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Strong Forms of $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen Sets and $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous Functions

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Abstract In this paper, we introduce two strong forms of $\alpha_{[\gamma,\gamma']}$-semiopen sets called $\alpha_{[\gamma,\gamma']}$-semiregular sets and $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen sets. we also introduce a new class of functions called $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous functions. Moreover, we obtain some characterizations and several properties of such functions.

1 Introduction

In 1965, Njastad [4] defined α-open sets in a space X and discussed many of its properties. Ibrahim [3] defined the concept of an operation γ on $\alpha O(X,\tau)$ and introduced α_γ-open sets in topological spaces and studied some of their basic properties. Khalaf, et. al. [1] introduced the notion of $\alpha O(X,\tau)_{[\gamma,\gamma']}$, which is the collection of all $\alpha_{[\gamma,\gamma']}$-open sets in a topological space (X,τ). In [2] the authors, introduced the notion of $\alpha_{[\gamma,\gamma']}$-semiopen sets in a topological space and studied some of its properties. In this paper, we introduce and study the notion of $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed sets. We also introduce $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous functions and investigate some important properties.

2 Preliminaries

Throughout the present paper, (X,τ) and (Y,σ) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The closure and the interior of a subset A of X are denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 2.1. [4] A subset A of a topological space (X,τ) is called α-open if $A \subseteq Int(Cl(Int(A)))$.

The family of all α-open sets in a topological space (X,τ) is denoted by $\alpha O(X,\tau)$ (or $\alpha O(X)$).

Definition 2.2. [3] Let (X,τ) be a topological space. An operation γ on the topology $\alpha O(X)$ is a mapping from $\alpha O(X)$ into the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \alpha O(X)$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \alpha O(X) \to P(X)$.

Definition 2.3. [3] An operation γ on $\alpha O(X,\tau)$ is said to be α-regular if for every α-open sets U and V containing $x \in X$, there exists an α-open set W of X containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Definition 2.4. [1] Let (X,τ) be a topological space and γ, γ' be operations on $\alpha O(X,\tau)$. A subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-open if for each $x \in A$ there exist α-open sets U and V of X containing x such that $U^\gamma \cap V^{\gamma'} \subseteq A$. A subset of (X,τ) is said to be $\alpha_{[\gamma,\gamma']}$-closed if its complement is $\alpha_{[\gamma,\gamma']}$-open.

The family of all $\alpha_{[\gamma,\gamma']}$-open sets of (X,τ) is denoted by $\alpha O(X,\tau)_{[\gamma,\gamma']}$.

Definition 2.5. [2] A subset A of X is said to be $\alpha_{[\gamma,\gamma']}$-semiopen, if there exists an $\alpha_{[\gamma,\gamma']}$-open set U of X such that $U \subseteq A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(U)$. A subset A of X is $\alpha_{[\gamma,\gamma']}$-semiclosed if and only if $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-semiopen.

The family of all $\alpha_{[\gamma,\gamma']}$-semiopen sets of a topological space (X, τ) is denoted by $\alpha SO(X, \tau)_{[\gamma,\gamma']}$, the family of all $\alpha_{[\gamma,\gamma']}$-semiopen sets of (X, τ) containing x is denoted by $\alpha SO(X, x)_{[\gamma,\gamma']}$. Also the family of all $\alpha_{[\gamma,\gamma']}$-semiclosed sets of a topological space (X, τ) is denoted by $\alpha SC(X, \tau)_{[\gamma,\gamma']}$.

Definition 2.6. Let A be a subset of a topological space (X, τ). Then:

(i) $\alpha_{[\gamma,\gamma']}$-$Cl(A) = \bigcap\{F : F$ is $\alpha_{[\gamma,\gamma']}$-closed and $A \subseteq F\}$ [1].

(ii) $\alpha_{[\gamma,\gamma']}$-$Int(A) = \bigcup\{U : U$ is $\alpha_{[\gamma,\gamma']}$-open and $U \subseteq A\}$ [1].

(iii) $\alpha_{[\gamma,\gamma']}$-$sCl(A) = \bigcap\{F : F$ is $\alpha_{[\gamma,\gamma']}$-semiclosed and $A \subseteq F\}$ [2].

(iv) $\alpha_{[\gamma,\gamma']}$-$sInt(A) = \bigcup\{U : U$ is $\alpha_{[\gamma,\gamma']}$-semiopen and $U \subseteq A\}$ [2].

3 $\alpha_{[\gamma,\gamma']}$-semiregular Sets and $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen Sets

Definition 3.1. A subset A of a topological space (X, τ) is said to be $\alpha_{[\gamma,\gamma']}$-semiregular, if it is both $\alpha_{[\gamma,\gamma']}$-semiopen and $\alpha_{[\gamma,\gamma']}$-semiclosed.

The family of all $\alpha_{[\gamma,\gamma']}$-semiregular sets in X is denoted by $\alpha SR(X)_{[\gamma,\gamma']}$.

Lemma 3.2. *The following properties hold for a subset A of a topological space (X, τ):*

(i) If $A \in \alpha SO(X)_{[\gamma,\gamma']}$, then $\alpha_{[\gamma,\gamma']}$-$sCl(A) \in \alpha SR(X)_{[\gamma,\gamma']}$.

(ii) If $A \in \alpha SC(X)_{[\gamma,\gamma']}$, then $\alpha_{[\gamma,\gamma']}$-$sInt(A) \in \alpha SR(X)_{[\gamma,\gamma']}$.

