Palestine Journal of Mathematics

Vol. 7(1)(2018) , 227–233

On Global Bipartite Domination Polynomials

Latheeshkumar A. R. and Anil Kumar. V.

Communicated by Ayman Badawi

MSC 2010 Classifications: 05C31, 05C69

Keywords and phrases: Domination polynomial, global bipartite domination, global bipartite domination polynomial.

Abstract In this paper we introduce the concept of the *global bipartite domination polynomial* of a connected bipartite graph and study some of its general properties. We establish some relationships between domination polynomial and global bipartite domination polynomial of certain classes of graphs.

1 Introduction

In this paper we consider simple, connected and bipartite graphs. All notations and definitions not given here can be found in [2, 4]. A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite nonempty set and $E(G)$ is a collection of 2- point subsets of V. The sets $V(G)$ and $E(G)$ are the vertex set and edge set of G respectively. The *degree* of a vertex v in G is the number of edges incident at v. The set of all neighbors of v is the *open neighborhood* of v, denoted by $N(v)$. Let P_n, C_n and $K_{m,n}$ denote path, cycle and complete bipartite graph respectively. A set $A \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a *dominating set*, if every vertex $v \in V$ is either an element of A or adjacent to an element of A. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. *The domination polynomial* of a graph G of order n is the polynomial $\mathcal{D}(G, x) = \Sigma_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i [1].

2 Main Results

In this section we introduce a new concept, namely, **Global Bipartite Dominating Set** of a simple bipartite graph G. Then we define the **Global Bipartite Domination Polynomial of** G.

Definition 2.1. Let G be a connected bipartite graph with bipartition (X, Y), with $|X| = m$ and $|Y| = n$. The relative complement of G in $K_{m,n}$ denoted by $\widehat{G}$ is the graph obtained by deleting all edges of G from $K_{m,n}$ $(i.e., K_{m,n} \setminus G)$. A global bipartite dominating set (GBDS) of G is a set S of vertices of G such that it dominates G and its relative complement $\widehat{G}$. The global bipartite domination number, $\gamma_{gb}(G)$ is the minimum cardinality of a global bipartite dominating set of G.

Definition 2.2. Let $\mathcal{D}_{gb}(G, i)$ be the family of global bipartite dominating sets of a simple connected bipartite graph G with cardinality i and let $d_{gb}(G, i) = |\mathcal{D}_{gb}(G, i)|$. Then the global bipartite domination polynomial $\mathcal{D}_{gb}(G, x)$ of G is defined as $\mathcal{D}_{gb}(G, x) = \Sigma_{i=\gamma_{gb}(G)}^{n} d_{gb}(G, i)x^i$

Theorem 2.3. *If G and $\widehat{G}$ are connected, then $\mathcal{D}_{gb}(G, x) = \mathcal{D}_{gb}(\widehat{G}, x)$.*

Proof. The proof follows immediately from the definitions of G.B.D.S and $\mathcal{D}_{gb}(G, x)$. □

Theorem 2.4. *For any positive integers m and n,*

(i) $\mathcal{D}_{gb}(K_{m,n}, x) = x^{m+n}$.

(ii) If $K_{m,n} \setminus e$ is connected, then $\mathcal{D}_{gb}(K_{m,n} \setminus e, x) = x^{m+n-1}(x + 2)$.

Proof. (i) Obviously $\gamma_{gb}(K_{m,n}) = m + n$. Therefore $\mathcal{D}_{gb}(K_{m,n}, x) = x^{m+n}$.

(ii) We have $\gamma_{gb}(K_{m,n} \setminus e) = m+n-1$. Since $d_{gb}((K_{m,n} \setminus e, m+n-1) = 2$ and $d_{gb}((K_{m,n} \setminus e, m+n) = 1$, the proof follows.

□

A bi-star graph $B_{(m,n)}$ is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in v.

Theorem 2.5. *The global bipartite domination polynomial of bi-star graph is*

$$\mathcal{D}_{gb}(B_{(m,n)}) = x^2 \left[x^m + x^n + [(1+x)^m - 1]\,[(1+x)^n - 1]\right]$$

Proof. Let U and V be the set of all pendant vertices in u and v respectively. Suppose S is a G.B.D.S of $B_{(m,n)}$. Since the vertices u and v are isolated in $\widehat{B}_{(m,n)}$, $\{u,v\} \subseteq S$. For $|S|-2 \neq m$ or n, $S \cap U \neq \phi$ and $S \cap V \neq \phi$. If $|S|-2 = m$, then $U \cup \{u,v\}$ and if $|S|-2 = n$, then $V \cup \{u,v\}$ are G.B.D.S of $B_{(m,n)}$. This completes the proof. □

The next theorem follows immediately from the definition of global bipartite domination polynomial.

