Palestine Journal of Mathematics

Vol. 7(1)(2018) , 191–201

Applications of symbolic operators to the Kampé de Fériets double hypergeometric series

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Communicated by P. K. Banerji

MSC 2010 Classifications: Primary 35K65; Secondary 33C20, 33C65.

Keywords and phrases: Generalized hypergeometric function; Kampé de Fériets functions; Decomposition formulas; Inverse pairs of symbolic operators.

Abstract In our present investigation we propose to present certain decomposition formulas for Kampé de Fériets series of double hypergeometric series $F^{p:q;k}_{l:m;n}$. Based upon the theory of symbolic operators, we introduce a new symbolic operational images and by means of these symbolic operational images a number of decomposition formulas for Kampé de Fériets series are then found. Other closely-related special cases are also considered.

1. Introduction and Preliminaries

The enormous success of the theory of hypergeometric series in a single variable, that are the Gaussian hypergeometric function ${}_2F_1$ and the generalized hypergeometric function ${}_pF_q$ [23, p.19(23)], has stimulated the development of a corresponding theory in two variables. In this regard we recall the definitions of the four Appell functions F_1, F_2, F_3 and F_4 as follows [23, p.22-23]:

$$F_1\left[a,b,c;e;x,y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_{r+s}(b)_r(c)_s x^r y^s}{(e)_{r+s} r!s!}, |x|<1, |y|<1, \tag{1.1}$$

$$F_2\left[a,b,c;e,f;x,y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_{r+s}(b)_r(c)_s x^r y^s}{(e)_r (f)_s r!s!}, |x|+|y|<1, \tag{1.2}$$

$$F_3\left[a,b,c,d;e;x,y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_r(b)_s(c)_r(d)_r x^r y^s}{(e)_{r+s} r!s!}, |x|<1, |y|<1, \tag{1.3}$$

and

$$F_4\left[a,b;e,f;x,y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_{r+s}(b)_{r+s} x^r y^s}{(e)_r (f)_s r!s!}, |\sqrt{x}|+|\sqrt{y}|<1, \tag{1.4}$$

where $(a)_n$ denotes the Pochhammer symbol given by $(a)_0 = 1$, $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ and Γ is the Gamma function. The above four Appell series were unified and generalized by Kampé de Fé(see [1] and [22]) who defined a general hypergeometric series of two variables as follows:

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{cc}(a_p):(b_q);(c_k); & \\ & x,y \\ (e_l):(f_m);(g_n); & \end{array}\right]=\sum_{r,s=0}^{\infty}\frac{((a_p))_{r+s}((b_q))_r((c_k))_s x^r y^s}{((e_l))_{r+s}(f_m)_r(g_n)_s r!s!}, \tag{1.5}$$

where for convergence

$$p+q<l+m+1, p+k<l+n+1, |x|<\infty, |y|<\infty,$$

or

$$p+q=l+m+1, p+k=l+n+1, \text{and}$$

$$|x|^{\frac{1}{(p-l)}}+|y|^{\frac{1}{(p-l)}}<1, \text{if} \quad p>l; \qquad max\{|x|,|y|\}<1, \text{if} \quad p\le l,$$

and

$$((a_p))_{r+s} = \prod_{j=1}^{p} (a_j)_{r+s} = (a_1)_{r+s} (a_2)_{r+s} \cdots (a_p)_{r+s},$$

