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Some common fixed point theorems using rational contraction in complex valued metric spaces

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Abstract In this paper, we prove some common fixed point theorems using rational contraction in the setting of complex valued metric spaces. The results presented in this paper extend and generalize the corresponding results given in the existing literature.

1 Introduction

Fixed point theory plays a very significant role in the development of nonlinear analysis. The Banach contraction principle [4] is a very popular tool in solving existence problems in many branches of mathematics. This famous theorem can be stated as follows.

Theorem 1.1. *([4]) Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \leq k\, d(x, y),\ \forall\, x, y \in X \tag{1.1}$$

where k is a constant in $[0, 1)$. Then T has a fixed point $x^* \in X$.

The Banach contraction principle with rational expressions have been expanded and some fixed point and common fixed point theorems have been obtained in [5], [6].

In the existing literature, there are a great number of generalizations of the Banach contraction principle (see [1, 2] and others).

In 2011, Azam et al. [3] introduced the concept of complex valued metric space and established some fixed point results for mappings satisfying a rational inequality. Complex-valued metric space is useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, mechanical engineering and electrical engineering, for some details, see ([8, 9]).

In this paper, we prove some common fixed point theorems using rational contraction in the framework of complex valued metric spaces.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;

(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \lesssim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

The following definition was introduced by Azam et al. in 2011 (see, [3]).

Definition 2.1. ([3]) Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to \mathbb{C}$ satisfies:

(C_1) $0 \precsim d(x,y)$ for all $x, y \in X$ with $x \neq y$ and $d(x,y) = 0 \ \Leftrightarrow\ x = y$;

(C_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(C_3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 2.2. Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{it}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $t \in [0, \frac{\pi}{2}]$. Then (X,d) is a complex valued metric space.

Definition 2.3. (i)A point $x \in X$ is called an interior point of a subset $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) = \{y \in X :\ d(x,y) \prec r\} \subseteq A.$$

(ii) A point $x \in X$ is called a limit of A whenever for every $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) \cap \Big(A - \{x\}\Big) \neq \emptyset.$$

(iii) The set A is called open whenever each element of A is an interior point of A. A subset B is called closed whenever each limit point of B belongs to B.

(iv)The family $\mathcal{F} := \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Definition 2.4. ([3]) Let (X,d) be a complex valued metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is called convergent, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. Also, $\{x_n\}$ converges to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) and x is the limit of $\{x_n\}$.

(ii) $\{x_n\}$ is called a Cauchy sequence in X, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$. If every Cauchy sequence converges in X, then X is called a complete complex valued metric space.

Definition 2.5. ([7]) Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are said to be pairwise commuting if (i) $T_iT_j = T_jT_i$, $i, j \in \{1, 2, \ldots, m\}$; (ii) $S_kS_l = S_lS_k$, $k, l \in \{1, 2, \ldots, n\}$; (iii) $T_iS_k = S_kT_i$, $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

Lemma 2.6. *([3]) Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if* $\lim_{n\to\infty} |d(x_n, x)| = 0$.

Lemma 2.7. *([3]) Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if* $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.

3 Main Results

In this section we shall prove some common fixed point theorems using rational contraction in the framework of complex valued metric spaces.

Theorem 3.1. *Let (X, d) be a complete complex valued metric space. Suppose that the mappings $S, T\colon X \to X$ satisfy:*

$$d(Sx, Ty) \precsim k\Big[\frac{d(x, Sx)d(x, Ty) + [d(x, y)]^2 + d(x, Sx)d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)}\Big] \tag{3.1}$$

for all $x, y \in X$ such that $x \neq y$, $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$, where $k \in [0, 1)$ is a constant or $d(Sx, Ty) = 0$ if $d(x, Sx) + d(x, y) + d(x, Ty) = 0$. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X and define

