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ON THE INJECTIVE DOMINATION OF GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph. A subset D of V is called injective dominating set (Inj-dominating set) if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $|\Gamma(u, v)| \geq 1$, where $|\Gamma(u, v)|$ is the number of common neighbors between the vertices u and v. The minimum cardinality of such dominating set denoted by $\gamma_{in}(G)$ and is called injective domination number (Inj-domination number) of G. In this paper, we introduce the injective domination of a graph G and analogous to that, we define the injective independence number (Inj-independence number) $\beta_{in}(G)$ and injective domatic number (Inj-domatic number) $d_{in}(G)$. Bounds and some interesting results are established.

1 Introduction

By a graph we mean a finite, undirected with no loops and multiple edges. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$, $N[v]$ denote the open and closed neighborhood of a vertex v, respectively. The distance between two vertices u and v in G is the number of edges in a shortest path connecting them, this is also known as the geodesic distance. The eccentricity of a vertex v is the greatest geodesic distance between v and any other vertex and denoted by $e(v)$.
A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. We denote to the smallest integer greater than or equal to x by $\lceil x \rceil$ and the greatest integer less than or equal to x by $\lfloor x \rfloor$. A strongly regular graph with parameters (n, k, λ, μ) is a graph with n vertices such that the number of common neighbors of two vertices u and v is k, λ or μ according to whether u and v are equal, adjacent or non-adjacent, respectively. When $\lambda = 0$ the strongly regular graph is called strongly regular graph with no triangles (SRNT graph). A strongly regular graph G is called primitive if G and $\overline{G}$ are connected.

For terminology and notations not specifically defined here we refer the reader to [5]. For more details about domination number and neighborhood number and their related parameters, we refer to [3], [4].

The common neighborhood domination in graph has introduced in [2]. A subset D of V is called common neighborhood dominating set (CN-dominating set) if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\Gamma(u, v)| \geq 1$, where $|\Gamma(u, v)|$ is the number of common neighborhood between the vertices u and v. The minimum cardinality of such dominating set denoted by $\gamma_{cn}(G)$ and is called common neighborhood domination number (CN-domination number) of G. The common neighborhood (CN-neighborhood) of a vertex $u \in V(G)$ denoted by $N_{cn}(u)$ is defined as $N_{cn}(u) = \{v \in V(G) : uv \in E(G)$ and $|\Gamma(u, v)| \geq 1\}$.

The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with the vertex set $v_1, v_2, \ldots, v_p$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G [1].

In this paper, we introduce the concept of injective domination in graphs. In ordinary domination between vertices is enough for a vertex to dominate another in practice. If the persons have common friend then it may result in friendship. Human beings have a tendency to move with others when they have common friends.

2 Injective Dominating Sets

In defence and domination problem in some situations there should not be direct contact between two individuals but can be linked by a third person this motivated us to introduced the concept of injective domination.

Definition 2.1 ([1]). Let G be simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. For $i \neq j$, the common neighborhood of the vertices v_i and v_j, denoted by $\Gamma(v_i, v_j)$, is the set of vertices, different from v_i and v_j , which are adjacent to both v_i and v_j.

Definition 2.2. Let $G = (V, E)$ be a graph. A subset D of V is called injective dominating set (Inj-dominating set) if for every vertex $v \in V$ either $v \in D$ or there exists a vertex $u \in D$ such that $|\Gamma(u, v)| \geq 1$. The minimum cardinality of Inj-dominating set of G denoted by $\gamma_{in}(G)$ and called injective domination number (Inj-domination number) of G.

For example consider a graph G in Figure 1. Then $\{2, 7\}$ is a minimum dominating set, $\{2, 3, 4, 5, 6\}$ is a minimum CN-dominating set and $\{1\}$ is a minimum injective dominating set of G. Thus $\gamma(G) = 2$, $\gamma_{cn}(G) = 5$ and $\gamma_{in}(G) = 1$.