with similar interpretations for $((e_r))_{r+s}$, et cetera.
A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of one and several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [23, p.47-48]. Also, in this regard, it is noticed that the general sextic equation can be solved in terms of Kampé de Fé function (see [5]). A decomposition formula for a hypergeometric function is the one which describes the hypergeometric function with a summation of same or other hypergeometric functions. It was started to study by Burchnall and Chaundy in 1940 for AppellŠs double hypergeometric functions ([3] and [4]) and Chaundy [6]. Recently, it has been studying for various special functions by many mathematicians (see [2],[7]-[9] and [12]- [20]). In particular, Hasanov and Srivastava [12] and [16] presented a number of decomposition formulas in terms of such simpler hypergeometric functions as the Gauss and Appell functions and Choi-Hasanov [7] gave a formula of an analytic continuation of the Clausen hypergeometric function ${}_3F_2$ as an application of their decomposition formula. In [2], using the differential operator D and its inverse D^{-1}(defined as the integral operator and setting the lower limit to 0), Bin-Saad develops techniques to represent hypergeometric functions and their generalizations with several summation quantifiers. Here the operator (and combinations of it) is applied to simpler expressions like, e.g., products such that its image produces the desired series. In addition, new decomposition formulas are presented. Various examples illustrate how these techniques can be applied (see[2]). In [7] the authors introduced the following symbolic operators:

$$H_{x_1,x_2,\ldots,x_r}(\alpha,\beta) = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\beta-\alpha)_{m_1+\ldots+m_r} (-\delta_1)_{m_1} (-\delta_2)_{m_2} \cdots (-\delta_{m_r})_{m_r}}{(\beta)_{m_1+\ldots+m_r} m_1!\ldots m_r!}, \tag{1.6}$$

$$\bar{H}_{x_1,x_2,\ldots,x_r}(\alpha,\beta) = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\beta-\alpha)_{m_1+\ldots+m_r} (-\delta_1)_{m_1} (-\delta_2)_{m_2} \cdots (-\delta_{m_r})_{m_r}}{(1-\alpha-\delta_1-\ldots-\delta_{m_r})_{m_1+\ldots+m_r} m_1!\ldots m_r!}, \tag{1.7}$$

$$\delta_i = x_i \frac{\partial}{\partial x_i}.$$

Based on the operators defined in (1.6) and (1.7), we aim in this paper to study similar type of decomposition formulas to Choi-HasanovŠs ones [7] for the Kampé de Fé double hypergeoemtric series (1.5) proved by using the theory of symbolic operators.

2. A set of operator identities

By using Burchnall and Chaundy [3] and [4] and Chaundy [6] method together with symbolic operators (1.6) and (1.7), we find the following set of operator identities and decomposition formulas for a number of interesting special cases of Kampé de Fé double hypergeometric function $F^{p:q;k}_{l:m;n}$:

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{lr} a : b, c, d; b_1, c_1, d_1; & \\ & x, y \\ e : f, g; \quad f_1, g_1; & \end{array}\right]$$

$$= H_y(d_1, g_1) F^{1:3;2}_{1:2;1}\left[\begin{array}{lr} a : b, c, d; b_1, c_1; & \\ & x, y \\ e : f, g; \quad f_1; & \end{array}\right], \tag{2.1}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad\qquad x,y \\ e:f,g; \qquad f_1,g_1; \end{array}\right]$$

$$=H_x(d,g)H_y(d_1,g_1)F^{1:2;2}_{1:1;1}\left[\begin{array}{l} a:b,c;b_1,c_1; \\ \qquad\qquad\qquad\qquad x,y \\ e:f; \qquad f_1; \end{array}\right], \tag{2.2}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad\qquad x,y \\ e:f,g; \qquad f_1,g_1; \end{array}\right]$$

$$=H_x(d,g)H_y(c_1,f_1)H_y(d_1,g_1)F^{1:2;1}_{1:1;0}\left[\begin{array}{l} a:b,c;b_1; \\ \qquad\qquad\qquad x,y \\ e:f;-; \end{array}\right], \tag{2.3}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad\qquad x,y \\ e:f,g; \qquad f_1,g_1; \end{array}\right]$$

$$=H_x(c,f)H_x(d,g)H_y(c_1,f_1)H_y(d_1,g_1)F_1\left[a,b;b_1;e;x,y\right], \tag{2.4}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad\qquad x,y \\ e:f,g; \qquad f_1,g_1; \end{array}\right]$$

