SPACELIKE CURVES OF CONSTANT BREADTH IN SEMI-RIEMANNIAN SPACE $E^4_2$

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Abstract In this paper, we investigate curves of constant breadth in $E^4_2$. Also, we obtain some characterizations according to the state of the spacelike curve in semi-Riemannian space $E^4_2$.

1 Introduction

The curves of constant breadth were first defined in 1778 by Euler. Then, Solow [12] and Blascke [1] investigated the curves of constant breadth. In Euclidean spaces $E^3$ and $E^4$, plane curves of constant breadth were studied by Kose [7], Magden and Yılmaz [8]. In [14], some geometric properties of plane curves of constant breadth in Minkowski 3-space were given. Also, these curves in Minkowski 4-space were obtained by Kazaz, Onder and Kocayigit [5]. A number of authors have, recently, studied the curves of constant breadth under different conditions (see [4, 6]).

In this study, we investigate the spacelike curves of constant breadth with timelike normal and first binormal and with timelike binormal and second binormal in $E^4_2$. Then we give some differential equations for these curves in semi-Riemannian space.

2 Preliminaries

In this section, we provide a brief view of the theory of curves in the semi-Riemannian space $E^4_2$. This space is an Euclidean space $E^4$ provided with the standard flat metric given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$$

where $(x_1, x_2, x_3, x_4)$ is rectangular coordinate system in $E^4_2$ [14]. An any vector $\mathbf{v} \in E^4_2$ can have one of the three causal characters; it is spacelike if $g(\mathbf{v}, \mathbf{v}) > 0$ or $\mathbf{v} = 0$, timelike if $g(\mathbf{v}, \mathbf{v}) < 0$ and null or lightlike if $g(\mathbf{v}, \mathbf{v}) = 0$ and $\mathbf{v} \neq 0$. Similarly, any an curve $\alpha(s)$ in $E^2_2$ can locally be spacelike, timelike or null if its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively. Furthermore, the norm of a vector $\mathbf{v}$ is given by $\|\mathbf{v}\| = \sqrt{g(\mathbf{v}, \mathbf{v})}$. Thus, $\mathbf{v}$ is a unit vector if $g(\mathbf{v}, \mathbf{v}) = \pm 1$. The velocity of the curve $\alpha$ is given by $\|\alpha'(s)\|$. Thus, a spacelike or a timelike $\alpha$ is said to be parametrized by arclength function $s$, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ be the moving Frenet frame along the curve $\alpha$ in $E^4_2$. Here $\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2$ are the tangent, the principal normal, the first binormal and the second binormal vector fields, respectively. Recall that a spacelike curve $\alpha$ with timelike principal normal $\mathbf{N}$ and second binormal $\mathbf{B}_2$. Then the following Frenet equations for the curve $\alpha$ are given by

$$
\begin{bmatrix}
T' \\
N' \\
B'_1 \\
B'_2
\end{bmatrix} =
\begin{bmatrix}
0 & k_1 & 0 & 0 \\
k_1 & 0 & k_2 & 0 \\
0 & k_2 & 0 & k_3 \\
0 & 0 & k_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}
$$
where \( T, N, B_1 \) and \( B_2 \) are mutually orthogonal vectors satisfying equation \( g(T, T) = 1, \ g(N, N) = -1, \ g(B_1, B_1) = 1, \ g(B_2, B_2) = -1 \) and \( g(T, N) = 0, \ g(T, B_1) = 0, \ g(T, B_2) = 0, \ g(N, B_1) = 0, \ g(N, B_2) = 0, \ g(B_1, B_2) = 0. \)

If \( \alpha \) is a spacelike curve with a timelike first binormal \( B_1 \) and second binormal \( B_2 \), then we write

\[
\begin{bmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{bmatrix} =
\begin{bmatrix}
k_1 & 0 & 0 \\
-k_1 & k_2 & 0 \\
0 & k_2 & k_3 \\
0 & 0 & -k_3
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}
\]

where \( g(T, T) = 1, \ g(N, N) = 1, \ g(B_1, B_1) = -1, \ g(B_2, B_2) = -1 \) and \( g(T, N) = 0, \ g(T, B_1) = 0, \ g(T, B_2) = 0, \ g(N, B_1) = 0, \ g(N, B_2) = 0, \ g(B_1, B_2) = 0. \) Also, here, \( k_1, k_2 \) and \( k_3 \) are first, second and third curvature of the curve \( \alpha \), respectively.

