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FUNCTIONS ON STATIONARY SETS

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Abstract A characterization of stationary sets is established using regressive functions, selection property, continuous functions on ordinals and real continuous functions.

1 Introduction

Whenever a number of pigeons, to be put into cages, is greater that the number of cages considered for this purpose; one cage, at least, has to contain more than one pigeon. This self-evident principle inspired many mathematicians over time and has been a basis for many generalizations indeed. The first extension of this principle to ordinals dates back to P. Alexandroff and P. Uryshon (1926), see [1] : A *regressive* function f on limit ordinals (i.e., $f(\alpha) \not\geq \alpha$) has a constant value on some uncountable set of ordinal numbers. Four years later, Ben Dushnik (1930), see [3], gave a more explicit generalization: Any regressive function on $\omega_{\nu+1}$ into itself is constant on some set of size $\aleph_{\nu+1}$. In 1950 P. Erdös extended Ben Dushnik result to any ω_ν of uncountable cofinality, see [4]. In the same year (1950) J. Novak, see [9], generalized Alexandroff-Uryshon's theorem to closed (under the order topology) and unbounded subsets of ω_1: Any regressive function on a *club*, closed and unbounded, C of ω_1 into itself is constant on some uncountable subset of C. H. Bachman, see [2], strengthened Novak's result by showing that, in fact, whenever a cardinal κ has no countable cofinal subset every regressive function on a club of κ into κ is constant on some subset of size κ. It is with W. Neumer (1951), see [8], that the terminology of "stationary" set came into use (a set $S \subseteq \kappa$ is *stationary* in κ whenever S meets all clubs in κ): If κ is an uncountable regular cardinal and f is a regressive function on a stationary set S in κ, then f is constant on some cofinal subset of S.

2 Stationary sets and regressive functions

Throughout this paper κ shall denote any regular uncountable cardinal, ordinals are assumed to be endowed with the order topology and by *club* we mean a set of κ that is closed under the order topology and unbounded in κ. Now we say that a set $S \subseteq \kappa$ is *stationary* in κ whenever it meets all clubs of κ. Otherwise we say that S is *non-stationary*. The ideal $\mathcal{NS}_\kappa$ of non-stationary sets of κ is indeed λ-complete for any $\lambda < \kappa$, see Corollary 2.2 below. Next, the assumption of studying stationary sets in regular cardinals is not restrictive. For let α be a limit ordinal and denote by cf(α) the least cardinal $\mu \geq \omega$ that μ is unbounded in α. Now, putting $\mu = \text{cf}(\alpha) > \omega$, then μ may be regarded as a club in α since one can choose $(x_\nu)_{\nu<\mu}$ a strictly increasing and continuous cofinal set in α. Now notice that $S \cap \mu = T \cap \mu$ modulo $\mathcal{NS}_\mu$ implies that there is a club $D \subseteq \mu$ so that $((S \cap \mu) \vartriangle (T \cap \mu)) \cap D = \emptyset$, where $\vartriangle$ is the symmetric difference. Thus $(S \vartriangle T) \cap (D \cap \mu) = \emptyset$, where $D \cap \mu$ is a club of α. Thus $S = T$ modulo $\mathcal{NS}_\alpha$. Now the mapping θ defined by $\theta(\overline{S}) = \overline{S \cap \mu}$ put $P(\alpha)/\mathcal{NS}_\alpha$ and $P(\mu)/\mathcal{NS}_\mu$ in one-to-one correspondence.

Proposition 2.1. *Any intersection of less than κ clubs of κ is a club of κ.*

Proof. Let λ less than κ, and let $\{C_\nu : \nu < \lambda\}$ a family of clubs in κ. We shall use induction on λ. For set $D_{\lambda_\alpha} := \cap\{C_\nu : \nu < \lambda_\alpha\}$, where $\lambda_\alpha < \lambda_{\alpha+1}$ for all $\alpha < \text{cf}(\lambda)$. By the induction hypothesis $\{D_{\lambda_\alpha} : \alpha < \mathrm{cf}(\lambda)\}$ is a decreasing sequence of clubs in κ. Next, pick any $\xi \in \kappa$ and construct an increasing sequence $\{x_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ so that $x_0 > \xi$ and $x_\alpha \in D_{\lambda_\alpha}$. Thus $\sup\{x_\alpha : \alpha < \mathrm{cf}(\lambda)\} \in D_{\lambda_\alpha}$ for each $\alpha < \mathrm{cf}(\lambda)$ and therefore $\sup\{x_\alpha : \alpha < \mathrm{cf}(\lambda)\} \in \cap\{D_{\lambda_\alpha} : \alpha < \mathrm{cf}(\lambda)\}(= \cap\{C_\nu : \nu < \lambda\})$; moreover $\xi < \sup\{x_\alpha : \alpha < \mathrm{cf}(\lambda)\}$. This shows that $\cap\{C_\nu : \nu < \lambda\}$ is a club in κ. □