$$=H_{x,y}(a,e){}_3F_2\left[b,c,d;f,g;x\right]{}_3F_2\left[b_1,c_1,d_1;f_1,g_1;y\right], \tag{2.5}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad\qquad x,y \\ -:e,f,g;e_1,f_1,g_1; \end{array}\right]$$

$$=H_y(d_1,g_1)F^{1:3;2}_{0:3;2}\left[\begin{array}{l} a:b,c,d;b_1,c_1; \\ \qquad\qquad\qquad\qquad x,y \\ -:e,f,g;e_1,f_1; \end{array}\right], \tag{2.6}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad\qquad x,y \\ -:e,f,g;e_1,f_1,g_1; \end{array}\right]$$

$$=H_x(d,g)H_y(d_1,g_1)F^{1:2;2}_{0:2;2}\left[\begin{array}{l} a:b,c;b_1,c_1; \\ \qquad\qquad\qquad\qquad x,y \\ -:e,f;e_1,f_1; \end{array}\right], \tag{2.7}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad\qquad x,y \\ -:e,f,g;e_1,f_1,g_1; \end{array}\right]$$

$$= H_x(d,g)H_y(c_1,f_1)H_y(d_1,g_1)F^{1:2;1}_{0:2;1}\left[\begin{array}{ll} a:b,c;b_1; & \\ & x,y \\ -:e,f;e_1; & \end{array}\right], \tag{2.8}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{ll} a:b,c,d;b_1,c_1,d_1; & \\ & x,y \\ -:e,f,g;e_1,f_1,g_1; & \end{array}\right]$$
$$= H_x(c,f)H_x(d,g)H_y(c_1,f_1)H_y(d_1,g_1)F_2\left[a,b;b_1;e;e_1;x,y\right], \tag{2.9}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$
$$= H_{x,y}(\lambda,e)F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ \lambda \quad :f,g; \qquad f_1,g_1; & \end{array}\right], \tag{2.10}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$
$$= H_y(d_1,g_1)F^{0:4;3}_{1:2;1}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1; & \\ & x,y \\ \lambda \quad :f,g; \qquad f_1; & \end{array}\right], \tag{2.11}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$
$$= H_x(d,g)H_y(d_1,g_1)F^{0:3;3}_{1:1;1}\left[\begin{array}{ll} -:a,b,c,;a_1,b_1,c_1; & \\ & x,y \\ \lambda \quad :f; \qquad f_1; & \end{array}\right], \tag{2.12}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$
$$= H_x(d,g)H_y(c_1,f_1)H_y(d_1,g_1)F^{0:3;2}_{1:1;0}\left[\begin{array}{ll} -:a,b,c,;a_1,b_1; & \\ & x,y \\ \lambda \quad :f;--; & \end{array}\right], \tag{2.13}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$
$$= H_x(c,f)H_x(d,g)H_y(c_1,f_1)H_y(d_1,g_1)F_3\left[a,a_1,b,b_1;e;x,y\right], \tag{2.14}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{l} a,b,c:d;d_1; \\ \qquad\qquad\qquad\qquad x,y \\ e:f,g;\ f_1,g_1; \end{array}\right]$$
$$=H_{x,y}(c,e)F^{2:1;1}_{0:2;2}\left[\begin{array}{l} a,b:d;d_1; \\ \qquad\qquad\qquad\qquad x,y \\ -:f,g;\quad f_1,g_1; \end{array}\right], \tag{2.15}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{l} a,b,c:d;d_1; \\ \qquad\qquad\qquad\qquad x,y \\ e:f,g;\ f_1,g_1; \end{array}\right]$$
$$=H_x(d,g)H_y(d_1,g_1)F^{3:0;0}_{1:1;1}\left[\begin{array}{l} a,b,c:-;-; \\ \qquad\qquad\qquad\qquad x,y \\ e:f;\quad f_1; \end{array}\right], \tag{2.16}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{l} a,b,c:d;d_1; \\ \qquad\qquad\qquad\qquad x,y \\ e:f,g;\ f_1,g_1; \end{array}\right]$$
$$=H_x(d,g)F^{3:0;1}_{1:1;2}\left[\begin{array}{l} a,b,c:-;d_1; \\ \qquad\qquad\qquad\qquad x,y \\ e:f;\quad f_1,g_1; \end{array}\right], \tag{2.17}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{l} a,b,c:d;d_1; \\ \qquad\qquad\qquad\qquad x,y \\ e:f,g;\ f_1,g_1; \end{array}\right]$$
$$=H_{x,y}(c,e)H_x(d,g)H_y(d_1,g_1)F_4\left[a,b;f,f_1;x,y\right], \tag{2.18}$$