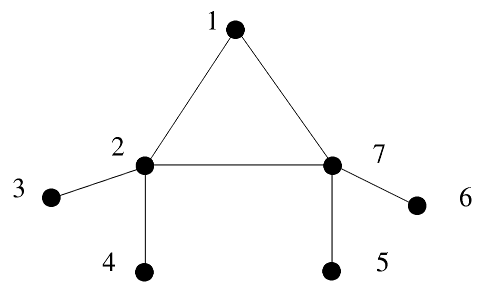

Figure 1. Graph with $\gamma_{in}(G) = 1$

Obviously, for any graph G, the vertex set $V(G)$ is an Inj-dominating set, that means any graph G has an Inj-dominating set and hence Inj-domination number. An injective dominating set D is said to be a minimal Inj-dominating set if no proper subset of D is an Inj-dominating set. Clearly each minimum Inj-dominating set is minimal but the converse is not true in general, for example let G be a graph as in Figure 1. Then $\{2, 3\}$ is a minimal Inj-dominating set but not minimum Inj-dominating set.

Let $u \in V$. The Inj-neighborhood of u denoted by $N_{in}(u)$ is defined as $N_{in}(u) = \{v \in V(G) : |\Gamma(u, v)| \geq 1\}$. The cardinality of $N_{in}(u)$ is called the injective degree of the vertex u and denoted by $deg_{in}(u)$ in G, and $N_{in}[u] = N_{in}(u) \cup \{u\}$. The maximum and minimum injective degree of a vertex in G are denoted respectively by $\Delta_{in}(G)$ and $\delta_{in}(G)$. That is $\Delta_{in}(G) = \max_{u \in V} |N_{in}(u)|$, $\delta_{in}(G) = \min_{u \in V} |N_{in}(u)|$. The injective complement of G denoted by $\overline{G}^{inj}$ is the graph with same vertex set $V(G)$ and any two vertices u and v in $\overline{G}^{inj}$ are adjacent if and only if they are not Inj-adjacent in G. A subset S of V is called an injective independent set (Inj-independent set), if for every $u \in S, v \notin N_{in}(u)$ for all $v \in S - \{u\}$. An injective independent set S is called maximal if any vertex set properly containing S is not Inj-independent set, the maximum cardinality of Inj-independent set is denoted by β_{in}, and the lower Inj-independence number i_{in} is the minimum cardinality of the Inj-maximal independent set. As usual P_p, C_p, K_p and W_p are the p-vertex path, cycle, complete, and wheel graph respectively, $K_{r,m}$ is the complete bipartite graph on $r + m$ vertices and S_p is the star with p vertices.

Proposition 2.3. *Let $G = (V, E)$ be a graph and $u \in V$ be such that $|\Gamma(u, v)| = 0$ for all $v \in V(G)$. Then u is in every injective dominating set, such vertices are called injective isolated vertices.*

Proposition 2.4. *Let $G = (V, E)$ be strongly regular graph with parameters (n, k, λ, μ). Then $\gamma_{in}(G) = 1$ or 2.*

Proposition 2.5. *For any graph G, $\gamma_{in}(G) \leq \gamma_{cn}(G)$*

Proof. From the definition of the CN-dominating set of a graph G. For any graph G any CN-dominating set D is also Inj-dominating set. Hence $\gamma_{in}(G) \leq \gamma_{cn}(G)$. □

We note that the Inj-domination number of a graph G may be greater than, smaller than or equal to the domination number of G.

Example 2.6.

(i) $\gamma_{in}(P_2) = 2; \gamma(P_2) = 1$.

(ii) $\gamma_{in}(C_5) = 2; \gamma(C_5) = 2$.

(iii) If G is the Petersen graph, then $\gamma_{in}(G) = 2; \gamma(G) = 3$.

Proposition 2.7.

(i) For any complete graph K_p, where $p \neq 2$, $\gamma_{in}(K_p) = 1$.

(ii) For any wheel graph $G \cong W_p$, $\gamma_{in}(G) = 1$.

(iii) For any complete bipartite graph $K_{r,m}$, $\gamma_{in}(K_{r,m}) = 2$.

