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WEIGHTED SHARING AND UNIQUENESS OF CERTAIN TYPE OF DIFFERENTIAL-DIFFERENCE POLYNOMIALS

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Abstract. With the notion of weighted sharing of values we investigate the uniqueness problems of certain type of differential-difference polynomials sharing a small function with finite weight. The research findings also include IM analogues of the theorem in which the small function is allowed to be shared ignoring multiplicities. The results of the paper improve, supplement and rectify the recent results due to K. Zhang and H.X. Yi [Acta Mathematica Scientia 34, 719-728 (2014)].

1 Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [8], [10] and [17]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f - a$ and $g - a$ coincide in locations and multiplicities, we say that f and g share the value a CM (counting multiplicities). On the other hand, if the zeros of $f - a$ and $g - a$ coincide only in locations, then we say that f and g share the value a IM (ignoring multiplicities). We say f and g sharing a function h CM or IM if $f - h$ and $g - h$ share 0 CM or IM respectively. For a positive integer p, we denote by $N_p(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. We say that a is a small function of f, if a is a meromorphic function satisfying $T(r, a) = S(r, f)$. We define difference operators $\triangle_c f(z) = f(z + c) - f(z)$, $\triangle_c^n f(z) = \triangle_c^{n-1}(\triangle_c f(z))$, where c is a nonzero complex number and $n \geq 2$ is a positive integer. If $c = 1$, we denote $\triangle_c f(z) = \triangle f(z)$.

Many research works on meromorphic functions whose differential polynomials share certain value or fixed point have been done (see [5], [11], [13], [14], [16]). Now it is an increasing interest to the difference equations and difference product in the complex plane. In 2006 R.G. Halburd and R.J. Korhonen [6] established a version of Nevanlinna theory based on difference operators. The difference logarithmic derivative lemma, given by R.G. Halburd and R.J. Korhonen [7] in 2006, Y.M. Chiang and S.J. Feng [3] in 2008 independently plays an important role in considering the difference analogues of Nevanlinna theory. With the development of difference analogues of Nevanlinna theory, many mathematicians of the world paid their attention on the distribution of zeros of difference polynomials. In this direction, we recall the following uniqueness result of X.G. Qi, L.Z. Yang and K. Liu [12].

Theorem A. Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n f(z + c)$ and $g^n g(z + c)$ share 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 that satisfy $t_1^{n+1} = t_2^{n+1} = 1$.

In 2010 J.L. Zhang [18] replaced value sharing as sharing of small function and obtained the following result.

Theorem B. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(f-1)f(z+c)$ and $g^n(g-1)g(z+c)$ share $\alpha(z)$ CM, then $f = g$.

In 2012 M.R. Chen and Z.X. Chen [2] considered the zeros of the difference polynomial of the form $f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j}$ and obtained the following uniqueness theorem.

Theorem C. Let f and g be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g, $c_j(j=1,2,...,d)$ be distinct finite complex numbers. If $n \geq m + 8\sigma$, n, m, d and $\nu_j(j=1,2,...,d)$ are integers, $\sigma = \Sigma_{j=1}^{d}\nu_j$, and $f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j}$ and $g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j}$ share $\alpha(z)$ CM, then $f = tg$, where $t^m = t^{n+\sigma} = 1$.

Recently K. Zhang and H.X. Yi [20] considered the zeros of more general difference polynomials of the form $(f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)}$ and $(f^n(f-1)^m\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)}$ where f is a transcendental entire function of finite order, n, m, d, k and $\nu_j(j=1,2,...,d)$ are nonnegative integers and $c_j(j=1,2,...,d)$ are distinct finite complex numbers. They proved the following uniqueness results which extend and improve many previous results in this direction.

Theorem D. Let f and g be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g, $c_j(j=1,2,...,d)$ be distinct finite complex numbers, and n, m, d, k and $\nu_j(j=1,2,...,d)$ be nonnegative integers. If $n \geq 2k+m+\sigma+5$ and the differential-difference polynomials $(f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)}$ and $(g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j})^{(k)}$ share $\alpha(z)$ CM, then $f = tg$, where $t^m = t^{n+\sigma} = 1$.

Theorem E. Let f and g be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g, $c_j(j=1,2,...,d)$ be distinct finite complex numbers, and n, m, d, k and $\nu_j(j=1,2,...,d)$ be nonnegative integers. If $n \geq 4k-m+\sigma+9$ and the differential-difference polynomials $(f^n(f-1)^m\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)}$ and $(g^n(g-1)^m\prod_{j=1}^{d} g(z+c_j)^{\nu_j})^{(k)}$ share $\alpha(z)$ CM, then $f = g$.