3. Decomposition formulas via operator identities

In [21, p. 93], it is proved that, for any analytic functions f, the following equalities hold true:

$$(\delta+\alpha)_n\{f(\xi)\}=\xi^{1-\alpha}\frac{d}{d\xi}\{\xi^{\alpha+n-1}f(\xi)\}, \tag{3.1}$$

$$(-\delta)_n\{f(\xi)\}=(-1)^n\xi^n\frac{d}{d\xi}\{f(\xi)\}, \tag{3.2}$$

where $\left(\delta\equiv\xi\frac{d}{d\xi};n\in N_0=N\cup\{0\}\right)$.
In view of formulas (3.1) and (3.2), and taking into account the differentiation formula for hypergeometric functions, from operator identities (2.1) to (2.18), we find the following decomposition formulas:

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{l} a:b,c,d;b_1,c_1,d_1; \\ \qquad\qquad\qquad\qquad x,y \\ e:f,g;\quad f_1,g_1; \end{array}\right]$$
$$=\sum_{i=0}^{\infty}\frac{(-1)^i(a)_i(c)_i(g_1-d_1)_i}{i!(e)_i(f_1)_i(g_1)_i}y^iF^{1:3;2}_{1:2;1}\left[\begin{array}{l} a+i:b,c,d;b_1+i,c_1+i; \\ \qquad\qquad\qquad\qquad x,y \\ e+i:f,g;\quad f_1+i; \end{array}\right], \tag{3.3}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{ll} a:b,c,d;b_1,c_1,d_1; & \\ & x,y \\ e:f,g; \quad f_1,g_1; & \end{array}\right]$$

$$\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a)_{i+j}(b)_i(b_1)_j(c)_i(c_1)_j(g-d)_i(g_1-d_1)_j}{i!j!(e)_{i+j}(f)_i(f_1)_j(g)_i(g_1)_j}x^iy^j$$

$$F^{1:2;2}_{1:1;1}\left[\begin{array}{ll} a+i+j:b+i,c+i;b_1+j,c_1+j; & \\ & x,y \\ e+i+j:f+i; \quad f_1+j; & \end{array}\right], \tag{3.4}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{ll} a:b,c,d;b_1,c_1,d_1; & \\ & x,y \\ e:f,g; \quad f_1,g_1; & \end{array}\right]$$

$$=\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}(a)_{i+j+k}(b)_i(b_1)_{j+k}(c)_i(d_1)_j(g-d)_i(f_1-c_1)_j(g_1-d_1)_k}{i!J!k!(e)_{i+j+k}(f)_i(f_1)_j(g)_i(g_1)_j}x^iy^{j+k}$$

$$F^{1:2;1}_{1:1;0}\left[\begin{array}{ll} a+i+j+k:b+i,c+i;b_1+j+k; & \\ & x,y \\ e+i+j+k:f+i;-; & \end{array}\right], \tag{3.5}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{ll} a:b,c,d;b_1,c_1,d_1; & \\ & x,y \\ e:f,g; \quad f_1,g_1; & \end{array}\right]$$

$$=\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+l}(a)_{i+j+k+l}(b)_{i+j}(b_1)_{k+l}(d)_i(d_1)_k(g-d)_j(f_1-c_1)_k(g_1-d_1)_l}{i!j!k!l!(e)_{i+j}(g)_{i+j}(g_1)_{k+l}(f)_i(f_1)_k}x^{i+j}y^{j+k}$$