Proof. (i) Since $\alpha_{[\gamma,\gamma']}$-$sCl(A)$ is $\alpha_{[\gamma,\gamma']}$-semiclosed, we show that $\alpha_{[\gamma,\gamma']}$-$sCl(A) \in \alpha SO(X)_{[\gamma,\gamma']}$. Since $A \in \alpha SO(X)_{[\gamma,\gamma']}$, then for $\alpha_{[\gamma,\gamma']}$-open set U of X, $U \subseteq A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(U)$. Therefore we have, $U \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(U) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(\alpha_{[\gamma,\gamma']}$-$Cl(U)) = \alpha_{[\gamma,\gamma']}$-$Cl(U)$ or $U \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(U)$ and hence $\alpha_{[\gamma,\gamma']}$-$sCl(A) \in \alpha SO(X)_{[\gamma,\gamma']}$.

(ii) This follows from (1).

□

Definition 3.3. A point $x \in X$ is said to be $\alpha_{[\gamma,\gamma']}$-$\theta$-semiadherent point of a subset A of X if $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap A \neq \phi$ for every $\alpha_{[\gamma,\gamma']}$-semiopen set U containing x. The set of all $\alpha_{[\gamma,\gamma']}$-$\theta$-semiadherent points of A is called the $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosure of A and is denoted by $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. A subset A is called $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed if $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A) = A$. A subset A is called $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen if and only if $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed.

Definition 3.4. A point $x \in X$ is said to be $\alpha_{[\gamma,\gamma']}$-$\theta$-adherent point of a subset A of X if $\alpha_{[\gamma,\gamma']}$-$Cl(U) \cap A \neq \phi$ for every $\alpha_{[\gamma,\gamma']}$-open set U containing x. The set of all $\alpha_{[\gamma,\gamma']}$-$\theta$-adherent points of A is called the $\alpha_{[\gamma,\gamma']}$-$\theta$-closure of A and is denoted by $\alpha_{[\gamma,\gamma']}$-$Cl_\theta(A)$. A subset A is called $\alpha_{[\gamma,\gamma']}$-$\theta$-closed if $\alpha_{[\gamma,\gamma']}$-$Cl_\theta(A) = A$. The complement of an $\alpha_{[\gamma,\gamma']}$-$\theta$-closed set is called an $\alpha_{[\gamma,\gamma']}$-$\theta$-open set.

Corollary 3.5. *Let $x \in X$ and $A \subseteq X$. If $x \in \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$, then $x \in \alpha_{[\gamma,\gamma']}$-$Cl_\theta(A)$.*

Proof. Let $x \in \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$, then $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap A \neq \phi$ for every $\alpha_{[\gamma,\gamma']}$-semiopen set U containing x. Since $\alpha_{[\gamma,\gamma']}$-$sCl(U) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(U)$, so we have $\phi \neq \alpha_{[\gamma,\gamma']}$-$sCl(U) \cap A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(U) \cap A$. Hence, $\alpha_{[\gamma,\gamma']}$-$Cl(U) \cap A \neq \phi$ for every $\alpha_{[\gamma,\gamma']}$-open set U containing x. Therefore, $x \in \alpha_{[\gamma,\gamma']}$-$Cl_\theta(A)$. □

Lemma 3.6. *The following properties hold for a subset A of a topological space (X, τ):*

(i) If $A \in \alpha SO(X)_{[\gamma,\gamma']}$, then $\alpha_{[\gamma,\gamma']}$-$sCl(A) = \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$.

(ii) If $A \in \alpha SR(X)_{[\gamma,\gamma']}$ if and only if A is both $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed and $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen.

(iii) If $A \in \alpha O(X)_{[\gamma,\gamma']}$, then $\alpha_{[\gamma,\gamma']}$-$Cl(A) = \alpha_{[\gamma,\gamma']}$-$Cl_\theta(A)$.

Proof. (i) Clearly $\alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. Suppose that $x \notin \alpha_{[\gamma,\gamma']}$-$sCl(A)$. Then, for some $\alpha_{[\gamma,\gamma']}$-semiopen set U, $A \cap U = \phi$ and hence $A \cap \alpha_{[\gamma,\gamma']}$-$sCl(U) = \phi$, since $A \in \alpha SO(X)_{[\gamma,\gamma']}$. This shows that $x \notin \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. Therefore $\alpha_{[\gamma,\gamma']}$-$sCl(A) = \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$.

(ii) Let $A \in \alpha SR(X)_{[\gamma,\gamma']}$, then $A \in \alpha SO(X)_{[\gamma,\gamma']}$, by (1), we have $A = \alpha_{[\gamma,\gamma']}$-$sCl(A) = \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. Therefore, A is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed. Since $X \setminus A \in \alpha SR(X)_{[\gamma,\gamma']}$, by the argument above, $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed and hence A is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen. The converse is obvious.

(iii) This similar to (1).