(iv) For any graph G, $\gamma_{in}(K_p + G) = 1$, where $p \geq 2$.

Proposition 2.8. *For any graph G with p vertices, $1 \leq \gamma_{in}(G) \leq p$.*

Proposition 2.9. *Let G be graph with p vertices. Then $\gamma_{in}(G) = p$ if and only if G is a forest with $\Delta(G) \leq 1$.*

Proof. Let G be a forest with $\Delta(G) \leq 1$. Then we have two cases.
Case 1. If G is connected, then either $G \cong K_2$ or $G \cong K_1$. Hence, $\gamma_{in}(G) = p$.
Case 2. If G is disconnected, then $G \cong n_1K_2 \cup n_2K_1$, thus $\gamma_{in}(G) = p$.
Conversely, if $\gamma_{in}(G) = p$, then all the vertices of G are Inj-isolated, that means G is isomorphic to K_1 or K_2 or to the disjoint union of K_1 and K_2, that is $G \cong n_1K_2 \cup n_2K_1$ for some n_1,$n_2 \in \{0, 1, 2, ...\}$. Hence, G is a forest with $\Delta(G) \leq 1$. □

Proposition 2.10. *Let G be a nontrivial connected graph. Then $\gamma_{in}(G) = 1$ if and only if there exists a vertex $v \in V(G)$ such that $N(v) = N_{cn}(v)$ and $e(v) \leq 2$.*

Proof. Let $v \in V(G)$ be any vertex in G such that $N(v) = N_{cn}(v)$ and $e(v) \leq 2$. Then for any vertex $u \in V(G) - \{v\}$ if u is adjacent to v, since $N(v) = N_{cn}(v)$, then obvious $u \in N_{in}(v)$. If u is not adjacent to v, then $|\Gamma(u, v)| \geq 1$. Thus for any vertex $u \in V(G) - \{v\}$, $|\Gamma(u, v)| \geq 1$. Hence, $\gamma_{in}(G) = 1$.
Conversely, if G is a graph with p vertices and $\gamma_{in}(G) = 1$, then there exist at least one vertex $v \in V(G)$ such that $deg_{in}(v) = p - 1$, then any vertex $u \in V(G) - \{v\}$ either contained in a triangle with v or has distance two from v. Hence, $N(v) = N_{cn}(v)$ and $e(v) \leq 2$. □

Theorem 2.11 ([6]). *For any path P_p and any cycle C_p, where $p \geq 3$, we have,*

$$\gamma(P_p) = \gamma(C_p) = \left\lceil \frac{p}{3} \right\rceil.$$

Proposition 2.12 ([2]). *For any path P_p and any cycle C_p*

(i) $con(P_p) \cong P_{\lceil \frac{p}{2} \rceil} \cup P_{\lfloor \frac{p}{2} \rfloor}$.

(ii) $con(C_p) \cong \begin{cases} C_p, & \text{if } p \text{ is odd and } p \geq 3; \\ P_2 \cup P_2, & \text{if } p = 4; \\ C_{\frac{p}{2}} \cup C_{\frac{p}{2}}, & \text{if } p \text{ is even.} \end{cases}$

From the definition of the common neighborhood graph and the Inj-domination in a graph the following proposition can easily verified.

Proposition 2.13. *For any graph G, $\gamma_{in}(G) = \gamma(con(G))$.*

The proof of the following proposition is straightforward from Theorem 2.11 and Proposition 2.12.

Proposition 2.14. *For any cycle C_p with odd number of vertices $p \geq 3$,*

$$\gamma_{in}(C_p) = \gamma(C_p) = \left\lceil \frac{p}{3} \right\rceil$$

Theorem 2.15. *For any cycle C_p with even number of vertices $p \geq 3$,*

$$\gamma_{in}(C_p) = 2\left\lceil \frac{p}{6} \right\rceil.$$

Proof. From Proposition 2.13, Theorem 2.11 and Proposition 2.12, if p is even, then $\gamma_{in}(C_p) = \gamma((C_{\frac{p}{2}}) \cup (C_{\frac{p}{2}})) = 2\gamma(C_{\frac{p}{2}}) = 2\lceil \frac{p}{6} \rceil$. □