$$F_1\left[a+i+j+k+l,b+i+j;b_1+k+l;e+i+j+k+l;x,y\right], \tag{3.6}$$

$$F^{1:3;3}_{1:2;2}\left[\begin{array}{ll} a:b,c,d;b_1,c_1,d_1; & \\ & x,y \\ e:f,g; \quad f_1,g_1; & \end{array}\right]$$

$$=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(b)_i(b_1)_j(c)_i(c_1)_j(e-a)_{i+j}}{i!j!(e)_{i+j}(f)_i(f_1)_j(g)_i(g_1)_j}x^iy^j$$

$${}_3F_2\left[b+i,c+i,d+i;f+i,g+i;x\right]\times {}_3F_2\left[b_1+j,c_1+j,d_1+j;f_1+j,g_1+j;y\right], \tag{3.7}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{ll} a:b,c,d;b_1,c_1,d_1; & \\ & x,y \\ -:e,f,g;e_1,f_1,g_1; & \end{array}\right]=\sum_{i=0}^{\infty}\frac{(-1)^i(a)_i(c_1)_i(d_1)_i(g_1-d_1)_i}{i!(e)_i(f_1)_i(g_1)_i}y^i$$

$$F^{1:3;2}_{0:3;2}\left[\begin{array}{ll} a+i:b,c,d;b_1+i,c_1+i; & \\ & x,y \\ -:e,f,g;e_1+i,f_1+i; & \end{array}\right], \tag{3.8}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{l} a: b, c, d; b_1, c_1, d_1; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ -: e, f, g; e_1, f_1, g_1; \end{array}\right]$$

$$= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(a)_{i+j}(b)_i(b_1)_j(c)_i(c_1)_j(g-d)_i(g_1-d_1)_j}{i!j!(e)_i(e_1)_i(f)_j(f_1)_i(g)_i(g_1)_i} x^i y^j$$

$$F^{1:2;2}_{0:2;2}\left[\begin{array}{l} a+i+j: b+i, c+i; b_1+j, c_1+j; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ -: e+i, f+i; e_1+j, f_1+j; \end{array}\right], \tag{3.9}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{l} a: b, c, d; b_1, c_1, d_1; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ -: e, f, g; e_1, f_1, g_1; \end{array}\right]$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}(a)_{i+j+j}(b)_i(b_1)_{j+k}(c)_i(d_1)_j(g-d)_i(g_1-d_1)_k(f_1-c_1)_j}{i!j!k!(e)_i(e_1)_{j+k}(f)_i(f_1)_j(g)_i(g_1)_{j+k}} x^i y^{j+k}$$

$$F^{1:2;1}_{0:2;1}\left[\begin{array}{l} a+i+j+k: b+i, c+i; b_1+j+k; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ -: e+i, f+i; e_1+j+k; \end{array}\right], \tag{3.10}$$

$$F^{1:3;3}_{0:3;3}\left[\begin{array}{l} a: b, c, d; b_1, c_1, d_1; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ -: e, f, g; e_1, f_1, g_1; \end{array}\right]$$

$$= \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+j+k+l}(a)_{i+j+j+l}(b)_{i+j}(b_1)_{k+l}(d)_i(d_1)_k(g-d)_j(g_1-d_1)_l(f-c)_i(f_1-c_1)_k}{i!j!k!l!(e)_{i+j}(e_1)_{k+l}(f)_i(f_1)_k(g)_{i+j}(g_1)_{k+l}} x^{i+j} y^{k+l}$$

$$F_2\left[a+i+j+k+l, b+i+j; b_1+k+l; e+i+j; e_1+k+l; x, y\right], \tag{3.11}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{l} -: a, b, c, d; a_1, b_1, c_1, d_1; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ e \quad : f, g; \qquad f_1, g_1; \end{array}\right]$$

