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On Locally ϕ-semisymmetric Kenmotsu Manifolds

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Abstract The object of the present paper is to study the locally ϕ- semisymmetric Kenmotsu manifolds along with the characterization of such notion.

1 Introduction

Let M be an n-dimensional, $n \geq 3$, connected smooth Riemannian manifold endowed with the Riemannian metric g. Let ∇, R, S and r be the Levi-Civita connection, curvature tensor, Ricci tensor and the scalar curvature of M respectively. The manifold M is called locally symmetric due to Cartan ([2], [3]) if the local geodesic symmetry at $p \in M$ is an isometry, which is equivalent to the fact that $\nabla R = 0$. Generalizing the concept of local symmetry, the notion of semisymmetric manifold was introduced by Cartan [4] and fully classified by Szabo ([11], [12], [13]). The manifold M is said to be semisymmetric if $(R(U,V).R)(X,Y)Z = 0$, for all vector fields X, Y, Z, U, V on M, where $R(U,V)$ is considered as the derivation of the tensor algebra at each point of M.

In 1977 Takahashi [14] introduced the notion of local ϕ- symmetry on a Sasakian manifold. A Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{1.1}$$

for all horizontal vector fields X, Y, Z, W on M that is all vector fields orthogonal to ξ, where ϕ is the structure tensor of the manifold M. The concept of local ϕ- symmetry on various structures and their generalizations or extension are studied in ([6], [7], [8], [9]). By extending the notion of semisymmetry and generalizing the concept of local ϕ- symmetry of Takahashi [14], the first author and his coauthor introduced [10] the notion of local ϕ-semisymmetry on a Sasakian manifold. A Sasakian manifold M, $n \geq 3$, is said to be locally ϕ-semisymmetric if

$$\phi^2((R(U,V).R)(X,Y)Z) = 0, \tag{1.2}$$

for all horizontal vector fields X, Y, Z, U, V on M. In the present paper we study locally ϕ-semisymmetric Kenmotsu manifolds. The paper is organized as follows:

In section 2 some rudimentary facts and curvature related properties of Kenmotsu manifolds are discussed. In section 3 we study locally ϕ-semisymmetric Kenmotsu manifolds and obtained the characterization of such notion.

2 Preliminaries

Let M be a $(2n+1)$-dimensional connected smooth manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [1]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y on M.
Then we have [1]

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X,\xi). \tag{2.3}$$

$$g(\phi X, X) = 0. \tag{2.4}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.5}$$

for all vector fields X, Y on M.
If

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.6}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.7}$$

holds on M, then it is called a Kenmotsu manifold [5].
In a Kenmotsu manifold the following relations hold [5]

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.8}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(X, \xi)Z = g(X, Z)\xi - \eta(Z)X, \tag{2.11}$$

$$R(X, \xi)\xi = \eta(X)\xi - X, \tag{2.12}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.13}$$

$$(\nabla_W R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W, \tag{2.14}$$

$$(\nabla_W R)(X, \xi)Z = g(X, Z)W - g(W, Z)X - R(X, W)Z, \tag{2.15}$$

for all vector fields X, Y, Z and W on M.
In a Kenmotsu manifold we also have [5]

$$\begin{aligned} R(X, Y)\phi W &= g(Y, W)\phi X - g(X, W)\phi Y + g(X, \phi W)Y - g(Y, \phi W)X \\ &+ \phi R(X, Y)W. \end{aligned} \tag{2.16}$$

Applying ϕ and using (2.1) we get from (2.16)

$$\begin{aligned} \phi R(X, Y)\phi W &= -g(Y, W)X + g(X, W)Y + g(X, \phi W)\phi Y - g(Y, \phi W)\phi X \\ &- R(X, Y)W. \end{aligned} \tag{2.17}$$

In view of (2.17) we obtain from (2.14)

$$(\nabla_W R)(X, Y)\xi = g(Y, \phi W)\phi X - g(X, \phi W)\phi Y + \phi R(X, Y)\phi W. \tag{2.18}$$

3 Locally ϕ-semisymmetric Kenmotsu Manifolds

Definition 3.1. A Kenmotsu manifold M is said to be locally ϕ-semisymmetric if

$$\phi^2((R(U, V).R)(X, Y)Z) = 0, \tag{3.1}$$

for all horizontal vector fields X, Y, Z, U, V on M.