□

Theorem 3.7. *Let (X, τ) be a topological space and $A \subseteq X$. Then, A is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen in X if and only if for each $x \in A$ there exists $U \in \alpha SO(X, x)_{[\gamma,\gamma']}$ such that $\alpha_{[\gamma,\gamma']}$-$sC(U) \subseteq A$.*

Proof. Let A be $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen and $x \in A$. Then, $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed and $X \setminus A = \alpha_{[\gamma,\gamma']}$-$sCl_\theta(X \setminus A)$. Hence, $x \notin \alpha_{[\gamma,\gamma']}$-$sCl_\theta(X \setminus A)$. Therefore, there exists $U \in \alpha SO(X, x)_{[\gamma,\gamma']}$ such that $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap (X \setminus A) = \phi$ and so $\alpha_{[\gamma,\gamma']}$-$sCl(U) \subseteq A$.

Conversely, let $A \subseteq X$ and $x \in A$. From hypothesis, there exists $U \in \alpha SO(X, x)_{[\gamma,\gamma']}$ such that $\alpha_{[\gamma,\gamma']}$-$sCl(U) \subseteq A$. Therefore, $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap (X \setminus A) = \phi$. Hence, $X \setminus A = \alpha_{[\gamma,\gamma']}$-$sCl_\theta(X \setminus A)$ and A is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen. □

Theorem 3.8. *For a subset A of a topological space (X, τ), we have $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A) = \cap\{V : A \subseteq V$ and $V \in \alpha SR(X)_{[\gamma,\gamma']}\}$.*

Proof. Let $x \notin \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. Then, there exists an $\alpha_{[\gamma,\gamma']}$-semiopen set U containing x such that $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap A = \phi$. Then $A \subseteq X \setminus \alpha_{[\gamma,\gamma']}$-$sCl(U) = V$ (say). Thus $V \in \alpha SR(X)_{[\gamma,\gamma']}$ such that $x \notin V$. Hence $x \notin \cap\{V : A \subseteq V$ and $V \in \alpha SR(X)_{[\gamma,\gamma']}\}$. Again, if $x \notin \cap\{V : A \subseteq V$ and $V \in \alpha SR(X)_{[\gamma,\gamma']}\}$, then there exists $V \in \alpha SR(X)_{[\gamma,\gamma']}$ containing A such that $x \notin V$. Then $(X \setminus V)$ $(= U$, say) is an $\alpha_{[\gamma,\gamma']}$-semiopen set containing x such that $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap V = \phi$. This shows that $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap A = \phi$, so that $x \notin \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. □

Corollary 3.9. *A subset A of X is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed if and only if $A = \cap\{V : A \subseteq V \in \alpha SR(X)_{[\gamma,\gamma']}\}$.*

Proof. Obvious. □

Theorem 3.10. *Let A and B be any subsets of a space X. Then, the following properties hold:*

(i) $x \in \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$ if and only if $U \cap A \neq \phi$ for each $U \in \alpha SR(X)_{[\gamma,\gamma']}$ containing x.

(ii) If $A \subseteq B$, then $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl_\theta(B)$.

Proof. Clear. □

Theorem 3.11. *For any subset A of X, $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)) = \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$.*

Proof. Obviously, $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl_\theta(\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A))$. Now, let $x \in \alpha_{[\gamma,\gamma']}$-$sCl_\theta(\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A))$ and $U \in \alpha SO(X, x)_{[\gamma,\gamma']}$. Then, $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A) \neq \phi$. Let $y \in \alpha_{[\gamma,\gamma']}$-$sCl(U) \cap \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. Since $\alpha_{[\gamma,\gamma']}$-$sCl(U) \in \alpha SO(X, y)_{[\gamma,\gamma']}$, then $\alpha_{[\gamma,\gamma']}$-$sCl(\alpha_{[\gamma,\gamma']}$-$sCl(U)) \cap A \neq \phi$, that is $\alpha_{[\gamma,\gamma']}$-$sCl(U) \cap A \neq \phi$. Thus, $x \in \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. □

Corollary 3.12. *For any $A \subseteq X$, $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed.*

Proof. Obvious. □

Theorem 3.13. *Intersection of arbitrary collection of $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed sets in X is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed.*

Proof. Let $\{A_i : i \in I\}$ be any collection of $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed sets in a topological space (X, τ) and $A = \cap_{i\in I} A_i$. Now, using Definition 3.3, $x \in \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$, in consequence, $x \in \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A_i)$ for all $i \in I$. Follows that $x \in A_i$ for all $i \in I$. Therefore, $x \in A$. Thus, $A = \alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$. □

Corollary 3.14. *For any $A \subseteq X$, $\alpha_{[\gamma,\gamma']}$-$sCl_\theta(A)$ is the intersection of all $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed sets each containing A.*

Proof. Obvious. □

Corollary 3.15. *Let A and A_i $(i \in I)$ be any subsets of a space X. Then, the following properties hold:*

(i) A is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen in X if and only if for each $x \in A$ there exists $U \in \alpha SR(X)_{[\gamma,\gamma']}$ such that $x \in U \subseteq A$.

(ii) If A_i is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen in X for each $i \in I$, then $\cup_{i\in I} A_i$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen in X.

Proof. Obvious. □

Remark 3.16. The following example shows that the union of $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed sets may fail to be $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed.

Example 3.17. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and γ', respectively, by $A^\gamma = Int(Cl(A))$ and

$$A^{\gamma'} = \begin{cases} X & \text{if } A = \{a, c\} \\ A & \text{if } A \neq \{a, c\}. \end{cases}$$

Then, the subsets $A = \{a\}$ and $B = \{c\}$ are $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed, but their union $\{a, c\} = A \cup B$ is not $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed.

