ on the curve meets the curve at a single point $X^*(s)$. If the curve (C) has parallel tangents $\vec{T}$ and $\vec{T}^*$ in opposite direction at the opposite points X and X^* of the curve and the distance between opposite points is always constant then the curve (C) is named a spacelike curve of constant breadth in E_2^4. Furthermore, a pair of spacelike curves (C) and (C^*), for which the tangent vectors at the corresponding points are in opposite directions and parallel, and the distance between corresponding points is always constant, is called a spacelike curve pair of constant breadth in E_2^4.

Assume that C and C^* be a pair of unit speed spacelike curves in E_2^4 with position vectors $\overrightarrow{X(s)}$ and $\vec{X}^*(s^*)$, where s and s^* are length parameters of the curves, respectively, and let C and C^* have parallel tangents in opposite directions at the opposite points. Then the curve C^* can be written by the following equation

$$X^*(s) = X(s) + m_1(s)T(s) + m_2(s)N(s) + m_3(s)B_1(s) + m_4(s)B_2(s) \tag{3.1}$$

where $m_i(s)$, $(1 \le i \le 4)$ are differentiable functions of s. Differentiating equation (3.1) with respect to s, we obtain

$$\begin{aligned} T^*\frac{ds^*}{ds} &= \left(1 + \frac{dm_1}{ds} + m_2k_1\right)T + \left(m_1k_1 + \frac{dm_2}{ds} + m_3k_2\right)N \\ &\quad + \left(m_2k_2 + \frac{dm_3}{ds} + m_4k_3\right)B_1 + \left(m_3k_3 + \frac{dm_4}{ds}\right)B_2. \end{aligned}$$

If we consider $T^* = -T$ at the corresponding points of C and C^*, we have

$$\begin{aligned} 1 + \frac{dm_1}{ds} + m_2k_1 &= -\frac{ds^*}{ds} \\ m_1k_1 + \frac{dm_2}{ds} + m_3k_2 &= 0 \\ m_2k_2 + \frac{dm_3}{ds} + m_4k_3 &= 0 \\ m_3k_3 + \frac{dm_4}{ds} &= 0 \end{aligned} \tag{3.2}$$

Since the curvature of the curve C is $\frac{d\phi}{ds} = k_1(s)$, where $\phi(s) = \int_0^s k_1 ds$ is the angle between tangent vectors of the curve C and a given fixed direction at the point $\alpha(s)$, from (3.2) we get

$$\begin{aligned}\frac{dm_1}{d\phi} &= -m_2 - f(\phi)\\ \frac{dm_2}{d\phi} &= -m_1 - m_3\sigma k_2\\ \frac{dm_3}{d\phi} &= -m_2\sigma k_2 - m_4\sigma k_3\\ \frac{dm_4}{d\phi} &= -m_3\sigma k_3\end{aligned} \tag{3.3}$$

Here, $f(\phi) = \sigma + \sigma^*$ and $\sigma = \frac{1}{k_1}$ and $\sigma^* = \frac{1}{k_1^*}$ are the radius of curvatures at the points $X(s)$ and $X^*(s^*)$, respectively. Using (3.3), we have following equation

$$\begin{aligned}&\frac{d}{d\phi}\left(\frac{1}{\sigma^2 k_2 k_3}\left(\frac{d^3 m_1}{d\phi^3} + \frac{d^2 f}{d\phi^2} - \frac{dm_1}{d\phi}\right)\right)\\ &-\frac{d}{d\phi}\left(\frac{1}{\sigma^3 k_2^2 k_3}\frac{d(\sigma k_2)}{d\phi}\left(\frac{d^2 m_1}{d_\phi^2} + \frac{df}{d\phi} - m_1\right)\right)\\ &-\frac{d}{d\phi}\left(\frac{k_2}{k_3}\left(\frac{dm_1}{d\phi} + f\right)\right) - \frac{k_3}{k_2}\left(\frac{d^2 m_1}{d\phi^2} + \frac{df}{d\phi} - m_1\right) = 0\end{aligned} \tag{3.4}$$