Corollary 2.2. *The ideal $\mathcal{NS}_\kappa$ is λ-complete for any $\lambda < \kappa$.*

Next, recall that a function $f : \omega_1 \to \omega_1$ is *regressive* whenever $f(\alpha) < \alpha$ for every non-zero ordinal α in ω_1. To have an idea of what Fodor's theorem on a stationary set of κ looks like, take $f : \omega_1 \to \omega_1$ a regressive function and suppose for all ξ's in ω_1 $f^{-1}(\xi) = \{\nu : f(\nu) = \xi\}$ is a non-stationary set of ω_1. So there is for each ξ a club C_ξ so that $f^{-1}(\xi) \cap C_\xi = \emptyset$. Hence, $\omega_1 = \cup f^{-1}(\xi)$. So, is there $\alpha \in \omega_1$ so that $f(\alpha) \geq \alpha$? This may be possible if e.g., $f(\alpha) \neq \xi$, for all $\xi < \alpha$. Now, if $\alpha \in C_\xi$ for all $\xi < \alpha$ this shall make $f(\alpha) \neq \xi$ for all $\xi < \alpha$ whenever α is a limit ordinal. The question now is to find α so that $\alpha \in \cap\{C_\xi : \xi < \alpha\}$. This fixe-point like situation is actually the key in proving Fodor's theorem. To finish up the proof here construct a sequence $(\beta_n)_{n<\omega}$ so that $\beta_{n+1} \in \cap\{C_\nu : \nu < \beta_n\}$. Then $\sup\{\beta_n : n < \omega\} = \beta \in \cap\{C_\nu : \nu < \beta\}$, and $f(\beta) \geq \beta$; contradiction. Next, the set of α so that $\alpha \in \cap\{C_\nu : \nu < \alpha\}$ is actually a club as showed by the next proposition.

Proposition 2.3. *Let $\left(C_\alpha\right)_{\alpha<\kappa}$ be a family of clubs in κ. The diagonal intersection of $\left(C_\alpha\right)_{\alpha<\kappa}$ is a club of κ denoted by $\triangle\{C_\alpha : \alpha < \kappa\}$ and defined by:*

$$\alpha \in \triangle\{C_\nu : \nu < \kappa\} \;\leftrightarrow\; \alpha \in \cap\{C_\nu : \nu < \alpha\}$$

Proof. First, to see that $\triangle\{C_\alpha : \alpha < \kappa\}$ is closed let $\langle \xi_\nu : \nu < \lambda\rangle$ be a strictly increasing sequence in $\triangle\{C_\alpha : \alpha < \kappa\}$. Put $\xi = \sup\{\xi_\nu : \nu < \lambda\}$. Now for each $\alpha < \xi$, there is ν_0 so that $\alpha < \xi_\nu < \xi$ for $\nu \geq \nu_0$. Thus $\xi_\nu \in C_\alpha$ for $\nu \geq \nu_0$. Hence $\xi \in C_\alpha$ and therefore $\xi \in \cap\{C_\alpha : \alpha < \xi\}$ i.e., $\xi \in \triangle\{C_\alpha : \alpha < \kappa\}$.

Second, to show that $\triangle\{C_\alpha : \alpha < \kappa\}$ is cofinal in κ, let $\beta < \kappa$ and construct an increasing sequence $\langle \beta_n : n < \omega\rangle$ so that $\beta_0 > \beta$, for some $\beta_0 \in C_0$ and $\beta_{n+1} \in \cap\{C_\alpha : \alpha < \beta_n\}$, for each $n < \omega$. Thus $\delta = \sup\{\beta_n : n < \omega\}$ is in $\triangle\{C_\alpha : \alpha < \kappa\}$ and is bigger than β. □

Theorem 2.4 (G. Fodor). *Let S be stationary in κ. Then any regressive function on S is constant on some stationary subset S_0 of S.*

Proof. Towards a contradiction assume $f^{-1}(\alpha)$ is non-stationary for all α's. For each $\alpha < \kappa$, pick a club C_α of κ so that $C_\alpha \cap f^{-1}(\alpha) = \emptyset$. By Proposition 2.3 pick ξ limit in $S \cap \triangle\{C_\alpha : \alpha < \kappa\}$. So $f(\xi) \neq \alpha$ for all $\alpha < \xi$ i.e., $f(\xi) \geq \xi$ which is a contradiction. □

The following is a converse of Fodor's theorem.