Proposition 2.16. *For any odd number $p > 3$,*

$$\gamma_{in}(P_p) = \left\lceil \frac{p+1}{6} \right\rceil + \left\lceil \frac{p-1}{6} \right\rceil.$$

Proof. From Proposition 2.13, Theorem 2.11 and Proposition 2.12, if p is odd then,

$$\gamma_{in}(P_p) = \gamma(P_{\lceil \frac{p}{2} \rceil} \cup P_{\lfloor \frac{p}{2} \rfloor}) = \gamma(P_{\frac{p+1}{2}} \cup P_{\frac{p-1}{2}}) = \left\lceil \frac{p+1}{6} \right\rceil + \left\lceil \frac{p-1}{6} \right\rceil.$$

□

Proposition 2.17. *For any even number $p \geq 4$,*

$$\gamma_{in}(P_p) = 2\left\lceil \frac{p}{6} \right\rceil.$$

Proof. From Proposition 2.13, Theorem 2.11 and Proposition 2.12, if p is even then, $\lceil \frac{p}{2} \rceil = \lfloor \frac{p}{2} \rfloor = \frac{p}{2}$. Hence $\gamma_{in}(P_p) = 2\lceil \frac{p}{6} \rceil$. □

Theorem 2.18. *Let $G = (V, E)$ be a graph without Inj-isolated vertices. If D is a minimal Inj-dominating set, then $V - D$ is an Inj-dominating set.*

Proof. Let D be a minimal Inj-dominating set of G. Suppose $V - D$ is not Inj-dominating set. Then there exists a vertex u in D such that u is not Inj-dominated by any vertex in $V - D$, that is $|\Gamma(u, v)| = 0$ for any vertex v in $V - D$. Since G has no Inj-isolated vertices, then there is at least one vertex in $D - \{u\}$ has common neighborhood with u. Thus $D - \{u\}$ is Inj-dominating set of G, which contradicts the minimality of the Inj–dominating set D. Thus every vertex in D has common neighborhood with at least one vertex in $V - D$. Hence $V - D$ is an Inj-dominating set. □

Theorem 2.19. *Let G be a graph. Then the injective dominating set D is minimal if and only if for every vertex $v \in D$, one of the following conditions holds*

(i) *v is Inj-isolated vertex.*

(ii) *There exists a vertex u in $V - D$ such that $N_{in}(u) \cap D = \{v\}$.*

Proof. Suppose D is a minimal Inj-dominating set of G. Then $D - \{v\}$ is not Inj-dominating set, then there exists at least one vertex $u \in (V - D) \cup \{v\}$ is not Inj-dominated by any vertex in D, so we have two cases.
Case 1. If $u \in D$, then u is Inj-isolated vertex.
Case 2. If $u \in V - D$, then u has common neighborhood with only one vertex v in D, that means $N_{in}(u) \cap D = \{v\}$.
Conversely, suppose D is an Inj-dominating set of G and for each vertex $v \in D$ one of the two conditions holds, we want to prove that D is a minimal Inj-dominating set. Suppose that D is not minimal. Then there is at least one vertex $v \in D$ such that $D - \{v\}$ is an Inj-dominating set. Thus v has common neighborhood with at least one vertex in $D - \{v\}$. Hence, condition (i) is not hold.
Also, $V - D$ is an Inj-dominating set, then every vertex in $V - D$ has common neighborhood with at least one vertex in $D - \{v\}$. Therefore condition (ii) is not hold. Hence, neither condition (i) nor condition (ii) holds, which is a contradiction. □

Theorem 2.20. *A graph G has a unique minimal Inj-dominating set if and only if the set of all Inj-isolated vertices forms an Inj-dominating set.*