This differential equation is a characterization of constant breadth spacelike curves with timelike principal normal and second binormal in E_2^4

If the distance between the opposite points of C and C^* is constant, from (3.1) we have

$$\|X^* - X\|^2 = m_1^2 - m_2^2 + m_3^2 - m_4^2 = k^2, k \in \mathbb{R}.$$

Thus, we write

$$m_1\frac{dm_1}{d\phi} - m_2\frac{dm_2}{d\phi} + m_3\frac{dm_3}{d\phi} - m_4\frac{dm_4}{d\phi} = 0$$

By using (3.3) we obtain

$$m_1\left(\frac{dm_1}{d\phi} + m_2\right) = 0.$$

Then we have $m_1 = 0$ or $\frac{dm_1}{d\phi} = -m_2$. Hence we can write following system of equations

$$\begin{aligned}m_1 &= 0,\\ \frac{dm_2}{d\phi} &= -m_3\sigma k_2,\\ \frac{dm_3}{d\phi} &= -m_2\sigma k_2 - m_4\sigma k_3,\\ \frac{dm_4}{d\phi} &= -m_3\sigma k_3\end{aligned} \tag{3.5}$$

or

$$\begin{aligned}\frac{dm_1}{d\phi} &= -m_2,\\ \frac{dm_2}{d\phi} &= -m_1 - m_3\sigma k_2,\\ \frac{dm_3}{d\phi} &= -m_2\sigma k_2 - m_4\sigma k_3,\\ \frac{dm_4}{d\phi} &= -m_3\sigma k_3.\end{aligned} \tag{3.6}$$

Suppose that m_1 is a constant in the system (3.6). Then we write following linear differential equations

$$\sigma k_3 \frac{d^2 m_3}{d\phi^2} - \frac{d(\sigma k_3)}{d\phi}\frac{dm_3}{d\phi} - m_3\left(\sigma k_3\right)^3 = 0 \tag{3.7}$$

$$\sigma k_3 \frac{d^2 m_4}{d\phi^2} - \frac{d(\sigma k_3)}{d\phi}\frac{dm_4}{d\phi} - m_4\left(\sigma k_3\right)^3 = 0 \tag{3.8}$$

Changing the variable ϕ of the form $\delta = \int_0^\phi \sigma k_3 dt$, we have

$$\frac{d^2 m_3}{d\delta^2} - m_3 = 0 \tag{3.9}$$

Thus, general solution of m_3 is

$$m_3 = c_1 \cosh \int_0^\phi \sigma k_3 dt + c_2 \sinh \int_0^\phi \sigma k_3 dt \tag{3.10}$$

Also, if we consider m_4, we obtain

$$m_4 = -c_2 \cosh \int_0^\phi \sigma k_3 dt - c_1 \sinh \int_0^\phi \sigma k_3 dt \tag{3.11}$$

where c_1 and c_2 are arbitrary constants. Thus the general solution is given by

$$\begin{aligned} m_1 &= c, \quad m_2 = 0 \\ m_3 &= c_1 \cosh \int_0^\phi \sigma k_3 dt + c_2 \sinh \int_0^\phi \sigma k_3 dt \\ m_4 &= -c_2 \cosh \int_0^\phi \sigma k_3 dt - c_1 \sinh \int_0^\phi \sigma k_3 dt. \end{aligned}$$

Therefore, the breadth of the curve is denoted with $k^2 = c^2 + c_1^2 - c_2^2$.