Note 1. There are some mistakes in case of lower bound of n in Theorems D and E. One can check it for $m = 1$, say. In addition to this it is not possible to conclude always that $f = g$ in Theorem E. It may happen under certain extra condition.

An increment to uniqueness theory has been to considering weighted sharing instead of sharing IM or CM, this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001, which measure how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

Definition 1.1. [9] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g *share* (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If $\alpha(z)$ is a small function of f and g, then f, g share $\alpha(z)$ with weight k means that $f - \alpha$, $g - \alpha$ share the value 0 with weight k. Naturally one may ask the following questions which are the motivation of the paper.

Question 1. Is it possible to relax in any way the nature of sharing the small function in Theorems D and E keeping the lower bound of n fixed ?

Question 2. What will be the IM analogue of Theorems D and E ?

In the paper, our main concern is to find the possible answer of the above questions. We prove following two theorems first one of which improves Theorem D and second one improves Theorem E. Moreover, the results of the paper rectify Theorems D and E. The following are the main results of the paper.

Theorem 1.2. *Let f and g be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g with finitely many zeros, $c_j (j = 1, 2, ..., d)$ be distinct finite complex numbers, and n, m, l, d, k and $\nu_j (j = 1, 2, ..., d)$ be nonnegative integers. If the differential-difference polynomials $(f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)}$ and $(g^n(g^m - 1)\prod_{j=1}^{d} g(z + c_j)^{\nu_j})^{(k)}$ share (α, l), and one of*
(a) $l \geq 2$ and $n \geq \max\{m + 5\sigma, 2k + m + \sigma + 5\}$;
(b) $l = 1$ and $n \geq \max\{m + 5\sigma, 5k/2 + 3m/2 + 3\sigma/2 + 5\}$;
(c) $l = 0$ and $n \geq \max\{m + 5\sigma, 5k + 4m + 4\sigma + 8\}$,
holds, then $f = tg$, where $t^m = t^{n+\sigma} = 1$.

Theorem 1.3. *Let f and g be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0)$ be a common small function with respect to f and g with finitely many zeros, $c_j (j = 1, 2, ..., d)$ be distinct finite complex numbers, and n, m, l, d, k and $\nu_j (j = 1, 2, ..., d)$ be nonnegative integers. Suppose that the differential-difference polynomials $(f^n(f - 1)^m \prod_{j=1}^{d} f(z + c_j)^{\nu_j})^{(k)}$ and $(g^n(g - 1)^m \prod_{j=1}^{d} g(z + c_j)^{\nu_j})^{(k)}$ share (α, l). If $m \leq k + 1$ and one of*
(a) $l \geq 2$ and $n \geq 2k + m + \sigma + 5$;
(b) $l = 1$ and $n \geq 5k/2 + 3m/2 + 3\sigma/2 + 5$;
(c) $l = 0$ and $n \geq 5k + 4m + 4\sigma + 8$,
holds or if $m > k + 1$ and one of
(a) $l \geq 2$ and $n \geq 4k - m + \sigma + 9$;
(b) $l = 1$ and $n \geq 5k - m + 3\sigma/2 + 10$;
(c) $l = 0$ and $n \geq 10k - m + 4\sigma + 15$,
holds, then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m \prod_{j=1}^{d} w_1(z + c_j)^{\nu_j} - w_2^n(w_2 - 1)^m \prod_{j=1}^{d} w_2(z + c_j)^{\nu_j}.$$

2 Lemmas

Let F and G be two nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. *[15] Let f be a transcendental meromorphic function, and let $P_n(f)$ be a polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \dots + a_1 f(z) + a_0,$$

where $a_n (\neq 0)$, a_{n-1}, ... , a_1, a_0 are complex numbers. Then

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

Lemma 2.2. *[19] Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \le T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.1}$$

$$N_p\left(r, 0; f^{(k)}\right) \le k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 2.3. *[3] Let f be a meromorphic function of finite order ρ, $c \neq 0$ be fixed. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f) + O\{r^{\rho-1+\varepsilon}\} + O\{\log r\}.$$

Lemma 2.4. *[2] Let f be an entire function of finite order and $F = f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j}$.*

Then

$$T(r, F) = (n + m + \sigma)T(r, f) + S(r, f).$$

Arguing in a like manner as in Lemma 2.6 [2] we obtain the following lemma.