Example 3.18. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A & \text{if } A \neq \{a, c\} \\ X & \text{if } A = \{a, c\}, \end{cases}$$

and

$$A^{\gamma'} = \begin{cases} A & \text{if } A \neq \{a, b\} \\ X & \text{if } A = \{a, b\}. \end{cases}$$

The subsets $\{b\}$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed, but not $\alpha_{[\gamma,\gamma']}$-semiregular.

Remark 3.19. From Lemma 3.6 (ii), we have $\alpha_{[\gamma,\gamma']}$-semiregular set is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed set. In the above example, $\{b\}$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-semiclosed, but not $\alpha_{[\gamma,\gamma']}$-semiregular. Again, for a subset A, we always have $A \subseteq \alpha_{[\gamma,\gamma']}$-$sCl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$sCl_\theta$(A). Therefore, every $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen set is $\alpha_{[\gamma,\gamma']}$-semiopen. The following example shows that the converse is not true in general.

Example 3.20. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ be a topology on X. For each $A \in \alpha O(X, \tau)$, we define two operations γ and γ', respectively, by

$$A^{\gamma} = \begin{cases} A & \text{if } A \neq \{a\} \\ X & \text{if } A = \{a\}, \end{cases}$$

and

$$A^{\gamma'} = \begin{cases} A & \text{if } A \neq \{b\} \\ X & \text{if } A = \{b\}. \end{cases}$$

Then, $\{a, b\}$ is $\alpha_{[\gamma,\gamma']}$-semiopen set but not an $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen set.

Remark 3.21. The notions $\alpha_{[\gamma,\gamma']}$-openness and $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopenness are independent. In Example 3.18, $\{a, b\}$ is an $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen set but not an $\alpha_{[\gamma,\gamma']}$-open set, whereas in Example 3.20, $\{a, b\}$ is an $\alpha_{[\gamma,\gamma']}$-open set but not an $\alpha_{[\gamma,\gamma']}$-$\theta$-semiopen set.

Remark 3.22. Every $\alpha_{[\gamma,\gamma']}$-$\theta$-open set is $\alpha_{[\gamma,\gamma']}$-open.

4 $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous Functions

Throughout this section, let $\gamma, \gamma^{'} : \alpha O(X) \to P(X)$ and $\beta, \beta^{'} : \alpha O(Y) \to P(Y)$ be operations on $\alpha O(X)$ and $\alpha O(Y)$, respectively.

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous if for each point $x \in X$ and each $\alpha_{[\beta,\beta']}$-semiopen set V of Y containing $f(x)$, there exists an $\alpha_{[\gamma,\gamma']}$-open set U of X containing x such that $f(U) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V)$.

Example 4.2. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{3\}, \{1, 2\}, Y\}$. For each $A \in \alpha O(X, \tau)$ and $B \in \alpha O(Y, \sigma)$, we define the operations $\gamma : \alpha O(X, \tau) \to P(X)$, $\gamma^{'} : \alpha O(X, \tau) \to P(X)$, $\beta : \alpha O(Y, \sigma) \to P(Y)$ and $\beta^{'} : \alpha O(Y, \sigma) \to P(Y)$, respectively, by

$$A^{\gamma} = \begin{cases} A & \text{if } c \in A \\ A \cup \{c\} & \text{if } c \notin A, \end{cases}$$

$$A^{\gamma^{'}} = \begin{cases} A & \text{if } b \in A \\ A \cup \{b\} & \text{if } b \notin A, \end{cases}$$

$$B^{\beta} = \begin{cases} Y & \text{if } 2 \notin B \\ B & \text{if } 2 \in B, \end{cases}$$

and

$$B^{\beta^{'}} = \begin{cases} Y & \text{if } 1 \notin B \\ B & \text{if } 1 \in B. \end{cases}$$

Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 1 & \text{if } x = b \\ 3 & \text{if } x = c. \end{cases}$$

Clearly, $\alpha O(X, \tau)_{[\gamma,\gamma']} = \{\phi, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\alpha SO(Y, \sigma)_{[\beta,\beta']} = \{\phi, \{1, 2\}, Y\}$. Then, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous.

Theorem 4.3. *The following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

(i) f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous.

(ii) For each $x \in X$ and $V \in \alpha SR(Y)_{[\beta,\beta']}$ containing $f(x)$, there exists an $\alpha_{[\gamma,\gamma']}$-open set U containing x such that $f(U) \subseteq V$.

(iii) $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-clopen (That is, $\alpha_{[\gamma,\gamma']}$-open as well as $\alpha_{[\gamma,\gamma']}$-closed) in X for every $V \in \alpha SR(Y)_{[\beta,\beta']}$.

(iv) $f^{-1}(V) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V)))$ for every $V \in \alpha SO(Y)_{[\beta,\beta']}$.

(v) $\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(V))) \subseteq f^{-1}(V)$ for every $\alpha_{[\beta,\beta']}$-semiclosed set V of Y.

(vi) $\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(V)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$ for every $V \in \alpha SO(Y)_{[\beta,\beta']}$.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and $V \in \alpha SR(Y)_{[\beta,\beta']}$ containing $f(x)$. By (1), there exists an $\alpha_{[\gamma,\gamma']}$-open set U containing x such that $f(U) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V) = V$.