Suppose that $m_1 = 0$. By changing the variable ϕ of the form $\xi = \int_0^\phi \sigma k_3 dt$, we obtain the following linear differential equation

$$\frac{d^2 m_3}{d\xi^2} + m_3 = \left(f \frac{k_2}{k_3}\right)' \tag{3.12}$$

which has the solutions as

$$m_3 = c_1 \cos \int_0^\phi \sigma k_3 dt + c_2 \sin \int_0^\phi \sigma k_3 dt + \int_0^\phi \cos[\xi(\phi) - \xi(t)]\sigma k_2 f(t) dt. \tag{3.13}$$

In a similar manner, we have

$$m_4 = c_2 \cos \int_0^\phi \sigma k_3 dt - c_1 \sin \int_0^\phi \sigma k_3 dt - \int_0^\phi \sin[\xi(\phi) - \xi(t)]\sigma k_2 f(t) dt \tag{3.14}$$

Furthermore, from (3.4) we can write

$$\begin{aligned} &\frac{d}{d\phi}\left(\frac{1}{\sigma^2 k_2 k_3}\left(\frac{d^2 f}{d\phi^2}\right)\right) - \frac{d}{d\phi}\left(\frac{1}{\sigma^3 k_2^2 k_3}\frac{d(\sigma k_2)}{d\phi}\left(\frac{df}{d\phi}\right)\right) \\ &\qquad - \frac{d}{d\phi}\left(\frac{k_2}{k_3} f\right) - \frac{k_3}{k_2}\left(\frac{df}{d\phi}\right) = 0 \end{aligned} \tag{3.15}$$

Remark 3.1. If $\frac{k_2}{k_3}$ is a constant in equation (3.15), we get

$$\begin{aligned} &\frac{d}{d\phi}\left(\frac{1}{\sigma^2 k_2 k_3}\left(\frac{d^2 f}{d\phi^2}\right)\right) - \frac{d}{d\phi}\left(\frac{1}{\sigma^3 k_2^2 k_3}\frac{d(\sigma k_2)}{d\phi}\left(\frac{df}{d\phi}\right)\right) \\ &\qquad - \left(\frac{a^2+1}{a}\right)\frac{df}{d\phi} = 0 \end{aligned} \tag{3.16}$$

where $\frac{k_2}{k_3} = a$.

Now, suppose that α is a spacelike curve with timelike first binormal and second binormal, then we obtain

$$\begin{aligned}\frac{dm_1}{d\phi} &= m_2 - f(\phi)\\ \frac{dm_2}{d\phi} &= -m_1 - m_3 k_2 \sigma\\ \frac{dm_3}{d\phi} &= -m_2 k_2 \sigma + m_4 k_3 \sigma\\ \frac{dm_4}{d\phi} &= -m_3 k_3 \sigma\end{aligned} \tag{3.17}$$

From (3.17), we arrive at the following differential equation characterizing constant breadth spacelike curves in E_2^4.

$$\begin{aligned}&-\frac{d}{d\phi}\left(\frac{1}{\sigma^2 k_2 k_3}\left(\frac{d^3 m_1}{d\phi^3}+\frac{d^2 f}{d\phi^2}+\frac{dm_1}{d\phi}\right)\right)\\ &+\frac{d}{d\phi}\left(\frac{1}{\sigma^3 k_2^2 k_3}\frac{d(\sigma k_2)}{d\phi}\left(\frac{d^2 m_1}{d\phi^2}+\frac{df}{d\phi}+m_1\right)\right)\\ &+\frac{d}{d\phi}\left(\frac{k_2}{k_3}\left(\frac{dm_1}{d\phi}+f\right)\right)-\frac{k_3}{k_2}\left(\frac{d^2 m_1}{d\phi^2}+\frac{df}{d\phi}+m_1\right)=0.\end{aligned} \tag{3.18}$$

Also from (3.1), we can write

$$m_1 = 0,\ \frac{dm_2}{d\phi} = -m_3 k_2 \sigma,\ \frac{dm_3}{d\phi} = -m_2 k_2 \sigma + m_4 k_3 \sigma,\ \frac{dm_4}{d\phi} = -m_3 k_3 \sigma$$

and

$$\frac{dm_1}{d\phi} = m_2,\ \frac{dm_2}{d\phi} = -m_1 - m_3 k_2 \sigma,\ \frac{dm_3}{d\phi} = -m_2 k_2 \sigma + m_4 k_3 \sigma,\ \frac{dm_4}{d\phi} = -m_3 k_3 \sigma.$$