Proposition 2.5. *For each non-stationary set in κ, there is a regressive function f so that $|f^{-1}(\xi)| < \kappa$ for all $\xi < \kappa$.*

Proof. Assume S is a non-stationary set in κ and pick C a club disjoint from S; then write $\kappa \setminus C = \cup\{(x_\alpha, x_{\alpha+1}) : \alpha < \kappa\}$ where $(x_\alpha, x_{\alpha+1})$ is a maximal open interval and $(x_\alpha)_\alpha$ is an increasing continuous enumeration of C. Now since $S \subseteq \cup\{(x_\alpha, x_{\alpha+1}) : \alpha < \kappa\}$, define $f : S \to \kappa$ by $f(\xi) = x_{\alpha(\xi)}$ with $\alpha(\xi)$ is the unique ν so that $\xi \in (x_\nu, x_{\nu+1})$. f is regressive since S and C are disjoint sets, and $|f^{-1}(\xi)| \leq |(x_{\alpha(\xi)}, x_{\alpha(\xi)+1})| < \kappa$. □

3 Stationary sets and selection

Definition 3.1. We say that $\Sigma(S)$ is a *selector* of S whenever $S = \cup\{S_\alpha : \alpha < \kappa\}$ implies $|\Sigma(S) \cap S_\alpha| = 1$ for any $\alpha < \kappa$.

Next, we give two consequences of Fodor's theorem.

Theorem 3.2 (Selection). *S is stationary in κ iff any partition of S in non-stationary sets of κ has a stationary selector.*

Proof. One direction is obvious. To see the other one, assume that $\Sigma(S) = \{\min(S_\alpha) : \alpha < \kappa\}$ is non-stationary, where $S = \cup\{S_\alpha : \alpha < \kappa\}$ such that $S_\alpha \in \mathcal{NS}_\kappa$ for each $\alpha < \kappa$. Pick then C a club of κ so that $C \cap \Sigma(S) = \emptyset$. Next, define $\psi : S \cap C \to \kappa$ by $\psi(x) = \min(S_{\alpha(x)})$ whenever $x \in S_{\alpha(x)}$. ψ is a regressive function on $S \cap C$; and thus $\psi \restriction S_0$ is constant, where S_0 is stationary included in $S \cap C$. Hence, $x \in S_0$ implies $\psi(x) = \min(S_{\alpha(0)})$. Thus $S_0 \subseteq S_{\alpha(0)}$ and this contradicts the fact that $S_{\alpha(0)}$ is non-stationary. □

Proposition 3.3. *Let S be a stationary in κ and f be a function from S into κ. Then there is a stationary set S_0 so that at least one of the following three statements holds:*

(a) $f \restriction S_0$ is a constant function;

(b) $f \restriction S_0$ is the identity;

(c) $f \restriction S_0$ is strictly increasing and thus injective.

Moreover $f(\alpha) \geq \alpha$ for all $\alpha \in S_0$ and $f''S_0$ is non stationary.

Proof. Spilt S into three sets S_1, S_2, S_3 defined as follows:

$$S_1 = \{\alpha \in S : f(\alpha) < \alpha\}, S_2 = \{\alpha \in S : f(\alpha) = \alpha\}, \text{and } S_3 = \{\alpha \in S : f(\alpha) > \alpha\}.$$

Now, if S_1 is stationary then by Fodor's theorem f is constant on some stationary set of S_1. Hence we may assume $S_1 = \emptyset$. If S_2 is stationary then $f \restriction S_2$ is the identity and thus $S_0 = S_2$. So, we may assume $S = S_3$. Now, put

$$\begin{aligned}
A(\alpha_0) &= \{\nu \in S : f(\nu) \leq f(\alpha_0)\},\ \alpha_0 = \min S;\\
A(\alpha_1) &= \{\nu \in S : f(\nu) \leq f(\alpha_1)\},\ \alpha_1 = \min(S \setminus A(\alpha_0));\\
&\vdots\\
A(\alpha_\xi) &= \{\nu \in S : f(\nu) \leq f(\alpha_\xi)\},\ \alpha_\xi = \min(S \setminus \cup\{A(\alpha_\nu) : \nu < \xi\}).
\end{aligned}$$