(2) $\Rightarrow$ (3): Let $V \in \alpha SR(Y)_{[\beta,\beta']}$ and $x \in f^{-1}(V)$. Then, $f(U) \subseteq V$ for some $\alpha_{[\gamma,\gamma']}$-open set U of X containing x, hence $x \in U \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-open in X. Since $Y \setminus V \in \alpha SR(Y)_{[\beta,\beta']}$, $f^{-1}(Y \setminus V)$ is also $\alpha_{[\gamma,\gamma']}$-open and hence $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-clopen in X.

(3) $\Rightarrow$ (4): Let $V \in \alpha SO(Y)_{[\beta,\beta']}$. Since $V \subseteq \alpha_{[\beta,\beta']}$-$sCl(V)$ and by Lemma 3.2, we have $\alpha_{[\beta,\beta']}$-$sCl(V) \in \alpha SR(Y)_{[\beta,\beta']}$. By (3), we have $f^{-1}(V) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$ and $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$ is $\alpha_{[\gamma,\gamma']}$-open in X. Therefore, we obtain $f^{-1}(V) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V)))$.

(4) $\Rightarrow$ (5): Let V be an $\alpha_{[\beta,\beta']}$-semiclosed subset of Y. By (4), we have $f^{-1}(Y \setminus V) \subseteq \alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(Y \setminus V))) = \alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(Y \setminus \alpha_{[\beta,\beta']}$-$sInt(V))) = X \setminus \alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(V)))$. Therefore, we obtain $\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(V))) \subseteq f^{-1}(V)$.

(5) $\Rightarrow$ (6): Let $V \in \alpha SO(Y)_{[\beta,\beta']}$. By Lemma 3.2, $\alpha_{[\beta,\beta']}$-$sCl(V) \in \alpha SR(Y)_{[\beta,\beta']}$. By (5), we obtain $\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(V)) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))) = \alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V)))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and $V \in \alpha SO(Y, f(x))_{[\beta,\beta']}$. By Lemma 3.2, we have $\alpha_{[\beta,\beta']}$-$sCl(V) \in \alpha SR(Y)_{[\beta,\beta']}$ and $f(x) \notin Y \setminus \alpha_{[\beta,\beta']}$-$sCl(V) = \alpha_{[\beta,\beta']}$-$sCl(Y \setminus \alpha_{[\beta,\beta']}$-$sCl(V))$. Thus, by (6) we obtain $x \notin \alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(Y \setminus \alpha_{[\beta,\beta']}$-$sCl(V)))$. There exists an $\alpha_{[\gamma,\gamma']}$-open set U of X containing x such that $U \cap f^{-1}(Y \setminus \alpha_{[\beta,\beta']}$-$sCl(V)) = \phi$. Therefore, we have $f(U) \cap (Y \setminus \alpha_{[\beta,\beta']}$-$sCl(V)) = \phi$ and hence $f(U) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V)$. This shows that f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Theorem 4.4. *The following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

(i) f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous.

(ii) For each $x \in X$ and $V \in \alpha SR(Y)_{[\beta,\beta']}$ containing $f(x)$, there exists an $\alpha_{[\gamma,\gamma']}$-clopen set U containing x such that $f(U) \subseteq V$.

(iii) For each $x \in X$ and $V \in \alpha SO(Y)_{[\beta,\beta']}$ containing $f(x)$, there exists an $\alpha_{[\gamma,\gamma']}$-open set U containing x such that $f(\alpha_{[\gamma,\gamma']}$-$Cl(U)) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V)$.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and $V \in \alpha SR(Y)_{[\beta,\beta']}$ containing $f(x)$. By Theorem 4.3, $f^{-1}(V)$ is $\alpha_{[\gamma,\gamma']}$-clopen in X. Put $U = f^{-1}(V)$, then $x \in U$ and $f(U) \subseteq V$.

(2) $\Rightarrow$ (3): Let $V \in \alpha SO(Y, f(x))_{[\beta,\beta']}$. By Lemma 3.2, we have $\alpha_{[\beta,\beta']}$-$sCl(V) \in \alpha SR(Y)_{[\beta,\beta']}$ and by (2), there exists an $\alpha_{[\gamma,\gamma']}$-clopen set U containing x such that $f(\alpha_{[\gamma,\gamma']}$-$Cl(U)) = f(U) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V)$.

(3) $\Rightarrow$ (1): Let $x \in X$ and $V \in \alpha SO(Y, f(x))_{[\beta,\beta']}$. By (3), there exists an $\alpha_{[\gamma,\gamma']}$-open set U containing x such that $f(\alpha_{[\gamma,\gamma']}$-$Cl(U)) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V)$ implies that $f(U) \subseteq f(\alpha_{[\gamma,\gamma']}$-$Cl(U)) \subseteq \alpha_{[\beta,\beta']}$-$sCl(V)$. This shows that f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Theorem 4.5. *The following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

(i) f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous.

(ii) $\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl_\theta(B))$ for every subset B of Y.

(iii) $f(\alpha_{[\gamma,\gamma']}$-$Cl(A)) \subseteq \alpha_{[\beta,\beta']}$-$sCl_\theta(f(A))$ for every subset A of X.

(iv) $f^{-1}(F)$ *is* $\alpha_{[\gamma,\gamma']}$*-closed in* X *for every* $\alpha_{[\beta,\beta']}$*-*θ*-semiclosed set* F *of* Y.