Proof. Let G has a unique minimal Inj-dominating set D, and suppose $S = \{u \in V : u$ is Inj-isolated vertex$\}$. Then $S \subseteq D$. Now suppose $D \neq S$, let $v \in D - S$, since v is not Inj-isolated vertex, then $V - \{v\}$ is an Inj-dominating set. Hence there is a minimal Inj-dominating set $D_1 \subseteq V - \{v\}$ and $D_1 \neq D$ a contradiction to the fact that G has a unique minimal Inj-dominating set. Therefore $S = D$.
Conversely, if the set of all Inj-isolated vertices in G forms an Inj-dominating set, then it is clear that G has a unique minimal Inj-dominating set. □

Theorem 2.21. *For any (p, q)-graph G, $\gamma_{in}(G) \geq p - q$.*

Proof. Let D be a minimum Inj-dominating set of G. Since every vertex in $V - D$ has common neighborhood with at least one vertex of D, then $q \geq |V - D|$. Hence, $\gamma_{in}(G) \geq p - q$. □

Theorem 2.22. *For any graph G with p vertices, $\lceil \frac{p}{1+\Delta_{in}(G)} \rceil \leq \gamma_{in}(G)$. Further, the equality holds if and only if for every minimum Inj-dominating set D in G the following conditions are satisfied:*

(i) for any vertex v in D, $deg_{in}(v) = \Delta_{in}(G)$;

(ii) D is Inj-independent set in G;

(iii) every vertex in $V - D$ has common neighborhood with exactly one vertex in D.

Proof. Let S be any minimum Inj-dominating set in G. Clearly each vertex in G will Inj-dominate at most $(\Delta_{in}(G) + 1)$ vertices, so $p = |N_{in}[S]| \leq \gamma_{in}(G)(\Delta_{in}(G) + 1)$, hence $\frac{p}{1+\Delta_{in}(G)} \leq \gamma_{in}(G)$. Therefore $\lceil \frac{p}{1+\Delta_{in}(G)} \rceil \leq \gamma_{in}(G)$.
Suppose the given conditions are hold for any minimum Inj-dominating set D in G. Then obviously $\gamma_{in}(G)\Delta_{in}(G) + \gamma_{in}(G) = p$. Hence, $\lceil \frac{p}{1+\Delta_{in}(G)} \rceil = \gamma_{in}(G)$.
Conversely, suppose the equality holds, and suppose that one from the conditions is not satisfied. Then $p < \gamma_{in}(G)\Delta_{in}(G) + \gamma_{in}(G)$, a contradiction. □

Example 2.23. Let $G = C_p$, where $p \geq 3$ and p is odd number. Then the equality in Theorem 2.22 is hold. Since $\lceil \frac{p}{1+\Delta_{in}(G)} \rceil = \lceil \frac{p}{3} \rceil$ and by Proposition 2.14 we have $\gamma_{in}(G) = \lceil \frac{p}{3} \rceil$.

Theorem 2.24. *Let G be a graph on p vertices and $\delta_{in}(G) \geq 1$. Then $\gamma_{in}(G) \leq \frac{p}{2}$.*

Proof. Let D be any minimal Inj-dominating set in G. Then by Theorem 2.18, $V - D$ is also an Inj-dominating set in G. Hence, $\gamma_{in}(G) \leq min\{|D|, |V - D|\} \leq p/2$. □

Theorem 2.25. *For any graph G on p vertices, $\gamma_{in}(G) \leq p - \Delta_{in}(G)$*

Proof. Let v be a vertex in G such that $deg_{in}(v) = \Delta_{in}(G)$. Then v has common neighborhood with $|N_{in}(v)| = \Delta_{in}(G)$ vertices. Thus, $V - N_{in}(v)$ is an Inj-dominating set. Therefore $\gamma_{in}(G) \leq |V - N_{in}(v)|$. Hence, $\gamma_{in}(G) \leq p - \Delta_{in}(G)$. □

Proposition 2.26. *For any graph G with diameter less than or equal three and maximum degree $\Delta(G)$, $\gamma_{in}(G) \leq \Delta(G) + 1$.*