3 Some characterizations of spacelike curves of constant breadth in \( E_2^4 \)

Let \( (C) \) be a unit speed regular spacelike curve in \( E_2^4 \), and \( \overrightarrow{X}(s) \) position vector of the curve \( (C) \). The normal plane at every point \( X(s) \) on the curve meets the curve at a single point \( X^*(s) \). If the curve \( (C) \) has parallel tangents \( \overrightarrow{T} \) and \( \overrightarrow{T^*} \) in opposite direction at the opposite points \( X \) and \( X^* \) of the curve and the distance between the opposite points is always constant then the curve \( (C) \) is named a spacelike curve of constant breadth in \( E_2^4 \). Furthermore, a pair of spacelike curves \( (C) \) and \( (C^*) \), for which the tangent vectors at the corresponding points are in opposite directions and parallel, and the distance between corresponding points is always constant, is called a spacelike curve pair of constant breadth in \( E_2^4 \).

Assume that \( C \) and \( C^* \) be a pair of unit speed spacelike curves in \( E_2^4 \) with position vectors \( \overrightarrow{X}(s) \) and \( \overrightarrow{X^*}(s^*) \), where \( s \) and \( s^* \) are length parameters of the curves, respectively, and let \( C \) and \( C^* \) have parallel tangents in opposite directions at the opposite points. Then the curve \( C^* \) can be written by the following equation

\[
X^*(s) = X(s) + m_1(s)T(s) + m_2(s)N(s) + m_3(s)B_1(s) + m_4(s)B_2(s)
\]  \hspace{1cm} (3.1)

where \( m_i(s), \ (1 \leq i \leq 4) \) are differentiable functions of \( s \). Differentiating equation (3.1) with respect to \( s \), we obtain

\[
T^* \frac{ds^*}{ds} = \left(1 + \frac{dm_1}{ds} + m_2k_1\right)T + \left(m_1k_1 + \frac{dm_2}{ds} + m_3k_2\right)N + \left(m_2k_2 + \frac{dm_3}{ds} + m_4k_3\right)B_1 + \left(m_3k_3 + \frac{dm_4}{ds}\right)B_2.
\]

If we consider \( T^* = -T \) at the corresponding points of \( C \) and \( C^* \), we have

\[
1 + \frac{dm_1}{ds} + m_2k_1 = \frac{ds^*}{ds},
\]

\[
m_1k_1 + \frac{dm_2}{ds} + m_3k_2 = 0,
\]

\[
m_2k_2 + \frac{dm_3}{ds} + m_4k_3 = 0
\]

\[
m_3k_3 + \frac{dm_4}{ds} = 0
\]  \hspace{1cm} (3.2)

Since the curvature of the curve \( C \) is \( \frac{d\phi}{ds} = k_1(s) \), where \( \phi(s) = \int_0^s k_1 ds \) is the angle between
tangent vectors of the curve \( C \) and a given fixed direction at the point \( \alpha(s) \), from (3.2) we get
\[
\frac{d m_1}{d \phi} = -m_2 - f(\phi) \\
\frac{d m_2}{d \phi} = -m_1 - m_3 \sigma k_2 \\
\frac{d m_3}{d \phi} = -m_2 \sigma k_2 - m_4 \sigma k_3 \\
\frac{d m_4}{d \phi} = -m_3 \sigma k_3
\] (3.3)