Notice that $\langle \alpha_\xi : \xi < \alpha \rangle$ is increasing, and $S = \cup\{A(\alpha_\xi) : \xi < \kappa\}$ and $A(\alpha_\xi)$ are non stationary sets. Hence by selection property (Theorem 3.2) pick $\Sigma(S)$ a selector of S. $\Sigma(S)$ is stationary, $\Sigma(S) = \{\min(A(\alpha_\xi)) : \xi < \kappa\} = \{\alpha_\xi : \xi < \kappa\}$ and $f \restriction \Sigma(S)$ is increasing. □

4 Stationary sets and continuous functions

We characterize stationary sets using continuous functions. This feature actually makes stationary sets a very important tool in set theory distinguishing between objects and therefore constructing, at will, incomparably many of them in many areas of mathematics see [6].

Theorem 4.1. *For any subset S of κ, the following statements are equivalent:*

(a) S is stationary in κ.

(b) Every continuous function f from S into κ is either constant on a final segment of S or $Fix(f) := \{\nu \in S : f(\nu) = \nu\}$ is cofinal in κ.

Proof. (a) implies (b). Suppose S stationary and let $f : S \to \kappa$ be a continuous function.

Case 1. $S_0 = \{\alpha \in S : f(\alpha) < \alpha\}$ is stationary in κ.
Find a stationary $\Sigma \subseteq S_0$, and $a \in \kappa$ so that $f''\Sigma = \{a\}$. Now since two clubs intersect in κ, it follows that for each $\nu < a$, $f^{-1}(\nu)$ is bounded in κ. Thus $\sup\{f^{-1}(\nu) : \nu < a\} = \delta < \kappa$. Hence for $t \in \Sigma \cap [\delta+1, \to)$, we have $f(t) \geq a$. Next, assume that $f^{-1}([a+1, \to))$ is cofinal in κ. It follows then that $f^{-1}([a+1, \to))$ is a club of S and thus there is $t \in \Sigma \cap f^{-1}([a+1, \to))$. Therefore $f(t) \geq a+1$: this contradicts $f''\Sigma = \{a\}$. So, there is $\delta_1 < \kappa$ so that $\sup(f^{-1}([a+1, \to))) = \delta_1$.

Hence, for $t \in S \cap [(\delta \vee \delta_1) + 1, \rightarrow)$, $f(t) = a$ i.e., $f \restriction S \cap [(\gamma, \rightarrow)$ is constant for some $\gamma < \kappa$.

Case 2. $S_1 = \{\alpha \in S : f(\alpha) \geq \alpha\}$ is stationary in κ.

Construct, by induction, a sequence of size κ, $\left(x_\xi\right)_{\xi<\kappa}$ so that : $x_\xi \leq f(x_\xi) \leq x_{\xi+1}$ for each $\xi < \kappa$. Now, denote by $\mathrm{cl}(X)$ and $\lim(X)$ respectively the closure and the set of limit points of X in the order topology on κ. Next, let $t \in S \cap \lim\left(\mathrm{cl}(S)\right) \cap \lim\left(\mathrm{cl}\{x_\xi : \xi < \kappa\}\right)$. So, pick $(x_{\xi(\eta)})_\eta$ so that : $x_{\xi(\eta)} \leq f(x_{\xi(\eta)}) \leq x_{\xi(\eta+1)}$ and $\sup_\eta x_{\xi(\eta)} = t$. By continuity of f we have $f(t) = t$, but this shows that $\mathrm{Fix}(f) \neq \emptyset$, and hence is cofinal in κ.

(b) implies (a). Assume that S is non stationary. So pick C a club set in κ disjoint from S and write $\kappa \setminus C = \cup\{(x_\alpha, x_{\alpha+1}) : \alpha < \kappa\}$, where $(x_\alpha, x_{\alpha+1})$ is a maximal open interval and $(x_\alpha)_\alpha$ is an increasing continuous enumeration of C. Now define $f : S \to \kappa$ by $f(\xi) = x_{\alpha(\xi)}$ with $\alpha(\xi)$ is the unique ν so that $\xi \in (x_\nu, x_{\nu+1})$.

Now, $\mathrm{Fix}(f) = \emptyset$ since $S \cap C = \emptyset$ and $|f^{-1}(\xi)| \leq |(x_{\alpha(\xi)}, x_{\alpha(\xi)+1})| < \kappa$ for all ξ's. □

5 Stationary sets and real continuous functions

The following theorem shows that modulo non-stationary sets constant functions are the only real continuous functions on stationary sets.