(v) $f^{-1}(V)$ *is* $\alpha_{[\gamma,\gamma']}$*-open in* X *for every* $\alpha_{[\beta,\beta']}$*-*θ*-semiopen set* V *of* Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y and $x \notin f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B))$. Then, $f(x) \notin \alpha_{[\beta,\beta']}\text{-}sCl_\theta(B)$ and there exists $V \in \alpha SO(Y, f(x))_{[\beta,\beta']}$ such that $\alpha_{[\beta,\beta']}\text{-}sCl(V) \cap B = \phi$. By (1), there exists an $\alpha_{[\gamma,\gamma']}$-open set U containing x such that $f(U) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl(V)$. Hence $f(U) \cap B = \phi$ and $U \cap f^{-1}(B) = \phi$. Consequently, we obtain $x \notin \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(B))$.

(2) $\Rightarrow$ (3): Let A be any subset of X. By (2), we have $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(f(A))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(f(A)))$ and hence $f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(f(A))$.

(3) $\Rightarrow$ (4): Let F be any $\alpha_{[\beta,\beta']}$-θ-semiclosed set of Y. Then, by (3), we have $f(\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(F))) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(f(f^{-1}(F))) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(F) = F$. Therefore, we have $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(F)) \subseteq f^{-1}(F)$ and hence $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $\alpha_{[\gamma,\gamma']}$-closed in X.

(4) $\Rightarrow$ (5): Obvious.

(5) $\Rightarrow$ (1): Let $x \in X$ and $V \in \alpha SO(Y, f(x))_{[\beta,\beta']}$. By Lemmas 3.2 and 3.6 (ii), $\alpha_{[\beta,\beta']}\text{-}sCl(V)$ is $\alpha_{[\beta,\beta']}$-θ-semiopen in Y. Put $U = f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V))$. Then by (5), U is $\alpha_{[\gamma,\gamma']}$-open containing x and $f(U) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl(V)$. Thus, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Theorem 4.6. *The following statements are equivalent for a function* $f : (X, \tau) \to (Y, \sigma)$*:*

(i) f *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-*$\theta$*-semicontinuous.*

(ii) $\alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(f^{-1}(B)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B))$ *for every subset* B *of* Y.

(iii) $f(\alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(f(A))$ *for every subset* A *of* X.

(iv) $f^{-1}(F)$ *is* $\alpha_{[\gamma,\gamma']}$*-*$\theta$*-closed in* X *for every* $\alpha_{[\beta,\beta']}$*-*θ*-semiclosed set* F *of* Y.

(v) $f^{-1}(V)$ *is* $\alpha_{[\gamma,\gamma']}$*-*$\theta$*-open in* X *for every* $\alpha_{[\beta,\beta']}$*-*θ*-semiopen set* V *of* Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y and $x \notin f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B))$. Then, $f(x) \notin \alpha_{[\beta,\beta']}\text{-}sCl_\theta(B)$ and there exists $V \in \alpha SO(Y, f(x))_{[\beta,\beta']}$ such that $\alpha_{[\beta,\beta']}\text{-}sCl(V) \cap B = \phi$. By Theorem 4.4 (iii), there exists an $\alpha_{[\gamma,\gamma']}$-open set U containing x such that $f(\alpha_{[\gamma,\gamma']}\text{-}Cl(U)) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl(V)$. Hence $f(\alpha_{[\gamma,\gamma']}\text{-}Cl(U)) \cap B = \phi$ and $\alpha_{[\gamma,\gamma']}\text{-}Cl(U) \cap f^{-1}(B) = \phi$. Consequently, we obtain $x \notin \alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(f^{-1}(B))$.

(2) $\Rightarrow$ (3): Let A be any subset of X. By (2), we have $\alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(A) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(f^{-1}(f(A))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(f(A)))$ and hence $f(\alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(f(A))$.

(3) $\Rightarrow$ (4): Let F be any $\alpha_{[\beta,\beta']}$-θ-semiclosed set of Y. Then, by (3), we have $f(\alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(f^{-1}(F))) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(f(f^{-1}(F))) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(F) = F$. Therefore, we have $\alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(f^{-1}(F)) \subseteq f^{-1}(F)$ and hence $\alpha_{[\gamma,\gamma']}\text{-}Cl_\theta(f^{-1}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $\alpha_{[\gamma,\gamma']}$-$\theta$-closed in X.

(4) $\Rightarrow$ (5): Obvious.

(5) $\Rightarrow$ (1): Let $x \in X$ and $V \in \alpha SO(Y, f(x))_{[\beta,\beta']}$. By Lemmas 3.2 and 3.6 (ii), $\alpha_{[\beta,\beta']}\text{-}sCl(V)$ is $\alpha_{[\beta,\beta']}$-θ-semiopen in Y. Put $U = f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V))$. Then by (5), U is $\alpha_{[\gamma,\gamma']}$-$\theta$-open containing x and by Remark 3.22, U is $\alpha_{[\gamma,\gamma']}$-open such that $f(U) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl(V)$. Thus, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Proposition 4.7. *The following statements are equivalent for a function* $f : (X, \tau) \to (Y, \sigma)$*:*

(i) f *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-*$\theta$*-semicontinuous.*

(ii) $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl(B)))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B))$*, for every subset* B *of* Y*.*

(iii) $f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(B)) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(\alpha_{[\beta,\beta']}\text{-}sInt(B))))$*, for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let B be any subset of Y. Then, $\alpha_{[\beta,\beta']}\text{-}sCl(B)$ is $\alpha_{[\beta,\beta']}$-semiclosed in Y and by Theorem 4.3 (v), we have that if $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl(B))))$, then $x \in f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B))$.