Lemma 2.5. *Let f be an entire function of finite order and $F = f^n(f-1)^m\prod_{j=1}^{d} f(z+c_j)^{\nu_j}$.*

Then

$$T(r, F) = (n + m + \sigma)T(r, f) + S(r, f).$$

Lemma 2.6. *[9] Let f and g be two nonconstant meromorphic functions sharing $(1, 2)$. Then one of the following cases holds:*

(i) $T(r) \le N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,

(ii) $f \equiv g$,

(iii) $fg \equiv 1$,

where $T(r) = \max\{T(r, f), T(r, g)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.7. *[1] Let F and G be two nonconstant meromorphic functions sharing $(1, 1)$ and $H \not\equiv 0$. Then*
$T(r, F) \le N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G)$,
and the same inequality holds for $T(r, G)$.

Lemma 2.8. *[1] Let F and G be two nonconstant meromorphic functions sharing $(1, 0)$ and $H \not\equiv 0$. Then*
$T(r, F) \le N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G)$,
and the same inequality holds for $T(r, G)$.

Lemma 2.9. *Let f and g be entire functions, n, m, k be positive integers, and let*

$$F = (f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)},\ G = (g^n(g^m - 1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j})^{(k)}.$$

If there exists nonzero constants a_1 and a_2 such that $\overline{N}(r, a_1; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, a_2; G) = \overline{N}(r, 0; F)$, then $n \le 2k + m + \sigma + 2$.

Proof. We put $F_1 = f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j}$ and $G_1 = g^n(g^m - 1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j}$. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F) + \overline{N}(r,a_1;F) + S(r,F) \\ &\le \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \end{aligned} \tag{2.3}$$

Using (2.1), (2.2), (2.3) and Lemma 2.4 we obtain

$$\begin{aligned} (n+m+\sigma)T(r,f) &\le T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\le \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\le N_{k+1}(r,0;F_1) + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g) \\ &\le (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \end{aligned} \tag{2.4}$$

Similarly we obtain

$$(n+m+\sigma)T(r,g) \le (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{2.5}$$

Combining (2.4) and (2.5) we obtain

$$(n - 2k - m - \sigma - 2)(T(r,f) + T(r,g)) \le S(r,f) + S(r,g),$$

which gives $n \le 2k + m + \sigma + 2$. This proves the lemma. □

Lemma 2.10. *Let f and g be entire functions, n, m, k be positive integers, and let*

$$F = (f^n(f - 1)^m\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)},\ G = (g^n(g - 1)^m\prod_{j=1}^{d} g(z+c_j)^{\nu_j})^{(k)}.$$

If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r, c_1; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, c_2; G) = \overline{N}(r, 0; F)$, then $n \le 2k + m + \sigma + 2$ for $m \le k + 1$ and $n \le 4k - m + \sigma + 4$ for $m > k + 1$.

Proof. Arguing similarly as in the proof of Lemma 2.9 above we can deduce the result. Here we omit the details. □

Lemma 2.11. *[2] Suppose that f and g are transcendental entire functions of finite order, $c_j(j = 1, 2, ..., d)$ are distinct finite complex numbers, and n, m, d, $\nu_j(j = 1, 2, ..., d)$ are integers. If $n \ge m + 5\sigma$ and*

$$f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j} = g^n(g^m - 1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j}$$

then $f = tg$, where $t^m = t^{n+\sigma} = 1$.

3 Proof of the Theorem

Proof of Theorem 1.2. Let $F_1 = f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j}$, $G_1 = g^n(g^m - 1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j}$, $F = \frac{(F_1)^{(k)}}{\alpha(z)}$ and $G = \frac{(G_1)^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share $(1, l)$ except the zeros and poles of $\alpha(z)$. Using (2.1) and Lemma 2.4 we get

$$\begin{aligned} N_2(r, 0; F) &\leq N_2(r, 0; (F_1)^{(k)}) + S(r, f) \\ &\leq T(r, (F_1)^{(k)}) - (n + m + \sigma)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f) \\ &\leq T(r, F) - (n + m + \sigma)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f). \end{aligned} \tag{3.1}$$

Again by (2.2) we have

$$\begin{aligned} N_2(r, 0; F) &\leq N_2(r, 0; (F_1)^{(k)}) + S(r, f) \\ &\leq N_{k+2}(r, 0; F_1) + S(r, f). \end{aligned} \tag{3.2}$$

From (3.1) we get

$$(n + m + \sigma)T(r, f) \leq T(r, F) + N_{k+2}(r, 0; F_1) - N_2(r, 0; F) + S(r, f). \tag{3.3}$$

We now discuss the following three cases separately.