Theorem 2.6. *For any spanning subgraph G of $K_{m,n}$,*

(i) $d_{gb}(G, m+n) = 1$.

(ii) $d_{gb}(G,i) = 0$ *if and only if* $i < \gamma_{gb}(G)$ *or* $i > m+n$

(iii) $\mathcal{D}_{gb}(G,x)$ *has no constant term.*

(iv) $\mathcal{D}_{gb}(G,x)$ *is a strictly increasing function in* $[0,\infty)$.

(v) If H is an induced subgraph of G, then $deg(\mathcal{D}_{gb}(G,x)) \geq deg(\mathcal{D}_{gb}(H,x))$

(vi) Zero is a root of $\mathcal{D}_{gb}(G,x)$ with multiplicity $\gamma_{gb}(G)$.

Theorem 2.7. *Let G be a graph with bipartition (X,Y). If G has a γ-set $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ then S is a γ_{gb}-set of G if and only if $\bigcap_{x \in V_1} N(x) \subseteq V_2$ and $\bigcap_{y \in V_2} N(y) \subseteq V_1$.*

Proof. Let $\bigcap_{x \in V_1} N(x) \subseteq V_2$ and $\bigcap_{y \in V_2} N(y) \subseteq V_1$. Since S is a γ- set of G, it suffices to show that S dominates the relative compliment of G. Let $u \in X$. If $u \in \bigcap_{y \in V_2} N(y)$, then $u \in V_1$. If $u \notin \bigcap_{y \in V_2} N(y)$ then u is adjacent to atleast one vertex of V_2 in $\widehat{G}$. Similarly, we can prove that if $v \in Y$ then $v \in V_2$ or v is adjacent to atleast one vertex of V_1 in $\widehat{G}$. Conversely, let S dominates $\widehat{G}$. Let x be an arbitrary vertex in X. If $x \in \bigcap_{y \in V_2} N(y)$, then in $\widehat{G}$, x is not adjacent to any vertex of V_2. Since S dominates $\widehat{G}$, we can deduce that $x \in V_1$. If $x \notin \bigcap_{y \in V_2} N(y)$, then x is adjacent to at least one element of V_2 in $\widehat{G}$. Hence the proof. □

Corollary 2.8. *For $n \geq 10$, $\gamma_{gb}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let $V(P_n) = \{1,2,3,\ldots,n\}$. Then $X = \{x : x \text{ is even}, x \leq n\}, Y = \{y : y \text{ is odd}, y \leq n\}$ is the bipartition of P_n. Let $S_1 = \{i : i \equiv 1(mod\ 3), i \leq n\}$ and $S_2 = \{i : i+1 \equiv 0(mod\ 3), i \leq n\}$. Then either S_1 or S_2 is a γ-set of P_n. Also for $i = 1,2$, $\bigcap_{x \in S_i \cap X} N(x) = \phi$ and $\bigcap_{y \in S_i \cap Y} N(y) = \phi$. Thus the proof follows from Theorem 2.7. □

Corollary 2.9. *For an even integer $n \geq 10$, $\gamma_{gb}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

Proof. The proof is exactly similar to corollary 2.8. □

Corollary 2.10. *If G is an $n-1$-regular connected bipartite graph, then*

$$\mathcal{D}_{gb}(G,x) = [x\,(x+2)]^n - 2nx^n.$$

Proof. Since G is $n-1$ regular, each component of $\widehat{G}$ is P_2. Therefore a G.B.D.S of G contains at least one vertex from each component of $\widehat{G}$. So $\gamma_{gb}(G) = n$ and for $1 \le i \le n$, $d_{gb}(G, n+i) = \binom{n}{i} 2^{n-i}$. It follows from Theorem 2.7 that $d_{gb}(G,n) = 2^n - 2n$. This completes the proof. □

Next, we shall study the relation between domination polynomials and global bipartite domination polynomials of paths. For, we need the following:

Theorem 2.11. *[1] For every $n \ge 4$,*
$\mathcal{D}(P_n,x) = x[\mathcal{D}(P_{n-1},x) + \mathcal{D}(P_{n-2},x) + \mathcal{D}(P_{n-3},x)$, *with initial values* $\mathcal{D}(P_1,x) = x$, $\mathcal{D}(P_2,x) = x^2+2x$, $\mathcal{D}(P_3,x) = x^3+3x^2+x$.