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ n = 0, 1, 2, \ldots.$$

Then from (3.1), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim k\Big[\Big(d(x_{2n}, Sx_{2n})d(x_{2n}, Tx_{2n+1}) + [d(x_{2n}, x_{2n+1})]^2 \\ &\quad + d(x_{2n}, Sx_{2n})d(x_{2n}, x_{2n+1})\Big) \\ &\quad \times\Big(d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tx_{2n+1})\Big)^{-1}\Big] \\ &= k\Big[\Big(d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2}) + [d(x_{2n}, x_{2n+1})]^2 \\ &\quad + d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1})\Big) \\ &\quad \times\Big(d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})\Big)^{-1}\Big] \\ &= k\, d(x_{2n}, x_{2n+1}) \\ &\quad \times\Big[\frac{d(x_{2n}, x_{2n+2}) + 2d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+2}) + 2d(x_{2n}, x_{2n+1})}\Big] \\ &= k\, d(x_{2n}, x_{2n+1}). \end{aligned} \tag{3.2}$$

Similarly, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Sx_{2n-1}, Tx_{2n}) \\ &\precsim k\Big[\Big(d(x_{2n-1}, Sx_{2n-1})d(x_{2n-1}, Tx_{2n}) + [d(x_{2n-1}, x_{2n})]^2 \\ &\quad + d(x_{2n-1}, Sx_{2n-1})d(x_{2n-1}, x_{2n})\Big) \\ &\quad \times\Big(d(x_{2n-1}, Sx_{2n-1}) + d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, Tx_{2n})\Big)^{-1}\Big] \\ &= k\Big[\Big(d(x_{2n-1}, x_{2n})d(x_{2n-1}, x_{2n+1}) + [d(x_{2n-1}, x_{2n})]^2 \\ &\quad + d(x_{2n-1}, x_{2n})d(x_{2n-1}, x_{2n})\Big) \\ &\quad \times\Big(d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, x_{2n+1})\Big)^{-1}\Big] \\ &= k\, d(x_{2n-1}, x_{2n}) \\ &\quad \times\Big[\frac{d(x_{2n-1}, x_{2n+1}) + 2d(x_{2n-1}, x_{2n})}{d(x_{2n-1}, x_{2n+1}) + 2d(x_{2n-1}, x_{2n})}\Big] \\ &= k\, d(x_{2n-1}, x_{2n}). \end{aligned} \tag{3.3}$$

By induction, we have

$$\begin{aligned} d(x_{n+1},x_n) &\precsim k\,d(x_n,x_{n-1}) \precsim k^2\,d(x_{n-1},x_{n-2}) \precsim \cdots \\ &\precsim k^n\,d(x_1,x_0). \end{aligned} \tag{3.4}$$

Let $m,n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n,x_m) &\precsim d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + d(x_{n+2},x_{n+3}) \\ &\quad + \cdots + d(x_{n+m-1},x_m) \\ &\precsim [k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+m-1}]d(x_1,x_0) \\ &\precsim \Big[\frac{k^n}{1-k}\Big]d(x_1,x_0) \end{aligned}$$

and so

$$|d(x_n,x_m)| \leq \Big[\frac{k^n}{1-k}\Big]|d(x_1,x_0)| \to 0 \text{ as } m,n\to\infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. It follows that $p = Sp$, otherwise $d(p,Sp) = z > 0$ and we would then have

$$\begin{aligned} z &\precsim d(p,x_{2n+2}) + d(x_{2n+2},Sp) \\ &= d(p,x_{2n+2}) + d(Sp,Tx_{2n+1}) \\ &\precsim d(p,x_{2n+2}) \\ &\quad +k\Big[\frac{d(p,Sp)d(p,Tx_{2n+1}) + [d(p,x_{2n+1})]^2 + d(p,Sp)d(p,x_{2n+1})}{d(p,Sp) + d(p,x_{2n+1}) + d(p,Tx_{2n+1})}\Big] \\ &= d(p,x_{2n+2}) \\ &\quad +k\Big[\frac{d(p,Sp)d(p,x_{2n+2}) + [d(p,x_{2n+1})]^2 + d(p,Sp)d(p,x_{2n+1})}{d(p,Sp) + d(p,x_{2n+1}) + d(p,x_{2n+2})}\Big]. \end{aligned}$$

This implies that

$$\begin{aligned} |z| &\leq |d(p,x_{2n+2})| \\ &\quad +k\Big[\frac{|z||d(p,x_{2n+2})| + [|d(p,x_{2n+1})|]^2 + |z||d(p,x_{2n+1})|}{|z| + |d(p,x_{2n+1})| + |d(p,x_{2n+2})|}\Big]. \end{aligned}$$

Letting $n \to \infty$, it follows that

$$|z| \leq 0$$

which is a contradiction and so $|z| = 0$, that is, $p = Sp$.