First we suppose that M is a Kenmotsu manifold such that

$$\phi^2((R(U, V).R)(X, Y)\xi) = 0, \tag{3.2}$$

for all horizontal vector fields X, Y, U and V on M.

Differentiating (2.18) covariantly with respect to a horizontal vector field U, we get

$$\begin{aligned} & (\nabla_U \nabla_V R)(X,Y)\xi \\ = & [g(X,\phi V)g(U,\phi Y) - g(Y,\phi V)g(U,\phi X) + g(\phi U, R(X,Y)\phi V)]\xi \\ + & \phi(\nabla_U R)(X,Y)\phi V. \end{aligned} \tag{3.3}$$

Using (2.16) we obtain from (3.3)

$$\begin{aligned} (\nabla_U \nabla_V R)(X,Y)\xi &= [g(Y,V)g(U,X) - g(X,V)g(U,Y) + g(R(X,Y)V,U)]\xi \\ &+ \phi(\nabla_U R)(X,Y)\phi V. \end{aligned} \tag{3.4}$$

Interchanging U and V on (3.4) we get

$$\begin{aligned} (\nabla_V \nabla_U R)(X,Y)\xi &= [g(Y,U)g(V,X) - g(X,U)g(V,Y) + g(R(X,Y)U,V)]\xi \\ &+ \phi(\nabla_V R)(X,Y)\phi U. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$\begin{aligned} (R(U,V).R)(X,Y)\xi &= 2[g(Y,V)g(U,X) - g(X,V)g(U,Y) - R(X,Y,U,V)]\xi \\ &+ \phi\{(\nabla_U R)(X,Y)\phi V - (\nabla_V R)(X,Y)\phi U\}. \end{aligned} \tag{3.6}$$

Again from (3.2) we have

$$(R(U,V).R)(X,Y)\xi = 0, \tag{3.7}$$

From (3.6) and (3.7) we have

$$\begin{aligned} & 2[g(Y,V)g(U,X) - g(X,V)g(U,Y) - R(X,Y,U,V)]\xi \\ + & \phi\{(\nabla_U R)(X,Y)\phi V - (\nabla_V R)(X,Y)\phi U\} \\ = & 0. \end{aligned} \tag{3.8}$$

Applying ϕ on (3.8) and using (2.16), (2.18) and (2.3) we get

$$(\nabla_U R)(X,Y)\phi V - (\nabla_V R)(X,Y)\phi U = 0. \tag{3.9}$$

In view of (3.8) and (3.9) we get

$$R(X,Y,U,V) = g(Y,V)g(U,Y) - g(X,V)g(U,Y), \tag{3.10}$$

$$R(X,Y,U,V) = -\{g(X,V)g(U,Y) - g(Y,V)g(U,X)\}, \tag{3.11}$$

for all horizontal vector fields X, Y, U and V on M. Hence M is of constant ϕ-holomorphic sectional curvature -1 and hence of constant curvature -1. This leads to the following:

Theorem 3.2. *If a Kenmotsu manifold M satisfies the condition $\phi^2((R(U,V).R)(X,Y)\xi) = 0$, for all horizontal vector fields X, Y, Z, U and V on M, then M is a manifold of constant curvature -1.*

We consider a Kenmotsu manifold which is locally ϕ-semisymmetric. Then from (3.1) we have

$$(R(U,V).R)(X,Y)Z = g((R(U,V).R)(X,Y)Z,\xi)\xi, \tag{3.12}$$

from which we get

$$(R(U,V).R)(X,Y)Z = -g((R(U,V).R)(X,Y)\xi, Z)\xi \tag{3.13}$$

for all horizontal vector fields X, Y, Z, U, V on M.