Here, \( f(\phi) = \sigma + \sigma^* \) and \( \sigma = \frac{1}{k_2} \) and \( \sigma^* = \frac{1}{k^*} \) are the radius of curvatures at the points \( X(s) \) and \( X^*(s^*) \), respectively. Using (3.3), we have following equation
\[
\frac{d}{d \phi} \left( \frac{1}{\sigma^2 k_2 k_3} \left( \frac{d^3 m_1}{d \phi^3} + \frac{d^2 f}{d \phi^2} - \frac{d m_1}{d \phi} \right) \right) \\
- \frac{d}{d \phi} \left( \frac{1}{\sigma^3 k_2^2 k_3} \frac{d (\sigma k_2)}{d \phi} \left( \frac{d^2 m_1}{d \phi^2} + \frac{d f}{d \phi} - m_1 \right) \right) \\
- \frac{d}{d \phi} \left( \frac{k_2}{k_3} \left( \frac{d m_1}{d \phi} + f \right) \right) - \frac{k_3}{k_2} \left( \frac{d^2 m_1}{d \phi^2} + \frac{d f}{d \phi} - m_1 \right) = 0
\] (3.4)

This differential equation is a characterization of constant breadth spacelike curves with timelike principal normal and second binormal in \( E^d_4 \).

If the distance between the opposite points of \( C \) and \( C^* \) is constant, from (3.1) we have
\[
\|X^* - X\|^2 = m_1^2 - m_2^2 + m_3^2 - m_4^2 = k^2, \ k \in \mathbb{R}.
\]

Thus, we write
\[
m_1 \frac{d m_1}{d \phi} - m_2 \frac{d m_2}{d \phi} + m_3 \frac{d m_3}{d \phi} - m_4 \frac{d m_4}{d \phi} = 0
\]

By using (3.3) we obtain
\[
m_1 \left( \frac{d m_1}{d \phi} + m_2 \right) = 0.
\]

Then we have \( m_1 = 0 \) or \( \frac{d m_2}{d \phi} = -m_2 \). Hence we can write following system of equations
\[
m_1 = 0, \\
\frac{d m_2}{d \phi} = -m_3 \sigma k_2, \\
\frac{d m_3}{d \phi} = -m_2 \sigma k_2 - m_4 \sigma k_3, \\
\frac{d m_4}{d \phi} = -m_3 \sigma k_3
\] (3.5)
or
\[
\frac{d m_1}{d \phi} = -m_2, \\
\frac{d m_2}{d \phi} = -m_1 - m_3 \sigma k_2, \\
\frac{d m_3}{d \phi} = -m_2 \sigma k_2 - m_4 \sigma k_3, \\
\frac{d m_4}{d \phi} = -m_3 \sigma k_3
\] (3.6)
Suppose that $m_1$ is a constant in the system (3.6). Then we write following linear differential equations

$$
\sigma k_3 \frac{d^2 m_3}{d\phi^2} - \frac{d(\sigma k_3)}{d\phi} \frac{dm_3}{d\phi} - m_3 (\sigma k_3)^3 = 0
$$

(3.7)

$$
\sigma k_3 \frac{d^2 m_4}{d\phi^2} - \frac{d(\sigma k_3)}{d\phi} \frac{dm_4}{d\phi} - m_4 (\sigma k_3)^3 = 0
$$

(3.8)

Changing the variable $\phi$ of the form $\delta = \int_0^\phi \sigma k_3 dt$, we have

$$
\frac{d^2 m_3}{d\delta^2} - m_3 = 0
$$

(3.9)

Thus, general solution of $m_3$ is

$$
m_3 = c_1 \cosh \int_0^\phi \sigma k_3 dt + c_2 \sinh \int_0^\phi \sigma k_3 dt
$$

(3.10)

Also, if we consider $m_4$, we obtain

$$
m_4 = -c_2 \cosh \int_0^\phi \sigma k_3 dt - c_1 \sinh \int_0^\phi \sigma k_3 dt
$$

(3.11)

where $c_1$ and $c_2$ are arbitrary constants. Thus the general solution is given by

$$
m_3 = c_1 \cosh \int_0^\phi \sigma k_3 dt + c_2 \sinh \int_0^\phi \sigma k_3 dt
$$

$$
m_4 = -c_2 \cosh \int_0^\phi \sigma k_3 dt - c_1 \sinh \int_0^\phi \sigma k_3 dt.
$$

Therefore, the breadth of the curve is denoted with $k^2 = c_1^2 + c_2^2$.