Proof. Let $diam(G) \leq 3$ and $v \in V(G)$ such that $deg(v) = \Delta(G)$. Clearly that, if $diam(G) = 1$, then G is a complete graph and the result holds. Suppose $diam(G) = 2$ or 3. Let $V_i(G) \subseteq V(G)$ be the sets of vertices of G which have distance i from v, where $i = 1, 2, 3$. Obviously, the set $S = V_1(G) \cup \{v\}$ is an Inj-dominating set of G of order $\Delta(G)+1$. Hence, $\gamma_{in}(G) \leq \Delta(G)+1$. □

The Cartesian product $G \square H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ and edge set $E(G \square H) = \{((u, u'), (v, v')) : u = v$ and $(u', v') \in E(H)$, or $u' = v'$ and $(u, v) \in E(G)\}$.

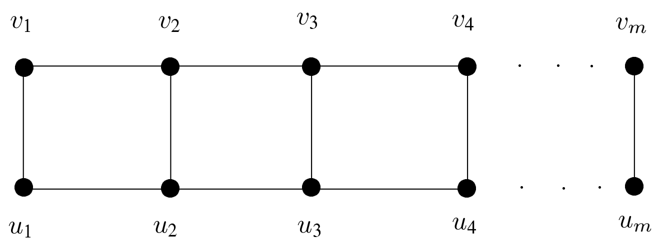

Figure 2. $P_m \square P_2$

Proposition 2.27. *For any graph $G \cong P_m \square P_2$, $\gamma_{in}(G) = 2\lceil \frac{m}{5} \rceil$.*

Proof. Let $G \cong P_m \square P_2$. From Figure 2, it is easy to see that any two adjacent vertices v_i, u_i can Inj-dominate all the vertices of distance less than or equal two from v_i or u_i, then $\gamma_{in}(G) \leq 2\lceil \frac{m}{5} \rceil$. obviously from Figure 2, $\Delta_{in}(G) = 4$, then by Theorem 2.22, $\gamma_{in}(G) \geq \lceil \frac{2m}{5} \rceil$. Now, if $m < 5$ or $m \equiv 0$ $(mod\ 5)$, then $2\lceil \frac{m}{5} \rceil = \lceil \frac{2m}{5} \rceil$ and hence $\gamma_{in}(G) = 2\lceil \frac{m}{5} \rceil$. Otherwise, $2\lceil \frac{m}{5} \rceil = \lceil \frac{2m}{5} \rceil + 1$, but in this case the equality of Theorem 2.22 does not hold because the third condition is not satisfied. Hence, $\gamma_{in}(G) = 2\lceil \frac{m}{5} \rceil$. □

Proposition 2.28. *For any graph $G \cong C_m \square P_2$, $\gamma_{in}(G) = 2\lceil \frac{m}{5} \rceil$.*

Proof. The proof is same as in Proposition 2.27. □

The Composition $G \cdot H$ or $G[H]$ has its vertex set $V(G) \times V(H)$, with (u, u') is adjacent to (v, v') if either u is adjacent to v in G or $u = v$ and u' is adjacent to v' in H.

Proposition 2.29. *For any graph G isomorphic to $P_m \cdot P_n$ or $P_m \cdot C_n$ or $C_m \cdot P_n$ or $C_m \cdot C_n$, $\gamma_{in}(G) = \lceil \frac{m}{5} \rceil$.*

Proof. Let G be a graph isomorphic to $P_m \cdot P_n$ or $P_m \cdot C_n$ or $C_m \cdot P_n$ or $C_m \cdot C_n$. Then from the definition of the Composition product, $N(w) = N_{cn}(w)$, $\forall w \in V(G)$, then each vertex $w = (u, v)$ in G Inj-dominates its neighbors and all the vertices of distance two of it, then $\gamma_{in}(G) \leq \lceil \frac{m}{5} \rceil$, but in this graph $\Delta_{in}(G) = 5n - 1$, so by Theorem 2.22, $\gamma_{in}(G) \geq \lceil \frac{mn}{5n} \rceil = \lceil \frac{m}{5} \rceil$. Hence, $\gamma_{in}(G) = \lceil \frac{m}{5} \rceil$. □

Definition 2.30. Let $G = (V, E)$ be a graph. $S \subseteq V(G)$ is called Inj-independent set if no two vertices in S have common neighbor. An Inj-independent set S is called maximal Inj-independent set if no superset of S is Inj-independent set. The Inj-independent set with maximum size called the maximum Inj-independent set in G and its size called the Inj-independence number of G and denoted by $\beta_{in}(G)$.