Now taking inner product on both side of (3.6) with a horizontal vector field Z, we obtain

$$g((R(U,V).R)(X,Y)\xi, Z) = g(\phi(\nabla_U R)(X,Y)\phi V, Z) - g(\phi(\nabla_V R)(X,Y)\phi U, Z). \tag{3.14}$$

Using (2.5) and (3.13) we get from (3.14)

$$(R(U,V).R)(X,Y)Z = [g((\nabla_U R)(X,Y)\phi V, \phi Z) - g((\nabla_V R)(X,Y)\phi U, \phi Z)]\xi \tag{3.15}$$

Differentiating (2.16) covariantly with respect to a horizontal vector field V, we get

$$\begin{aligned} & (\nabla_V R)(X,Y)\phi Z \\ = \; & [-g(Y,Z)g(V,\phi X) + g(X,Z)g(V,\phi Y) - g(V, R(X,Y)Z)]\xi \\ + \; & \phi(\nabla_V R)(X,Y)Z. \end{aligned} \tag{3.16}$$

Taking inner product on both sides of (3.16) with a horizontal vector field U, we obtain

$$g\{(\nabla_V R)(X,Y)\phi Z, U\} = g\{\phi(\nabla_V R)(X,Y)Z, U\}. \tag{3.17}$$

Using (2.5) we get from above

$$g\{(\nabla_V R)(X,Y)\phi Z, U\} = -g\{(\nabla_V R)(X,Y)Z, \phi U\}. \tag{3.18}$$

In view of (3.18) we obtain from (3.15)

$$(R(U,V).R)(X,Y)Z = [-g((\nabla_U R)(X,Y)V, \phi^2 Z) + g((\nabla_V R)(X,Y)U, \phi^2 Z)]\xi, \tag{3.19}$$

which implies that

$$(R(U,V).R)(X,Y)Z = [g((\nabla_U R)(X,Y)V, Z) - g((\nabla_V R)(X,Y)U, Z)]\xi, \tag{3.20}$$

i.e.

$$(R(U,V).R)(X,Y)Z = [-(\nabla_U R)(X,Y,Z,V) + (\nabla_V R)(X,Y,Z,U)]\xi, \tag{3.21}$$

for any horizontal vector field X, Y, Z, U, V on M. Hence we can state the following:

Theorem 3.3. *A Kenmotsu manifold M, $n \geq 3$, is locally ϕ-semisymmetric if and only if the relation (3.21) holds for all horizontal vector fields X, Y, Z, U, V on M.*

4 Characterization of Locally ϕ-semisymmetric Kenmotsu Manifolds

In this section we investigate the condition of local ϕ-semisymmetry of a Kenmotsu manifold for arbitrary vector fields on M. To find this we need the following results.

Lemma 4.1. *For any horizontal vector field X, Y and Z on a Kenmotsu manifold M, we have*

$$(\nabla_\xi R)(X,Y)Z = (\ell_\xi R)(X,Y)Z + 2R(X,Y)Z.$$

Proof. Let X^*, Y^* and Z^* be ξ- invariant horizontal vector field extensions on X, Y and Z respectively. Since X^* is ξ- invariant of X, we get by using (2.7)

$$\nabla_\xi X^* = \nabla_{X^*}\xi = X^* \tag{4.2}$$

Now making use of invariance of X^*, Y^* and Z^* by ξ and using (4.2) we get

$$\begin{aligned} (\ell_\xi R)(X^*,Y^*)Z^* &= [\xi, R(X^*,Y^*)Z^*] \\ &= \nabla_\xi(R(X^*,Y^*)Z^*) - \nabla_{R(X^*,Y^*)Z^*}\xi \\ &= (\nabla_\xi R)(X^*,Y^*)Z^* + R(\nabla_\xi X^*,Y^*)Z^* + R(X^*,\nabla_\xi Y^*)Z^* \\ &+ R(X^*,Y^*)\nabla_\xi Z^* - R(X^*,Y^*)Z^* \\ &= (\nabla_\xi R)(X^*,Y^*)Z^* + R(X^*,Y^*)Z^* + R(X^*,Y^*)Z^* \\ &+ R(X^*,Y^*)Z^* - R(X^*,Y^*)Z^* \\ &= (\nabla_\xi R)(X^*,Y^*)Z^* + 2R(X^*,Y^*)Z^* \end{aligned} \tag{4.3}$$