Suppose that $m_1 = 0$. By changing the variable $\phi$ of the form $\xi = \int_0^\phi \sigma k_3 dt$, we obtain the following linear differential equation

$$
\frac{d^2 m_3}{d\xi^2} + m_3 = \left( \frac{k_2}{k_3} \right)'
$$

(3.12)

which has the solutions as

$$
m_3 = c_1 \cos \int_0^\phi \sigma k_3 dt + c_2 \sin \int_0^\phi \sigma k_3 dt + \int_0^\phi \cos [\xi(\phi) - \xi(t)]\sigma k_2 f(t) dt.
$$

(3.13)

In a similar manner, we have

$$
m_4 = c_2 \cos \int_0^\phi \sigma k_3 dt - c_1 \sin \int_0^\phi \sigma k_3 dt - \int_0^\phi \sin [\xi(\phi) - \xi(t)]\sigma k_2 f(t) dt.
$$

(3.14)

Furthermore, from (3.4) we can write

$$
\frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \left( \frac{d^2 f}{d\phi^2} \right) \right) - \frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \frac{d(\sigma k_2)}{d\phi} \left( \frac{df}{d\phi} \right) \right) + \frac{d}{d\phi} \left( \frac{k_2}{k_3} \frac{df}{d\phi} \right) - \frac{k_3}{k_2} \frac{df}{d\phi} = 0
$$

(3.15)

**Remark 3.1.** If $\frac{k_2}{k_3}$ is a constant in equation (3.15), we get

$$
\frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \left( \frac{d^2 f}{d\phi^2} \right) \right) - \frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \frac{d(\sigma k_2)}{d\phi} \left( \frac{df}{d\phi} \right) \right) + \left( \frac{c^2 + 1}{a} \right) \frac{df}{d\phi} = 0
$$

(3.16)

where $\frac{k_2}{k_3} = a$. 
Now, suppose that $\alpha$ is a spacelike curve with timelike first binormal and second binormal, then we obtain
\[
\frac{dm_1}{d\phi} = m_2 - f(\phi) \\
\frac{dm_2}{d\phi} = -m_1 - m_3 k_2 \sigma \\
\frac{dm_3}{d\phi} = -m_2 k_2 \sigma + m_4 k_3 \sigma \\
\frac{dm_4}{d\phi} = -m_3 k_3 \sigma 
\]  \hspace{1cm} (3.17)

From (3.17), we arrive at the following differential equation characterizing constant breadth spacelike curves in $E^4_2$:
\[
-\frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \left( \frac{d^2 m_1}{d\phi^2} + \frac{d^2 f}{d\phi^2} + \frac{dm_1}{d\phi} \right) \right) + \frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \left( \frac{d (\sigma k_2)}{d\phi} \left( \frac{d^2 m_1}{d\phi^2} + \frac{df}{d\phi} + m_1 \right) \right) \right) + \frac{d}{d\phi} \left( \frac{k_2}{k_3} \left( \frac{dm_1}{d\phi} + f \right) \right) - \frac{k_3}{k_2} \left( \frac{d^2 m_1}{d\phi^2} + \frac{df}{d\phi} + m_1 \right) = 0. \hspace{1cm} (3.18)
\]

Also from (3.1), we can write
\[
m_1 = 0, \quad \frac{dm_1}{d\phi} = -m_3 k_2 \sigma, \quad \frac{dm_2}{d\phi} = -m_2 k_2 \sigma + m_4 k_3 \sigma, \quad \frac{dm_4}{d\phi} = -m_3 k_3 \sigma
\]
and
\[
\frac{dm_1}{d\phi} = m_2, \quad \frac{dm_2}{d\phi} = -m_1 - m_3 k_2 \sigma, \quad \frac{dm_3}{d\phi} = -m_2 k_2 \sigma + m_4 k_3 \sigma, \quad \frac{dm_4}{d\phi} = -m_3 k_3 \sigma.
\]