In an exactly the similar way, we can prove that $p = Tp$. Hence $Sp = Tp = p$. This shows that p is a common fixed point of S and T.

To prove uniqueness of common fixed point of S and T, assume that p^* is another common fixed point of S and T, that is, $Sp^* = Tp^* = p^*$ such that $p \neq p^*$. Then

$$\begin{aligned} d(p,p^*) &= d(Sp,Tp^*) \\ &\precsim k\Big[\frac{d(p,Sp)d(p,Tp^*) + [d(p,p^*)]^2 + d(p,Sp)d(p,p^*)}{d(p,Sp) + d(p,p^*) + d(p,Tp^*)}\Big] \\ &= k\Big[\frac{d(p,p)d(p,p^*) + [d(p,p^*)]^2 + d(p,p)d(p,p^*)}{d(p,p) + d(p,p^*) + d(p,p^*)}\Big] \end{aligned}$$

so that $|d(p,p^*)| \leq \frac{k}{2}\,|d(p,p^*)| \leq k\,|d(p,p^*)| < |d(p,p^*)|$, since $0 < k < 1$, which is a contradiction and hence $d(p,p^*) = 0$. Thus $p = p^*$, which proves the uniqueness of common fixed point.

Secondly, we consider the case: $d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tx_{2n+1}) = 0$ (for any n) implies $d(Sx_{2n}, Tx_{2n+1}) = 0$, so that $x_{2n} = Sx_{2n} = x_{2n+1} = Tx_{2n+1} = x_{2n+2}$. Thus, we have $x_{2n+1} = Sx_{2n} = x_{2n}$, so there exist k_1 and l_1 such that $k_1 = Sl_1 = l_1$. Using the same arguments as above, one can also show that there exist k_2 and l_2 such that $k_2 = Tl_2 = l_2$. As $d(l_1, Sl_1) + d(l_1, l_2) + d(l_1, Tl_2) = 0$ (according to the definition) implies $d(Sl_1, Tl_2) = 0$, so that $k_1 = Sl_1 = Tl_2 = k_2$ which in turn yields that $k_1 = Sl_1 = Sk_1$. Similarly, one can also have $k_2 = Tk_2$. As $k_1 = k_2$ implies $Sk_1 = Tk_1 = k_1$, therefore $k_1 = k_2$ is a common fixed point of S and T.

We now prove that S and T have unique common fixed point. For this, assume that k_1^* in X is another common fixed point of S and T, that is, $Sk_1^* = Tk_1^* = k_1^*$ such that $k_1 \neq k_1^*$. As $d(k_1, Sk_1) + d(k_1, k_1^*) + d(k_1, Tk_1^*) = 0$, therefore $d(k_1, k_1^*) = d(Sk_1, Tk_1^*) = 0$. This implies that $k_1^* = k_1$. This completes the proof. □

Putting $S = T$ in Theorem 3.1, we have the following result.

Corollary 3.2. *Let (X, d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx, Ty) \precsim k\Big[\frac{d(x, Tx)d(x, Ty) + [d(x, y)]^2 + d(x, Tx)d(x, y)}{d(x, Tx) + d(x, y) + d(x, Ty)}\Big]$$

for all $x, y \in X$ such that $x \neq y$, $d(x, Tx) + d(x, y) + d(x, Ty) \neq 0$, where $k \in [0, 1)$ is a constant or $d(Tx, Ty) = 0$ if $d(x, Tx) + d(x, y) + d(x, Ty) = 0$. Then T has a unique fixed point in X.

Corollary 3.3. *Let (X, d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$d(T^n x, T^n y) \precsim k\Big[\frac{d(x, T^n x)d(x, T^n y) + [d(x, y)]^2 + d(x, T^n x)d(x, y)}{d(x, T^n x) + d(x, y) + d(x, T^n y)}\Big]$$

for all $x, y \in X$ such that $x \neq y$, $d(x, T^n x) + d(x, y) + d(x, T^n y) \neq 0$, where $k \in [0, 1)$ is a constant or $d(T^n x, T^n y) = 0$ if $d(x, T^n x) + d(x, y) + d(x, T^n y) = 0$. Then T has a unique fixed point in X.