Hence we get the conclusion. □

Lemma 4.2. *For any vector field X, Y and Z on a Kenmotsu manifold M we have*

$$\begin{aligned} R(\phi^2X, \phi^2Y)\phi^2Z &= -R(X,Y)Z + \eta(Z)\{\eta(X)Y - \eta(Y)X\} \\ &+ \{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi \end{aligned} \tag{4.4}$$

Now lemma (4.1) and lemma (4.2) together imply the following:

Lemma 4.3. *For any vector field X, Y, Z and U on a Kenmotsu manifold M, we have*

$$\begin{aligned} &(\nabla_{\phi^2U}R)(\phi^2X, \phi^2Y)\phi^2Z \\ = \; &(\nabla_U R)(X,Y)Z - \eta(X)H_1(Y,U)Z + \eta(Y)H_1(X,U)Z + \eta(Z)H_1(X,Y)U \\ + \; &\eta(U)[\eta(Z)\{\eta(X)\ell_\xi Y - \eta(Y)\ell_\xi X\} - (\ell_\xi R)(X,Y)Z] \\ + \; &2\eta(U)[R(X,Y)Z - \eta(Z)\{\eta(X)Y - \eta(Y)X\} \\ - \; &\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi]. \end{aligned}$$

where the tensor field H_1 of type (1, 3) is given by

$$H_1(X,Y)Z = R(X,Y)Z - g(X,Z)Y + g(Y,Z)X, \tag{4.5}$$

for all vector fields X, Y, Z on M.
Now let X, Y, Z, U, V be arbitrary vector fields on M.
Now we compute $(R(\phi^2U, \phi^2V).R)(\phi^2X, \phi^2Y)\phi^2Z$ in two different ways. Firstly from (3.21), (2.1) and (4.3) we get

$$\begin{aligned} &(R(\phi^2U, \phi^2V).R)(\phi^2X, \phi^2Y)\phi^2Z \\ = \; &\{(\nabla_U R)(X,Y,Z,V) - (\nabla_V R)(X,Y,Z,U)\}\xi \\ + \; &\{\eta(U)\eta\{(\nabla_V R)(X,Y)Z)\} - \eta(V)\eta\{(\nabla_U R)(X,Y)Z)\}\}\xi \\ - \; &\eta(X)\{H(Y,U,Z,V) - H(Y,V,Z,U)\}\xi \\ + \; &\eta(Y)\{H(X,U,Z,V) - H(X,V,Z,U)\}\xi \\ + \; &\eta(Z)\{H(X,Y,U,V) - H(X,Y,V,U)\}\xi \\ + \; &\eta(X)\eta(Z)\{\eta(U)g(\ell_\xi Y,V) - \eta(V)g(\ell_\xi Y,U)\}\xi \\ - \; &\eta(Y)\eta(Z)\{\eta(U)g(\ell_\xi X,V) - \eta(V)g(\ell_\xi X,U)\}\xi \\ + \; &2\{\eta(U)R(X,Y,Z,V) - \eta(V)R(X,Y,Z,U)\}\xi \\ + \; &2\eta(Z)\eta(V)\{\eta(X)g(Y,U) - \eta(Y)g(X,U)\}\xi \\ - \; &2\eta(Z)\eta(U)\{\eta(X)g(Y,V) - \eta(Y)g(X,V)\}\xi, \end{aligned} \tag{4.6}$$

where $H(X,Y,Z,U) = g(H_1(X,Y)Z,U)$ and the tensor field H_1 of type (1, 3) is given by (4.5)
Secondly we have

$$\begin{aligned} &(R(\phi^2U, \phi^2V).R)(\phi^2X, \phi^2Y)\phi^2Z = R(\phi^2U, \phi^2V)R(\phi^2X, \phi^2Y)\phi^2Z \\ - \; &R(R(\phi^2U, \phi^2V)\phi^2X, \phi^2Y)\phi^2Z - R(\phi^2X, R(\phi^2U, \phi^2V)\phi^2Y)\phi^2Z \\ - \; &R(\phi^2X, \phi^2Y)R(\phi^2U, \phi^2V)\phi^2Z. \end{aligned} \tag{4.7}$$