(2) ⇒ (3): Let B be any subset of Y and $x \in f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(B))$. Then we have $x \in f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(B)) = X \setminus f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(Y \setminus B))$. Then $x \notin f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(Y \setminus B))$ and by (2), we have $x \in X \setminus \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl(Y \setminus B)))) = \alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(\alpha_{[\beta,\beta']}\text{-}sInt(B))))$.

(3) ⇒ (1): Let V be any $\alpha_{[\beta,\beta']}$-semiopen set of Y. Suppose that $z \notin f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V))$. Then, $f(z) \notin \alpha_{[\beta,\beta']}\text{-}sCl(V)$ and there exists an $\alpha_{[\beta,\beta']}$-semiopen set W containing $f(z)$ such that $W \cap V = \phi$ and hence $\alpha_{[\gamma,\gamma']}\text{-}sCl(W) \cap V = \phi$. By (3), we have $z \in \alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(W)))$ and hence there exists $U \in \alpha O(X)_{[\gamma,\gamma']}$ such that $z \in U \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(W))$. Since $\alpha_{[\beta,\beta']}\text{-}sCl(W) \cap V = \phi$, $U \cap f^{-1}(V) = \phi$ and so, $z \notin \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(V))$. Therefore, $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl((V)))$ for every $V \in \alpha SO(Y)_{[\beta,\beta']}$. Hence, by Theorem 4.3, f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Proposition 4.8. *The following statements are equivalent for a function* $f : (X, \tau) \to (Y, \sigma)$*:*

(i) f *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-*$\theta$*-semicontinuous.*

(ii) $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B)))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B))$*, for every subset* B *of* Y*.*

(iii) $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl(B)))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B))$*, for every subset* B *of* Y*.*

(iv) $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl(O)))) \subseteq f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(O))$*, for every* $\alpha_{[\beta,\beta']}$*-semiopen set* O *of* Y*.*

Proof. (1) ⇒ (2): Let B be any subset of Y. Then, $\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B)$ is $\alpha_{[\beta,\beta']}$-semiclosed in Y. Then by Theorem 4.3 (v), if $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B))))$, then $x \in f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl_\theta(B))$.

(2) ⇒ (3): This is obvious since $\alpha_{[\beta,\beta']}\text{-}sCl(B) \subseteq \alpha_{[\beta,\beta']}\text{-}sCl_\theta(B)$ for every subset B.

(3) ⇒ (4): By Lemma 3.6 (i), we have $\alpha_{[\beta,\beta']}\text{-}sCl(O) = \alpha_{[\beta,\beta']}\text{-}sCl_\theta(O)$ for every $\alpha_{[\beta,\beta']}$-semiopen set O.

(4) ⇒ (1): Let V be any $\alpha_{[\beta,\beta']}$-semiopen set of Y and $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(V))$. Then, V is $\alpha_{[\beta,\beta']}$-semiopen and $x \in \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sInt(\alpha_{[\beta,\beta']}\text{-}sCl(V))))$. By (4), $x \in f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V))$. It follows from Theorem 4.3, that f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Proposition 4.9. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is* $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$*-*$\theta$*-semicontinuous if and only if* $f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V))$ *is* $\alpha_{[\gamma,\gamma']}$*-open set in* X*, for each* $\alpha_{[\beta,\beta']}$*-semiopen set* V *in* Y*.*

Proof. Let V be any $\alpha_{[\beta,\beta']}$-semiopen set in Y. We have to show that $f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V))$ is $\alpha_{[\gamma,\gamma']}$-open set in X. Let $x \in f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(V))$. Then, $f(x) \in \alpha_{[\beta,\beta']}\text{-}sCl(V)$ and $\alpha_{[\beta,\beta']}\text{-}sCl(V) \in \alpha SR(Y)_{[\beta,\beta']}$. Since f is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-$\theta$-semicontinuous, then by Theorem 4.3 (ii), there exists an $\alpha_{[\gamma,\gamma']}$-open set U of X containing x such that $f(U)\subseteq\alpha_{[\beta,\beta']}$-$sCl(V)$. Which implies that $x\in U\subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$. Therefore, $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$ is an $\alpha_{[\gamma,\gamma']}$-open set in X.

Conversely, let $x\in X$ and V be any $\alpha_{[\beta,\beta']}$-semiopen set of Y containing $f(x)$. Then $x\in f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$, by hypothesis $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$ is an $\alpha_{[\gamma,\gamma']}$-open set in X containing x, so clearly $f(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))\subseteq\alpha_{[\beta,\beta']}$-$sCl(V)$. Therefore, f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Proposition 4.10. *A function $f:(X,\tau)\to(Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous if and only if $f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(F))$ is an $\alpha_{[\gamma,\gamma']}$-closed set in X, for each $\alpha_{[\beta,\beta']}$-semclosed set F of Y.*

Proof. Let F be any $\alpha_{[\beta,\beta']}$-semclosed set of Y. Then, $Y\setminus F$ is an $\alpha_{[\beta,\beta']}$-semopen set of Y, since f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. Then by Proposition 4.9, $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(Y\setminus F))$ is an $\alpha_{[\gamma,\gamma']}$-open set in X and $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(Y\setminus F))=f^{-1}(Y\setminus\alpha_{[\beta,\beta']}$-$sInt(F))=X\setminus f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(F))$ is an $\alpha_{[\gamma,\gamma']}$-open set in X and hence $f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(F))$ is an $\alpha_{[\gamma,\gamma']}$-closed set in X.