Proof. By Corollary 3.2, there exists $v \in X$ such that $T^n v = v$. Then

$$\begin{aligned}
d(Tv, v) &= d(TT^n v, T^n v) = d(T^n Tv, T^n v) \\
&\precsim k\Big[\frac{d(Tv, T^n Tv)d(Tv, T^n v) + [d(Tv, v)]^2 + d(Tv, T^n Tv)d(Tv, v)}{d(Tv, T^n Tv) + d(Tv, v) + d(Tv, T^n v)}\Big] \\
&= k\Big[\frac{d(Tv, TT^n v)d(Tv, T^n v) + [d(Tv, v)]^2 + d(Tv, TT^n v)d(Tv, v)}{d(Tv, TT^n v) + d(Tv, v) + d(Tv, T^n v)}\Big] \\
&= k\Big[\frac{d(Tv, Tv)d(Tv, v) + [d(Tv, v)]^2 + d(Tv, Tv)d(Tv, v)}{d(Tv, Tv) + d(Tv, v) + d(Tv, v)}\Big] \\
&= \frac{k}{2}\, d(Tv, v) \\
&\precsim k\, d(Tv, v)
\end{aligned}$$

so that $|d(Tv, v)| \leq k\,|d(Tv, v)| < |d(Tv, v)|$, since $0 < k < 1$, which is a contradiction and so $d(Tv, v) = 0$, that is, $Tv = v$. This shows that T has a unique fixed point in X. This completes the proof. □

As an application of Theorem 3.1, we prove the following theorem for two finite families of mappings.

Theorem 3.4. *If $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are two finite pairwise commuting finite families of self-mappings defined on a complete complex valued metric space (X, d) such that S and T (with $T = T_1 T_2 \ldots T_m$ and $S = S_1 S_2 \ldots S_n$) satisfy the condition (3.1), then the component maps of the two families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ have a unique common fixed point.*

Proof. In view of Theorem 3.1 one can conclude that T and S have a unique common fixed point p, that is, $T(p) = S(p) = p$. Now we are required to show that p is a common fixed point of all the components maps of both the families. In view of pairwise commutativity of the families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$, (for every $1 \leq k \leq m$) we can write

$$T_k(p) = T_k S(p) = ST_k(p) \quad \text{and} \quad T_k(p) = T_k T(p) = TT_k(p)$$

which show that $T_k(p)$ (for every k) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_k(p_1) = p_1$ (for every k) which shows that p_1 is a common fixed point of the family $\{T_i\}_{i=1}^m$. Using the same arguments as above, one can also show that (for every $1 \leq k \leq n$) $S_k(p_1) = p_1$. This completes the proof. □

By taking $T_1 = T_2 = \cdots = T_m = G$ and $S_1 = S_2 = \cdots = S_n = F$, in Theorem 3.4, we derive the following result involving iterates of mappings.

Corollary 3.5. *If F and G are two commuting self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition:*

$$d(F^n x, G^m y) \precsim k\left[\frac{d(x, F^n x)d(x, G^m y) + [d(x, y)]^2 + d(x, F^n x)d(x, y)}{d(x, F^n x) + d(x, y) + d(x, G^m y)}\right]$$

for all $x, y \in X$ such that $x \neq y$, $d(x, F^n x) + d(x, y) + d(x, G^m y) \neq 0$, where $k \in [0, 1)$ is a constant or $d(F^n x, G^m y) = 0$ if $d(x, F^n x) + d(x, y) + d(x, G^m y) = 0$. Then F and G have a unique common fixed point in X.

By setting $m = n$ and $F = G = T$ in Corollary 3.5, we deduce the following result.

Corollary 3.6. *Let (X, d) be a complete complex valued metric space and let the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$d(T^n x, T^n y) \precsim k\left[\frac{d(x, T^n x)d(x, T^n y) + [d(x, y)]^2 + d(x, T^n x)d(x, y)}{d(x, T^n x) + d(x, y) + d(x, T^n y)}\right]$$

for all $x, y \in X$ such that $x \neq y$, $d(x, T^n x) + d(x, y) + d(x, T^n y) \neq 0$, where $k \in [0, 1)$ is a constant or $d(T^n x, T^n y) = 0$ if $d(x, T^n x) + d(x, y) + d(x, T^n y) = 0$. Then T has a unique fixed point in X.