By straightforward calculation from (4.7) we get

$$\begin{aligned} &(R(\phi^2U, \phi^2V).R)(\phi^2X, \phi^2Y)\phi^2Z \\ = \; &-(R(U,V).R)(X,Y)Z \\ + \; &\eta(X)\{\eta(V)H_1(U,Y)Z - \eta(U)H_1(V,Y)Z\} \\ + \; &\eta(Y)\{\eta(V)H_1(X,U)Z - \eta(U)H_1(X,V)Z\} \\ + \; &\eta(Z)\{\eta(V)H_1(X,Y)U - \eta(U)H_1(X,Y)V\} \\ + \; &\{\eta(V)g(H(X,Y,Z,U) - \eta(U)g(H(X,Y,Z,V)\}\xi, \end{aligned} \tag{4.8}$$

where $H(X,Y,Z,U) = g(H_1(X,Y)Z,U)$ and the tensor field H_1 of type (1, 3) is given by (4.5) From (4.6) and (4.8) we obtain

$$\begin{aligned}
& (R(U,V).R)(X,Y)Z \\
= & [-(\nabla_U R)(X,Y,Z,V) + (\nabla_V R)(X,Y,Z,U)]\xi \\
+ & [\eta(V)\eta\{(\nabla_U R)(X,Y)Z)\} - \eta(U)\eta\{(\nabla_V R)(X,Y)Z)\}]\xi \\
+ & \eta(X)[\{H(Y,U,Z,V) - H(Y,V,Z,U)\}\xi + \eta(V)H_1(U,Y)Z - \eta(U)H_1(V,Y)Z] \\
- & \eta(Y)[\{H(X,U,Z,V) - H(X,V,Z,U)\}\xi - \eta(V)H_1(X,U)Z + \eta(U)H_1(X,V)Z] \\
- & \eta(Z)[\{H(X,Y,U,V) - H(X,Y,V,U)\}\xi + \eta(V)H_1(X,U)Z - \eta(U)H_1(X,V)Z] \\
+ & \{\eta(V)H(X,Y,Z,U) - \eta(U)H(X,Y,Z,V)\}\xi \\
- & 2\{\eta(U)R(X,Y,Z,V) - \eta(V)R(X,Y,Z,U)\}\xi \\
+ & .\{\eta(U)(\ell_\xi R)(X,Y,Z,V) - \eta(V)(\ell_\xi R)(X,Y,Z,U)\}\xi \\
- & \eta(Z)\eta(X)\{\eta(U)g(\ell_\xi Y,V) - \eta(V)g(\ell_\xi Y,U)\}\xi \\
+ & \eta(Z)\eta(Y)\{\eta(U)g(\ell_\xi X,V) - \eta(V)g(\ell_\xi X,U)\}\xi \\
- & 2\eta(Z)\eta(V)\{\eta(X)g(Y,U) - \eta(Y)g(X,U)\}\xi \\
+ & 2\eta(Z)\eta(U)\{\eta(X)g(Y,V) - \eta(Y)g(X,V)\}\xi.
\end{aligned} \tag{4.9}$$

Thus in a locally ϕ-semisymmetric Kenmotsu manifold the relation (4.9) holds for all arbitrary vector fields X, Y, Z, U, V on M. Next if the relation (4.9) holds in a Kenmotsu manifold, then for any horizontal vector field X, Y, Z, U, V on M, we get the relation (3.21) and hence the manifold is locally ϕ-semisymmetric.
Thus we can state the following:

Theorem 4.4. *A Kenmotsu manifold M is locally ϕ-semisymmetric if and only if the relation (4.9) holds for any arbitrary vector field X, Y, Z, U, V on M.*

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