Case 1. Let $l \geq 2$. Suppose, if possible, that (i) of Lemma 2.6 holds. Then using (3.2) we obtain from (3.3)

$$\begin{aligned} (n + m + \sigma)T(r, f) &\leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + N_{k+2}(r, 0; F_1) \\ &\quad + S(r, f) + S(r, g) \\ &\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + S(r, f) + S(r, g) \\ &\leq (k + m + \sigma + 2)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \end{aligned} \tag{3.4}$$

In a similar way we obtain

$$(n + m + \sigma)T(r, g) \leq (k + m + \sigma + 2)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$(n - 2k - m - \sigma - 4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

contradicting with the fact that $n \geq 2k + m + \sigma + 5$. So by Lemma 2.6 either $FG = 1$ or $F = G$. Let $FG = 1$. Then

$$(f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)}(g^n(g^m - 1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j})^{(k)} = \alpha^2.$$

Since the number of zeros of $\alpha(z)$ is finite, it follows that f as well as g has finitely many zeros. We put $f(z) = h(z)e^{\beta(z)}$, where $h(z)$ is a nonzero polynomial and $\beta(z)$ is a nonconstant polynomial. Now replacing $\sum_{j=1}^{d} \nu_j\beta(z + c_j)$ by $\gamma(z)$ and $\prod_{j=1}^{d} h(z + c_j)^{\nu_j}$ by $\mu(z)$ we deduce that

$$
\begin{aligned}
& (f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)} \\
= \; & (h^n(z)e^{n\beta(z)}(h^m(z)e^{m\beta(z)}-1)\prod_{j=1}^{d} h(z+c_j)^{\nu_j}e^{\nu_j\beta(z+c_j)})^{(k)} \\
= \; & (h^n(z)\mu(z)e^{n\beta(z)+\gamma(z)}(h^m(z)e^{m\beta(z)}-1))^{(k)} \\
= \; & (h^{n+m}(z)\mu(z)e^{(n+m)\beta(z)+\gamma(z)} - h^n(z)\mu(z)e^{n\beta(z)+\gamma(z)})^{(k)} \\
= \; & e^{(n+m)\beta(z)+\gamma(z)}P_1(\beta(z),\gamma(z),h(z),\mu(z),\ldots,\ \beta^{(k)}(z),\gamma^{(k)}(z),h^{(k)}(z),\mu^{(k)}(z)) \\
& -e^{n\beta(z)+\gamma(z)}P_2(\beta(z),\gamma(z),h(z),\mu(z),\ldots,\ \beta^{(k)}(z),\gamma^{(k)}(z),h^{(k)}(z),\mu^{(k)}(z)) \\
= \; & e^{n\beta(z)+\gamma(z)}(P_1e^{m\beta(z)}-P_2).
\end{aligned}
$$

Obviously $P_1e^{m\beta(z)}-P_2$ has infinite number of zeros, which contradicts with the fact that g is an entire function. Now we assume that $F=G$. Then

$$(f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k)} = (g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j})^{(k)}.$$

Integrating we get

$$(f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j})^{(k-1)} = (g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j})^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1}\neq 0$, from Lemma 2.9 we obtain $n\leq 2k+m+\sigma$, a contradiction. Hence $c_{k-1}=0$. Repeating the process k-times, we obtain

$$f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\nu_j} = g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{\nu_j},$$

which by Lemma 2.11 gives $f=tg$, where $t^m=t^{n+\sigma}=1$.

Case 2. Let $l=1$ and $H\not\equiv 0$. Using Lemma 2.7 and (3.2) we obtain from (3.3)

$$
\begin{aligned}
(n+m+\sigma)T(r,f) \leq \; & N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F) \\
& +\frac{1}{2}\overline{N}(r,\infty;F)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,g) \\
\leq \; & N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+\frac{1}{2}N_{k+1}(r,0;F_1) \\
& +S(r,f)+S(r,g) \\
\leq \; & \frac{1}{2}(3k+3m+3\sigma+5)T(r,f)+(k+m+\sigma+2)T(r,g) \\
& +S(r,f)+S(r,g) \\
\leq \; & \frac{1}{2}(5k+5m+5\sigma+9)T(r)+S(r). \qquad (3.6)
\end{aligned}
$$

In a similar way we obtain

$$(n+m+\sigma)T(r,g)\leq \frac{1}{2}(5k+5m+5\sigma+9)T(r)+S(r). \qquad (3.7)$$

Combining (3.6) and (3.7) we obtain

$$\left(n-\frac{5k+3m+3\sigma+9}{2}\right)T(r)\leq S(r),$$

contradicting with the fact that $n \geq 5k/2 + 3m/2 + 3\sigma/2 + 5$. We now assume that $H = 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.8}$$

where $A(\neq 0)$ and B are constants. From (3.8) it is obvious that F, G share the value 1 CM and so they share $(1,2)$. Hence we have $n \geq 2k + m + \sigma + 5$. Now we discuss the following three subcases.