Lemma 2.12. *For a path P_n with bipartition (X,Y), let $S = V_1 \cup V_2$ where $V_1 \subseteq X$ and $V_2 \subseteq Y$ be a dominating set. If $|V_i| > 2, \forall i$ then S is a G.B.D.S. of P_n.*

Proof. In P_n if $|V_i| > 2$, then $\bigcap_{v \in V_i} N(v) = \phi$. Then by Theorem 2.7, S is a G.B.D.S of P_n. □

Theorem 2.13. *Let G be a connected bipartite graph with partite sets X and Y. Let $S = V_1 \cup V_2$ be a GBDS of G, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. Then if $V_1 = \phi$, then $V_2 = Y$ and if $V_2 = \phi$, then $V_1 = X$.*

Proof. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. If $V_1 = \phi$, then $S \subseteq Y$. Since G is bipartite, the vertices in Y are not adjacent and so $S \supseteq Y$. Therefore $S = V_2 = Y$. Similarly, we can prove that if $V_2 = \phi$ then $V_1 = X$. □

So for $n \ge 12$, to find $d(P_n,i) - d_{gb}(P_n,i)$ it suffices to consider the dominating sets $S = V_1 \cup V_2$ of P_n with $1 \le |V_1| \le 2$ or $1 \le |V_2| \le 2$. To prove theorems 2.14 to 2.17, we take $X = \{1,3,5,\ldots,2n-1\}$ and $Y = \{2,4,6,\ldots,2n\}$ be the bipartition of P_{2n} and $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ be a dominating set. Using the following theorems we can find the number of dominating sets which are not global bipartite dominating sets.

Theorem 2.14. *For $|V_1| = 1$, we have*

(i) $d(P_{2n},n) - d_{gb}(P_{2n},n) = 2n-2$.

(ii) $d(P_{2n},n-1) - d_{gb}(P_{2n},n-1) = n-2$.

Proof. Since a vertex in X is adjacent to atmost two vertices in Y, $n-2 \le |V_2| \le n$. If $|V_2| = n$, then $S = V_1 \cup V_2$ is a G.B.D.S and the proof is complete. So $|V_2| = n-2$ or $n-1$. We consider the following cases:

Case 1: $V_1 = \{1\}$.
Here $V_2 = \{4,6,8,\ldots,2n\}$. Since $N(1) = \{2\} \nsubseteq V_2$, S is not a G.B.D.S.

Case 2: $V_1 = \{3\}$.
Here also $|V_2| = n-1$ and $V_2 = \{2,6,8,\ldots,2n\}$. Since $N(3) = \{2,4\} \nsubseteq V_2$, S is not a G.B.D.S.

Case 3: $V_1 = \{i\}, i \ne 1,3$.
Then for each i, $V_1 \cup Y \setminus \{i-1,i+1\}$, $V_1 \cup Y \setminus \{i-1\}$ and $V_1 \cup Y \setminus \{i+1\}$ are dominating sets of P_{2n}. Since $N(i) = \{i-1,i+1\} \nsubseteq V_2$, these are not G.B.D.S of P_{2n}.

In cases 1 and 2 we have two dominating sets of order n. In case 3 we have $2(n-2)$ dominating sets of order n and $n-2$ dominating sets of order $n-1$. Therefore the result follows. □

Theorem 2.15. *For $|V_2| = 1$, we have*

(i) $d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 2.$

(ii) $d(P_{2n}, n-1) - d_{gb}(P_{2n}, n-1) = n - 2.$

Proof. The proof is exactly similar to Theorem 2.14. □

Theorem 2.16. *For* $|V_1| = 2$, *we have*

(i) $d(P_{2n}, n-1) - d_{gb}(P_{2n}, n-1) = n - 3.$

(ii) $d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 4.$

(iii) $d(P_{2n}, n+1) - d_{gb}(P_{2n}, n+1) = n - 1.$

Proof. Since $|V_1| = 2$, we have $n - 3 \le |V_2| \le n$. If $|V_2| = n$, then $S = V_1 \cup V_2$ is a G.B.D.S. So it suffices to consider the cases $|V_2| = n-3, n-2$ and $n-1$.

Case 1: $V_1 = \{1, 3\}$.

Subcase 1: $|V_2| = n - 2$.
Then $V_2 = \{6, 8, \dots, 2n\}$. Since $N(1) \cup N(3) = \{2\} \nsubseteq V_2$, S is not a G.B.D.S of P_{2n}.

Subcase 2: $|V_2| = n - 1$.
Then $V_2 = \{4, 6, 8, \dots, 2n\}$. Since $N(1) \cup N(3) = \{2\} \nsubseteq V_2$, the dominating set S is not a G.B.D.S.

Case 2: $V_1 = \{3, 5\}$.
As in case 1 we get two dominating sets which are not G.B.D.S of P_{2n}.