Theorem 2.31. *Let S be a maximal Inj-independent set. Then S is a minimal Inj-dominating set.*

Proof. Let S be a maximal Inj-independent set and $u \in V - S$. If $u \notin N_{in}(v)$ for every $v \in S$, then $S \cup \{u\}$ is an Inj-independent set, a contradiction to the maximality of S. Therefore $u \in N_{in}(v)$ for some $v \in S$. Hence, S is an Inj-dominating set. To prove that S is a minimal Inj-dominating set, suppose S is not minimal. Then for some $u \in S$ the set $S - \{u\}$ is an Inj-dominating set. Then there exist some vertex in S has a common neighborhood with u, a contradiction because S is an Inj-independent set. Therefore S is a minimal Inj-dominating set. □

Corollary 2.32. *For any graph G, $\gamma_{in}(G) \leq \beta_{in}(G)$.*

3 Injective domatic number in a graph

Let $G = (V, E)$ be a graph. A partition Δ of its vertex set $V(G)$ is called a domatic partition of G if each class of Δ is a dominating set in G. The maximum order of a partition of $V(G)$ into dominating sets is called the domatic number of G and is denoted by $d(G)$.

Analogously as to $\gamma(G)$ the domatic number $d(G)$ was introduced, we introduce the injective domatic number $d_{in}(G)$, and we obtain some bounds and establish some properties of the injective domatic number of a graph G.

Definition 3.1. Let $G = (V, E)$ be a graph. A partition Δ of its vertex set $V(G)$ is called an injective domatic (in short Inj-domatic) partition of G if each class of Δ is an Inj-dominating set in G. The maximum order of a partition of $V(G)$ into Inj-dominating sets is called the Inj-domatic number of G and is denoted by $d_{in}(G)$.

For every graph G there exists at least one Inj-domatic partition of $V(G)$, namely $\{V(G)\}$. Therefore $d_{in}(G)$ is well-defined for any graph G.

Theorem 3.2.

(i) For any complete graph K_p, $d_{in}(K_p) = d_{cn}(K_p) = d(K_p) = p$.

(ii) $d_{in}(G) = 1$ if and only if G has at least one Inj-isolated vertex.

(iii) For any wheel graph of p vertices, $d_{in}(W_p) = p$.

(iv) For any complete bipartite graph $K_{r,m}$,

$$d_{in}(K_{r,m}) = \begin{cases} min\{r, m\}, & \text{if } r, m \geq 2; \\ 1, & \text{otherwise.} \end{cases}$$

(v) For any graph G, if $N_{in}(v) = N(v)$ for any vertex v in $V(G)$, then

$$d_{in}(G) = d(G).$$

Proof.

(i) If $G = (V, E)$ is the complete graph K_p, then for any vertex v the set $\{v\}$ is a minimum CN-dominating set and also a minimum Inj-dominating set. Then the maximum order of a partition of $V(G)$ into Inj-dominating or CN-dominating sets is p. Hence, $d_{in}(K_p) = d_{cn}(K_p) = p$.

(ii) Let G be a graph which has an Inj-isolated vertex say v, then every Inj-dominating set of G must contain the vertex v. Then $d_{in}(G) = 1$.
Conversely, if $d_{in}(G) = 1$ and suppose G has no Inj-isolated vertex, then by Theorem 2.24, $\gamma_{in}(G) \leq \frac{p}{2}$, so if we suppose D is a minimal Inj-dominating set in G, then $V - D$ is also a minimal Inj-dominating set. Thus $d_{in}(G) \geq 2$, a contradiction. Therefore G has at least one Inj-isolated vertex.