Conversely, let V be any $\alpha_{[\beta,\beta']}$-semopen set of Y. Then $Y\setminus V$ is $\alpha_{[\beta,\beta']}$-semclosed, and by hypothesis $f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(Y\setminus V))=f^{-1}(Y\setminus\alpha_{[\beta,\beta']}$-$sCl(V))=X\setminus f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$ is an $\alpha_{[\gamma,\gamma']}$-closed set in X, so $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$ is an $\alpha_{[\gamma,\gamma']}$-open set in X. Therefore, by Proposition 4.9, f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Proposition 4.11. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. If $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl_\theta(B))$ is $\alpha_{[\gamma,\gamma']}$-closed in X for every subset B of Y, then f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous.*

Proof. Let $B\subseteq Y$. Since $f^{-1}(\alpha_{[\beta,\beta']}$-$sCl_\theta(B))$ is $\alpha_{[\gamma,\gamma']}$-closed in X, then $\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(B))\subseteq\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl_\theta(B)))=f^{-1}(\alpha_{[\beta,\beta']}$-$sCl_\theta(B))$. By Theorem 4.5, f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Proposition 4.12. *The following statements are equivalent for a function $f:(X,\tau)\to(Y,\sigma)$:*

(i) *f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous.*

(ii) *$\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(V))\subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$, for every $V\subseteq\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V))$.*

(iii) *$f^{-1}(V)\subseteq\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V)))$, for every $V\subseteq\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V))$.*

Proof. (1) ⇒ (2): Let $V\subseteq\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V))$ such that $x\in\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(V))$. Suppose that $x\notin f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$. Then there exists an $\alpha_{[\beta,\beta']}$-semopen set W containing $f(x)$ such that $W\cap V=\phi$. Hence, we have $W\cap\alpha_{[\beta,\beta']}$-$sCl(V)=\phi$ and hence $\alpha_{[\beta,\beta']}$-$sCl(W)\cap\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V))=\phi$. Since $V\subseteq\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V))$ and we have $V\cap\alpha_{[\beta,\beta']}$-$sCl(W)=\phi$. Since f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous at $x\in X$ and W is an $\alpha_{[\beta,\beta']}$-semopen set containing $f(x)$, there exists $U\in\alpha O(X)_{[\gamma,\gamma']}$ containing x such that $f(U)\subseteq\alpha_{[\beta,\beta']}$-$sCl(W)$. Then $f(U)\cap V=\phi$ and hence $U\cap f^{-1}(V)=\phi$. This shows that $x\notin\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(V))$. This is a contradiction. Therefore, we have $x\in f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V))$.

(2) ⇒ (3): Let $V\subseteq\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V))$ and $x\in f^{-1}(V)$. Then, we have $f^{-1}(V)\subseteq f^{-1}(\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V)))=X\setminus f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(Y\setminus\alpha_{[\beta,\beta']}$-$sCl(V)))$. Therefore, $x\notin f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(Y\setminus\alpha_{[\beta,\beta']}$-$sCl(V)))$. Then by (2), $x\notin\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(Y\setminus\alpha_{[\beta,\beta']}$-$sCl(V)))$. Hence, $x\in X\setminus\alpha_{[\gamma,\gamma']}$-$Cl(f^{-1}(Y\setminus\alpha_{[\beta,\beta']}$-$sCl(V)))=\alpha_{[\gamma,\gamma']}$-$Int(f^{-1}(\alpha_{[\beta,\beta']}$-$sCl(V)))$.

(3) ⇒ (1): Let V be any $\alpha_{[\beta,\beta']}$-semiopen set of Y. Then, $V=\alpha_{[\beta,\beta']}$-$sInt(V)\subseteq\alpha_{[\beta,\beta']}$-$sInt(\alpha_{[\beta,\beta']}$-$sCl(V))$. Hence, by (3) and Theorem 4.3, f is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous. □

Proposition 4.13. *If $f:(X,\tau)\to(Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']},\alpha_{[\beta,\beta']})$-$\theta$-semicontinuous at $x\in X$, then for each $\alpha_{[\beta,\beta']}$-semiopen set B containing $f(x)$ and each $\alpha_{[\gamma,\gamma']}$-open set A containing x, there* *exists a nonempty* $\alpha_{[\gamma,\gamma']}$*-open set* $U \subseteq A$ *such that* $U \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B)))$*. Where* γ *and* $\gamma^{'}$ *are* α*-regular operations.*

Proof. Let B be any $\alpha_{[\beta,\beta']}$-semiopen set containing $f(x)$ and A be an $\alpha_{[\gamma,\gamma']}$-open set of X containing x. By Lemma 3.2 and Theorem 4.3, $x \in \alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B)))$, then $A \cap \alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B))) \neq \phi$. Take $U = A \cap \alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B)))$. Thus, U is a nonempty $\alpha_{[\gamma,\gamma']}$-open set by [[1], Proposition 3.4], and hence $U \subseteq A$ and $U \subseteq \alpha_{[\gamma,\gamma']}\text{-}Int(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B))) \subseteq \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(\alpha_{[\beta,\beta']}\text{-}sCl(B)))$. □

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