Therefore we get
\[
\sigma k_3 \frac{d^2 m_3}{d\sigma^2} - \frac{d (\sigma k_3)}{d\phi} \frac{dm_3}{d\phi} + m_3 (\sigma k_3)^3 = 0 \hspace{1cm} (3.19)
\]
or
\[
\sigma k_3 \frac{d^2 m_4}{d\sigma^2} - \frac{d (\sigma k_3)}{d\phi} \frac{dm_4}{d\phi} + m_4 (\sigma k_3)^3 = 0. \hspace{1cm} (3.20)
\]

Changing the variable $\phi$ of the form $\xi$, we have
\[
\frac{d^2 m_3}{d\xi^2} + m_3 = 0 \quad \text{and} \quad \frac{d^2 m_4}{d\xi^2} + m_4 = 0 \hspace{1cm} (3.21)
\]

Using (3.21), the general solutions of the differential equations are
\[
m_3 = c_1 \cos \left( \int_0^\phi \sigma k_3 dt \right) + c_2 \sin \left( \int_0^\phi \sigma k_3 dt \right) \\
m_4 = -c_1 \cos \left( \int_0^\phi \sigma k_3 dt \right) + c_2 \sin \left( \int_0^\phi \sigma k_3 dt \right).
\]

Thus, the solution of the system (3.21) can be written as
\[
m_1 = c = \text{constant}, \quad m_2 = 0, \\
m_3 = c_1 \cos \left( \int_0^\phi \sigma k_3 dt \right) + c_2 \sin \left( \int_0^\phi \sigma k_3 dt \right) \\
m_4 = -c_1 \cos \left( \int_0^\phi \sigma k_3 dt \right) + c_2 \sin \left( \int_0^\phi \sigma k_3 dt \right).
\]
Here the breadth of the curve is denoted with $k^2 = c^2 - c_1^2 - c_2^2$. Also, for $m_1 = 0$, we arrive at the following linear differential equation

$$\frac{d^2 m_3}{d\xi^2} + m_3 = \left( \frac{f_{k_2}}{k_3} \right)'$$

having the solution as

$$m_3 = c_1 \cosh \int_0^\phi \sigma k_3 dt + c_2 \sinh \int_0^\phi \sigma k_3 dt - \int_0^\phi \cosh (\xi(\phi) - \xi(t)) \sigma k_2 f(t) dt$$

In a similar manner, we have

$$m_4 = -c_2 \cosh \int_0^\phi \sigma k_3 dt - c_1 \sinh \int_0^\phi \sigma k_3 dt + \int_0^\phi \sinh (\xi(\phi) - \xi(t)) \sigma k_2 f(t) dt$$

Furthermore, since $m_1 = 0$, we can write

$$-\frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \left( \frac{d^2 f}{d\phi^2} \right) \right) + \frac{d}{d\phi} \left( \frac{1}{\sigma^3 k_2^2 k_3} \frac{d}{d\phi} (\sigma k_2) \left( \frac{df}{d\phi} \right) \right)$$

$$+ \frac{d}{d\phi} \left( \frac{k_2}{k_3} (f) \right) - \frac{k_3}{k_2} \left( \frac{df}{d\phi} \right) = 0. \quad (3.22)$$

**Remark 3.2.** If $\frac{k_2}{k_3}$ is a constant in equation (3.22), then we write

$$-\frac{d}{d\phi} \left( \frac{1}{\sigma^2 k_2 k_3} \left( \frac{d^2 f}{d\phi^2} \right) \right) + \frac{d}{d\phi} \left( \frac{1}{\sigma^3 k_2^2 k_3} \frac{d}{d\phi} (\sigma k_2) \left( \frac{df}{d\phi} \right) \right)$$

$$+ \left( \frac{a^2 - 1}{a} \right) \frac{df}{d\phi} = 0.$$

where $\frac{k_2}{k_3} = a$.

**References**


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