$$= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(a)_i(a_1)_j(b)_i(b_1)_j(c)_i(c_1)_j(d)_i(d_1)_j(e-\lambda)_{i+j}}{i!j!(e)_{i+j}(\lambda)_{i+j}(f)_i(f_1)_j(g)_i(g_1)_j} x^i y^j$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{l} -: a+i, b+i, c+i, d+i; a_1+j, b_1+j, c_1+j, d_1+j; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x, y \\ \lambda+i \quad : f+i, g+i; \qquad f_1+j, g_1+j; \end{array}\right], \tag{3.12}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{l} -: a, b, c, d; a_1, b_1, c_1, d_1; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ e \quad : f, g; \qquad f_1, g_1; \end{array}\right]$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i(a_1)_i(b_1)_i(c_1)_i(g_1-d_1)_i}{i!(e)_i(f_1)_i(g_1)_i} y^i$$

$$F^{0:4;3}_{1:2;1}\left[\begin{array}{ll} -:a,b,c,d;a_1+i,b_1+i,c_1+i; & \\ & x,y \\ e+i \quad :f,g; \qquad f_1+i; & \end{array}\right], \tag{3.13}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$

$$=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a)_i(b)_i(c)_i(a_1)_j(b_1)_j(c_1)_j(g-d)_i(g_1-d_1)_j}{i!j!(e)_{i+j}(f)_i(f_1)_j(g)_i(g_1)_j}x^iy^j$$

$$F^{0:4;3}_{1:2;1}\left[\begin{array}{ll} -:a+i,b+i,c+i,;a_1+j,b_1+j,c_1+j; & \\ & x,y \\ e+i+j \quad :f+i; \qquad f_1+j; & \end{array}\right], \tag{3.14}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$

$$=\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}(a)_i(a)_{j+k}(b)_i(b_1)_{j+k}(c)_i(d_1)_j(g-d)_i(g_1-d_1)_k(f_1-c_1)_j}{i!j!k!(e)_{i+j+k}(f)_i(f_1)_j(g)_i(g_1)_{j+k}}x^iy^{j+k}$$

$$F^{0:3;2}_{1:1;0}\left[\begin{array}{ll} -:a+i,+i,c+i,;a_1+j+k,b_1+j+k; & \\ & x,y \\ e+j+k \quad :f+i;--; & \end{array}\right], \tag{3.15}$$

$$F^{0:4;4}_{1:2;2}\left[\begin{array}{ll} -:a,b,c,d;a_1,b_1,c_1,d_1; & \\ & x,y \\ e \quad :f,g; \qquad f_1,g_1; & \end{array}\right]$$

$$=\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+l}(a)_{i+j}(a_1)_{k+l}(b)_{i+j}(b_1)_{k+l}(d)_i(d_1)_k(g-d)_j(g_1-d_1)_l(f-c)_i(f_1-c_1)_k}{i!j!k!l!(e)_{i+j+k+l}(f)_i(f_1)_k(g)_{i+j}(g_1)_{k+l}}x^{i+j}y^{k+l}$$

$$F_3\left[a+i+j,a_1+k+l,b+i+j,b_1+k+l;e+i+j+k+l;x,y\right], \tag{3.16}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{ll} a,b,c:d;d_1; & \\ & x,y \\ e:f,g;\ f_1,g_1; & \end{array}\right]$$

$$=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a)_{i+j}(b)_{i+j}(d)_i(d_1)_j(e-c)_{i+j}}{i!j!(e)_{i+j}(f)_i(f_1)_j(g)_j(g_1)_j}x^iy^j$$

$$F^{2:1;1}_{0:2;2}\left[\begin{array}{ll} a+i+j,b+i+j:d+i;d_1+j; & \\ & x,y \\ -:f+i,g+i; \qquad f_1+j,g_1+j; & \end{array}\right], \tag{3.17}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{ll} a,b,c:d;d_1; & \\ & x,y \\ e:f,g;\ f_1,g_1; & \end{array}\right]$$