Therefore we get

$$\sigma k_3 \frac{d^2 m_3}{d\phi^2} - \frac{d(\sigma k_3)}{d\phi}\frac{dm_3}{d\phi} + m_3(\sigma k_3)^3 = 0 \tag{3.19}$$

or

$$\sigma k_3 \frac{d^2 m_4}{d\phi^2} - \frac{d(\sigma k_3)}{d\phi}\frac{dm_4}{d\phi} + m_4(\sigma k_3)^3 = 0. \tag{3.20}$$

Changing the variable ϕ of the form ξ , we have

$$\frac{d^2 m_3}{d\xi^2} + m_3 = 0 \text{ and } \frac{d^2 m_4}{d\xi^2} + m_4 = 0 \tag{3.21}$$

Using (3.21), the general solutions of the differential equations are

$$\begin{aligned}m_3 &= c_1 \cos\left(\int_0^\phi \sigma k_3 dt\right) + c_2 \sin\left(\int_0^\phi \sigma k_3 dt\right)\\ m_4 &= -c_1 \cos\left(\int_0^\phi \sigma k_3 dt\right) + c_2 \sin\left(\int_0^\phi \sigma k_3 dt\right).\end{aligned}$$

Thus, the solution of the system (3.21) can be written as

$$\begin{aligned}m_1 &= c = constant,\ m_2 = 0,\\ m_3 &= c_1 \cos\left(\int_0^\phi \sigma k_3 dt\right) + c_2 \sin\left(\int_0^\phi \sigma k_3 dt\right)\\ m_4 &= -c_1 \cos\left(\int_0^\phi \sigma k_3 dt\right) + c_2 \sin\left(\int_0^\phi \sigma k_3 dt\right)\end{aligned}$$

Here the breadth of the curve is denoted with $k^2 = c^2 - c_1^2 - c_2^2$.

Also, for $m_1 = 0$, we arrive the following linear differential equation

$$\frac{d^2 m_3}{d\xi^2} + m_3 = \left(f \frac{k_2}{k_3}\right)'$$

having the solution as

$$m_3 = c_1 cosh \int_0^{\phi} \sigma k_3 dt + c_2 \sinh \int_0^{\phi} \sigma k_3 dt - \int_0^{\phi} \cosh[\xi(\phi) - \xi(t)]\sigma k_2 f(t) dt$$

In a similar manner, we have

$$m_4 = -c_2 cosh \int_0^{\phi} \sigma k_3 dt - c_1 \sinh \int_0^{\phi} \sigma k_3 dt + \int_0^{\phi} \sinh[\xi(\phi) - \xi(t)]\sigma k_2 f(t) dt$$

Furthermore, since $m_1 = 0$, we can write

$$-\frac{d}{d\phi}\left(\frac{1}{\sigma^2 k_2 k_3}\left(\frac{d^2 f}{d\phi^2}\right)\right) + \frac{d}{d\phi}\left(\frac{1}{\sigma^3 k_2^2 k_3}\frac{d(\sigma k_2)}{d\phi}\left(\frac{df}{d\phi}\right)\right) + \frac{d}{d\phi}\left(\frac{k_2}{k_3}(f)\right) - \frac{k_3}{k_2}\left(\frac{df}{d\phi}\right) = 0. \tag{3.22}$$

Remark 3.2. If $\frac{k_2}{k_3}$ is a constant in equation (3.22), then we write

$$-\frac{d}{d\phi}\left(\frac{1}{\sigma^2 k_2 k_3}\left(\frac{d^2 f}{d\phi^2}\right)\right) + \frac{d}{d\phi}\left(\frac{1}{\sigma^3 k_2^2 k_3}\frac{d(\sigma k_2)}{d\phi}\left(\frac{df}{d\phi}\right)\right) + \left(\frac{a^2 - 1}{a}\right)\frac{df}{d\phi} = 0.$$

where $\frac{k_2}{k_3} = a$.

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