(iii) Since for every vertex v of the wheel graph the $deg_{in}(v) = p - 1$. Hence, $d_{in}(W_p) = p$.

(iv) and (v) the proof is obvious. □

Evidently each CN-dominating set in G is an Inj-dominating set in G, and any CN-domatic partition is an Inj-domatic partition. We have the following proposition.

Proposition 3.3. *For any graph G, $d_{in}(G) \geq d_{cn}(G)$.*

Theorem 3.4. *For any graph G with p vertices, $d_{in}(G) \leq \frac{p}{\gamma_{in}(G)}$.*

Proof. Assume that $d_{in}(G) = d$ and $\{D_1, D_2, ..., D_d\}$ is a partition of $V(G)$ into d numbers of Inj-dominating sets, clearly $|D_i| \geq \gamma_{in}(G)$ for $i = 1, 2, ..., d$ and we have $p = \sum_{i=1}^{d} |D_i| \geq d\gamma_{in}(G)$. Hence, $d_{in}(G) \leq \frac{p}{\gamma_{in}(G)}$. □

Theorem 3.5. *For any graph G with p vertices, $d_{in}(G) \geq \lfloor \frac{p}{p-\delta_{in}(G)} \rfloor$.*

Proof. Let D be any subset of $V(G)$ such that $|D| \geq p - \delta_{in}(G)$. For any vertex $v \in V - D$, we have $|N_{in}[v]| \geq 1 + \delta_{in}(G)$. Therefore $N_{in}(v) \cap D \neq \phi$. Thus D is an Inj-dominating set of G. So, we can take any $\lfloor \frac{p}{p-\delta_{in}(G)} \rfloor$ disjoint subsets each of cardinality $p - \delta_{in}(G)$. Hence,

$$d_{in}(G) \geq \left\lfloor \frac{p}{p - \delta_{in}(G)} \right\rfloor.$$

□

Theorem 3.6. *For any graph G with p vertices $d_{in}(G) \leq \delta_{in}(G) + 1$. Further the equality holds if G is complete graph K_p.*

Proof. Let G be a graph such that $d_{in}(G) > \delta_{in}(G) + 1$. Then there exists at least $\delta_{in}(G) + 2$ Inj-dominating sets which they are mutually disjoint. Let v be any vertex in $V(G)$ such that $deg_{in}(v) = \delta_{in}(G)$. Then there is at least one of the Inj-dominating sets which has no intersection with $N_{in}[v]$. Hence, that Inj-dominating set can not dominate v, a contradiction. Therefore $d_{in}(G) \leq \delta_{in}(G) + 1$. It is obvious if G is complete, then $d_{in}(G) = \delta_{in}(G) + 1$. □

Theorem 3.7. *For any graph G with p vertices, $d_{in}(G) + d_{in}(\overline{G}^{inj}) \leq p + 1$*

Proof. From Theorem 3.6, we have $d_{in}(G) \leq \delta_{in}(G) + 1$ and $d_{in}(\overline{G}^{inj}) \leq \delta_{in}(\overline{G}^{inj}) + 1$, and clearly $\delta_{in}(\overline{G}^{inj}) = p - 1 - \Delta_{in}(G)$. Hence,

$$d_{in}(G) + d_{in}(\overline{G}^{inj}) \leq \delta_{in}(G) + p - \Delta_{in}(G) + 1 \leq p + 1.$$

□

Theorem 3.8. *For any graph G with p vertices and without Inj-isolated vertices, $d_{in}(G) + \gamma_{in}(G) \leq p + 1$.*

Proof. Let G be a graph with p vertices. Then by Theorem 2.25, we have

$$\gamma_{in}(G) \leq p - \Delta_{in}(G) \leq p - \delta_{in}(G),$$

and also from Theorem 3.6, $d_{in}(G) \leq \delta_{in}(G) + 1$. Then

$$d_{in}(G) + \gamma_{in}(G) \leq \delta_{in}(G) + 1 + p - \delta_{in}(G).$$

Hence,

$$d_{in}(G) + \gamma_{in}(G) \leq p + 1.$$

□

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