Subcase 1. Let $B \neq 0$ and $A = B$. Then from (3.8) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.9}$$

If $B = -1$, then from (3.9) we obtain $FG = 1$, a contradiction as in Case 1. If $B \neq -1$, from (3.9), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G). \end{aligned}$$

Using (2.1) and (2.2) we obtain from above inequality

$$\begin{aligned} T(r,G) \leq\ & N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) \\ & -(n+m+\sigma)T(r,g) + S(r,g). \end{aligned}$$

Hence

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)\{T(r,f) + T(r,g)\} + S(r,g).$$

Thus we obtain

$$(n - 2k - m - \sigma - 2)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction as $n \geq 2k + m + \sigma + 5$.

Subcase 2. Let $B \neq 0$ and $A \neq B$. Then from (3.8) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding in a manner similar to Subcase 1 we arrive at a contradiction.

Subcase 3. Let $B = 0$ and $A \neq 0$. Then from (3.8) we get $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. Now applying Lemma 2.9 it can be shown that $n \leq 2k + m + \sigma + 2$, a contradiction. Thus $A = 1$ and then $F = G$. Now the result follows from Case 1.

Case 3. Let $l = 0$ and $H \not\equiv 0$. Using Lemma 2.8 and (3.2) we obtain from (3.3)

$$\begin{aligned} (n+m+\sigma)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) \\ &\quad + \overline{N}(r,0;G) + N_{k+2}(r,0;F_1) + 2\overline{N}(r,\infty;F) \\ &\quad + \overline{N}(r,\infty;G) + S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + 2N_{k+1}(r,0;F_1) \\ &\quad + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (3k+3m+3\sigma+4)T(r,f) + (2k+2m+2\sigma+3)T(r,g) \\ &\quad + S(r,f) + S(r,g) \\ &\leq (5k+5m+5\sigma+7)T(r) + S(r). \end{aligned} \tag{3.10}$$

Similarly it follows that

$$(n+m+\sigma)T(r,g) \quad \leq \quad (5k+5m+5\sigma+7)T(r)+S(r). \tag{3.11}$$

From (3.10) and (3.11) we get

$$(n-5k-4m-4\sigma-7)T(r) \leq S(r),$$

contradicts with the assumption that $n \geq 5k+4m+4\sigma+8$. Therefore $H=0$ and then proceeding in a manner similar to Case 2 the result follows.

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let $F_2 = f^n(f-1)^m \prod_{j=1}^{d} f(z+c_j)^{\nu_j}$, $G_2 = g^n(g-1)^m \prod_{j=1}^{d} g(z+c_j)^{\nu_j}$, $F = \frac{(F_2)^{(k)}}{\alpha(z)}$ and $G = \frac{(G_2)^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share $(1,l)$ except the zeros and poles of $\alpha(z)$. Arguing in a manner similar to the proof of Theorem 1.2 we obtain either $FG=1$ or $F=G$. If $F=G$, then applying the same method of Theorem 1.2 and using Lemma 2.10 we get

$$f^n(f-1)^m \prod_{j=1}^{d} f(z+c_j)^{\nu_j} = g^n(g-1)^m \prod_{j=1}^{d} g(z+c_j)^{\nu_j}. \tag{3.12}$$

Set $h = \frac{f}{g}$. If h is a constant, then substituting $f=gh$ in (3.12), we deduce that

$$g^m(h^{n+m+\sigma}-1) - {}^mC_1 g^{m-1}(h^{n+m+\sigma-1}-1) + ... + (-1)^m(h^{n+\sigma}-1) = 0,$$

which implies $h=1$ and hence $f=g$. If h is not a constant, then we know by (3.12) that f and g satisfy the algebraic equation $R(f,g)=0$ where $R(f,g)$ is given by

$$R(w_1,w_2) = w_1^n(w_1-1)^m \prod_{j=1}^{d} w_1(z+c_j)^{\nu_j} - w_2^n(w_2-1)^m \prod_{j=1}^{d} w_2(z+c_j)^{\nu_j}.$$

If $FG=1$, using the same method as in Theorem 1.2 we arrive at a contradiction. This completes the proof of Theorem 1.3. □

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