$$= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(a)_{i+j}(b)_{i+j}(c)_{i+j}(g-d)_i(g_1-d_1)_i}{i!j!(e)_{i+j}(f)_i(f_1)_j(g)_j(g_1)_j} x^i y^j$$

$$F^{3:0;0}_{1:1;1}\left[\begin{array}{l} a+i+j, b+i+j, c+i+j : -; -; \\ \qquad\qquad\qquad\qquad\qquad\qquad x, y \\ e+i+j : f+i; \quad f_1+j; \end{array}\right], \tag{3.18}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{l} a, b, c : d; d_1; \\ \qquad\qquad\qquad x, y \\ e : f, g; \ f_1, g_1; \end{array}\right]$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i (a)_i (b)_i (c)_i (g-d)_i}{i!(e)_i (f)_i (g)_i} x^i$$

$$F^{3:0;1}_{1:1;2}\left[\begin{array}{l} a+i, b+i, c+i : -; d_1; \\ \qquad\qquad\qquad\qquad\qquad x, y \\ e+i : f+i; \quad f_1, g_1; \end{array}\right], \tag{3.19}$$

$$F^{3:1;1}_{1:2;2}\left[\begin{array}{l} a, b, c : d; d_1; \\ \qquad\qquad\qquad x, y \\ e : f, g; \ f_1, g_1; \end{array}\right]$$

$$= \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+j+k+l}(a)_{i+j+k+l}(b)_{i+j+k+l}(e-c)_{i+j}(g-d)_k(g_1-d_1)_l}{i!j!k!l!(e)_{i+j}(g)_k(g_1)_l} x^{i+j} y^{k+l}$$

$$F_4\left[a+i+j+k+l, b+i+j+k+l; f+i+k, f_1+j+l; x, y\right], \tag{3.20}$$

Remark. By assigning suitable special values to the coefficients in (3.3) to (3.20), we can establish a number of decomposition formulas as special cases involving hypergeometric functions of one and two variables. For exampe , if in (3.4) and (3.5), we let $a = e$, we get the following two interesting decomposition formulas:

$${}_3F_2\left[b, c, d; f, g; x\right] \times {}_3F_2\left[b_1, c_1, d_1; f_1, g_1; x\right]$$

$$\sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(b)_i(b_1)_j(c)_i(c_1)_j(g-d)_i(g_1-d_1)_j}{i!j!(f)_i(f_1)_j(g)_i(g_1)_j} x^i y^j$$

$${}_2F_1\left[b+i, c+i; f+i; x\right] \times {}_2F_1\left[b_1+j, c_1+j; f_1+j; x\right], \tag{3.21}$$

$${}_3F_2\left[b, c, d; f, g; x\right] \times {}_3F_2\left[b_1, c_1, d_1; f_1, g_1; x\right]$$

$$= (1-y)^{-b_1} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}(b)_i(b_1)_{j+k}(c)_i(d_1)_j(g-d)_i(f_1-c_1)_j(g_1-d_1)_k}{i!J!k!(f)_i(f_1)_j(g)_i(g_1)_j} x^i \left(\frac{y}{1-y}\right)^{j+k}$$

$${}_2F_1\left[b+i, c+i; f+i; x\right], \tag{3.22}$$

respectively. Also, by assigning suitable special values to the coefficients in (3.3) to (3.20), we can derive decomposition formulas for Appell functions of two variables. The details involved in these derivations are fairly straightforward and are being left as an exercise for the interested reader. Finally, let us stress that the schema suggested in Sections 2 and 3 can be applied to find symbolic operational images and decomposition formulas for other hypergeometric functions of double series related to Kampé de Fériets series of double hypergeometric series $F^{p:q;k}_{l:m;n}$. In a forthcoming papers we will consider the problems of establishing symbolic operational images and decomposition formulas for other generalized hypergeometric functions of double series by following the technique discussed in this paper.

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Received: November 24, 2016.

Accepted: January 2, 2017.