Theorem 5.1. *For any subset S of κ, the following statements are equivalent:*

(a) S is stationary in κ.

(b) Every continuous function $f : S \to \mathbb{R}$ is constant on some cofinal segment of S.

Proof. (a) implies (b). Let $f : S \to \mathbb{R}$ be continuous. We claim that there is $n_0 \in \omega$ so that $f^{-1}([n_0, +\infty))$ is bounded in κ. Indeed, if $f^{-1}([n, +\infty))$ are clubs in κ, it follows then that $\cap\{f^{-1}([n, +\infty)) : n > 0\}$ is not empty and thus $f(t) \geq n$ for some $t < \kappa$ and all $n > 0$. This is impossible. Likewise there is $m_0 \in \omega$ so that $f^{-1}((-\infty, -m_0])$ is bounded in κ. Thus $\sup(f^{-1}([n_0, +\infty)) \cup f^{-1}((-\infty, -m_0])) = \delta < \kappa$. Now construct a sequence of closed intervals $(J_k)_{k\in\omega}$ so that $J_0 = [-m_0, m_0]$ and $J_{k+1} \subseteq J_k$, $d(J_k) = \frac{d(J_0)}{2^k}$. Set $A = \{k \in \omega : f^{-1}(J_k)$ is bounded in $\kappa\}$, $B = \{k \in \omega : f^{-1}(J_k)$ is unbounded in $\kappa\}$. Let $\delta_1 = \sup\{f^{-1}(J_k) : k \in A\}$. For any $t \in]\delta \vee \delta_1, \rightarrow) \cap S \cap (\cap\{f^{-1}(J_k) : k \in B\})$, we have $f(t) \in \cap\{J_k : k \in B\} = \{a\}$. Next set $T = S \setminus (S \cap (\cap\{f^{-1}(J_k) : k \in B\}))$. Assume that T is cofinal and write $T = \{t_\alpha : \alpha < \kappa\}$. Then for each $\alpha < \kappa$, $f(t_\alpha) \neq a$. Since κ is regular uncountable pick a cofinal set $D \subseteq \{t_\alpha : \alpha < \kappa\}$ and some $k_0 \in \omega$ so that $f(D) \subseteq J_{k_0} \setminus J_{k_0+1}$. Write $J_k = [a_k, b_k]$ for all $k \in \omega$. Now let $t \in S \cap \lim(\mathrm{cl}(D)) \cap (\cap\{f^{-1}(J_k) : k \geq k_0\})$. Pick $d_{\nu(\eta)} \in D$ so that $\lim_\eta d_{\nu(\eta)} = \sup_\eta d_{\nu(\eta)} = t$. It follows then that $f(t) = a$ and $f(\lim_\eta d_{\nu(\eta)}) = \lim_\eta f(d_{\nu(\eta)})$, where $f(d_{\nu(\eta)}) \in [a_{k_0}, a_{k_0+1}[\cup]b_{k_0+1}, b_{k_0}[$. Thus $f(t) \in [a_{k_0}, a_{k_0+1}] \cup [b_{k_0+1}, b_{k_0}[$ and $f(t) = a$: contradiction. Therefore $\gamma = \sup T < \kappa$. Thus for $t \in [\xi_0, \rightarrow) \cap S$, $f(t) = a$, where $\xi_0 > \max(\gamma, \delta, \delta_1)$.

(b) implies (a). Suppose S is non-stationary. So pick C a club set in κ disjoint from S and write $\kappa \setminus C = \cup\{(x_\alpha, x_{\alpha+1}) : \alpha < \kappa\}$, where $(x_\alpha, x_{\alpha+1})$ is a maximal open interval and $(x_\alpha)_\alpha$ is an increasing continuous enumeration of C. $S \subseteq \kappa \setminus C = \cup\{(x_\alpha, x_{\alpha+1}) : \alpha < \kappa\}$. Define $f : S \setminus C \to \mathbb{R}$ by:

$$f(\xi) := \begin{cases} 0 & \text{if } \xi \in (x_\alpha, x_{\alpha+1}) \text{ and } \alpha = \lambda + 2n, \\ \frac{1}{n} & \text{if } \xi \in (x_\alpha, x_{\alpha+1}) \text{ and } \alpha = \lambda + 2n + 1. \end{cases}$$

f is the continuous function that works. □

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