Proof. By Corollary 3.2, we obtain $w \in X$ such that $T^n w = w$. The rest of the proof is same as that of Corollary 3.3. This completes the proof. □

Example 3.7. Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $'\precsim'$ is defined as $x \precsim y$ iff $x \geq y$. Let the complex valued metric d be given as

$$d(x, y) = |x - y|\sqrt{2}e^{i\frac{\pi}{4}} = |x - y|(1 + i) \text{ for } x, y \in X.$$

Let $T\colon X \to X$ be defined as follows:

$$T(0) = 0, T(\frac{1}{2}) = 0, T(2) = \frac{1}{2}.$$

Case I. Take $x = \frac{1}{2}$, $y = 0$, $T(0) = 0$ and $T(\frac{1}{2}) = 0$ in Corollary 3.2, then we have

$$d(Tx, Ty) = 0 \leq k.\left(\frac{1 + i}{2}\right).$$

This implies that $k \geq 0$. If we take $0 < k < 1$, then all the conditions of Corollary 3.2 are satisfied and of course 0 is the unique fixed point of T.

Case II. Take $x = 2$, $y = \frac{1}{2}$, $T(2) = \frac{1}{2}$ and $T(\frac{1}{2}) = 0$ in Corollary 3.2, then we have

$$d(Tx, Ty) = \frac{1 + i}{2} \leq k.\frac{3(1 + i)}{2}.$$

This implies that $k \geq \frac{1}{3}$. If we take $0 < k < 1$, then all the conditions of Corollary 3.2 are satisfied and of course 0 is the unique fixed point of T.

Case III. Take $x = 2$, $y = 0$, $T(2) = \frac{1}{2}$ and $T(0) = 0$ in Corollary 3.2, then we have

$$d(Tx, Ty) = \frac{1+i}{2} \leq k.\frac{20(1+i)}{11}.$$

This implies that $k \geq \frac{11}{40}$. If we take $0 < k < 1$, then all the conditions of Corollary 3.2 are satisfied and of course 0 is the unique fixed point of T.

Example 3.8. Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $' \precsim'$ is defined as $x \precsim y$ iff $x \geq y$. Let the complex valued metric d be given as

$$d(x, y) = |x-y|\sqrt{2}e^{i\frac{\pi}{4}} = |x-y|(1+i) \text{ for } x, y \in X.$$

Let S, $T\colon X \to X$ be defined as follows:

$$S(0) = 0, S(\frac{1}{2}) = 0, S(2) = \frac{1}{2};$$

$$T(0) = 0, T(\frac{1}{2}) = 2, T(2) = 0.$$

Case I. Take $x = \frac{1}{2}$, $y = 0$, $S(\frac{1}{2}) = 0$ and $T(0) = 0$ in Theorem 3.1, then we have

$$d(Sx, Ty) = 0 \leq k.\Big(\frac{1+i}{2}\Big).$$

This implies that $k \geq 0$. If we take $0 < k < 1$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique common fixed point of S and T.

Case II. Take $x = 2$, $y = \frac{1}{2}$, $S(2) = \frac{1}{2}$ and $T(\frac{1}{2}) = 2$ in Theorem 3.1, then we have

$$d(Sx, Ty) = \frac{3(1+i)}{2} \leq k.3(1+i).$$

This implies that $k \geq \frac{1}{2}$. If we take $0 < k < 1$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique common fixed point of S and T.

Case III. Take $x = 2$, $y = 0$, $S(2) = \frac{1}{2}$ and $T(0) = 0$ in Theorem 3.1, then we have

$$d(Sx, Ty) = \frac{1+i}{2} \leq k.2(1+i).$$

This implies that $k \geq \frac{1}{4}$. If we take $0 < k < 1$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique common fixed point of S and T.

4 Conclusion

In this paper, we establish some common fixed point theorems using rational contraction in the setting of complex-valued metric spaces. Our results extend and generalize several known results from the current existing literature.

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