Case 3: $V_1 = \{i, i+2\}, i \neq 1, 3$.

Subcase 1: $|V_2| = n - 3$.
Then $V_2 = Y \setminus \{i-1, i+1, i+3\}$.

Subcase 2: $|V_2| = n - 2$.
In this case we have the possibilities, $V_2 = Y \setminus \{i-1, i+1\}$ and $V_2 = Y \setminus \{i+1, i+3\}$.

Subcase 3: $|V_2| = n - 1$.
Then $V_2 = Y \setminus \{i+1\}$.

In subcase 1,2 and 3, $S = V_1 \cup V_2$ is a dominating set but since $N(i) \cap N(i+1) = \{i+1\} \nsubseteq V_2$, S is not a G.B.D.S of P_{2n}.

In cases 1 and 2 we have two dominating sets of order n and $n+1$. In case 3 we have $n-3$ dominating sets of order $n-1$, $2(n-3)$ dominating sets of order n and $n-3$ dominating sets of order $n+1$. Hence the result follows. □

Theorem 2.17. *For* $|V_2| = 2$, *we have*

(i) $d(P_{2n}, n-1) - d_{gb}(P_{2n}, n-1) = n - 3.$

(ii) $d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 4.$

(iii) $d(P_{2n}, n+1) - d_{gb}(P_{2n}, n+1) = n - 1.$

Proof. The proof is exactly similar to Theorem 2.16. □

Theorem 2.18. *For* $n \ge 6$,

$$\mathcal{D}(P_{2n}, x) - \mathcal{D}_{gb}(P_{2n}, x) = (4n-10)x^{n-1} + (8n-12)x^n + (2n-2)x^{n+1}.$$

Proof. It follows from Theorems 2.14, 2.15, 2.16 and 2.17. □

Next, we find the relationship between domination polynomials and global bipartite domination polynomials of P_{2n+1}. To prove theorems 2.19 to 2.22, we take $X = \{1, 3, 5, \dots, 2n+1\}$ and $Y = \{2, 4, 6, \dots, 2n\}$ be the bipartition of P_{2n+1} and $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ be a dominating set of P_{2n+1}.

Theorem 2.19. *For* $|V_1| = 1$, *we have*

(*i*) $d(P_{2n+1}, n-1) - d_{gb}(P_{2n+1}, n-1) = n-3$.

(*ii*) $d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n-2$.

Proof. **Case 1:** $V_1 = \{1\}$. Let $V_2 = Y \setminus \{2\}$. Since $N(1) = \{2\}, S = V_1 \cup V_2$ is not a G.B.D.S. The case $V_1 = \{2n+1\}$ is similar.

Case 2: $V_1 = \{3\}$. Let $V_2 = Y \setminus \{4\}$. Since $N(3) = \{2,4\}, S = V_1 \cup V_2$ is not a G.B.D.S. The case $V_1 = \{2n-1\}$ is similar.

Case 3: $V_1 = \{i\}$, $i \notin \{1, 3, 2n-1, 2n+1\}$. In this case we have the possibilities, $V_2 = Y \setminus \{i-1, i+1\}$ or $V_2 = Y \setminus \{i-1\}$ and $V_2 = Y \setminus \{i+1\}$. Since $N(i) = \{i-1, i+1\}, S = V_1 \cup V_2$ is not a G.B.D.S.

In cases 1 and 2 we have four dominating sets of order n and in case 3 there are $n-3$ dominating sets of order $n-1$ and $2(n-3)$ dominating sets of order n. This completes the proof. □

Theorem 2.20. *For* $|V_2| = 1$, *we have*

(*i*) $d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = n$.

(*ii*) $d(P_{2n+1}, n+1) - d_{gb}(P_{2n+1}, n+1) = 2n$.

Proof. Let $V_2 = \{i\}, i \in Y \Rightarrow N(i) = \{i-1, i+1\}$. Then V_1 can be $X \setminus \{i-1\}$ or $X \setminus \{i+1\}$ or $X \setminus \{i-1, i+1\}$. Since i can be selected in n ways , we have $2n$ dominating sets of order $n+1$ and n dominating sets of order$n$. Since $N(i) = \{i-1, i+1\},\ S = V_1 \cup V_2$ is not a G.B.D.S. of P_{2n+1}. Hence the result follows. □

Theorem 2.21. *For* $|V_1| = 2$, *we have*

(*i*) $d(P_{2n+1}, n-1) - d_{gb}(P_{2n+1}, n-1) = n-4$.

(*ii*) $d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n-4$.

(*iii*) $d(P_{2n+1}, n+1) - d_{gb}(P_{2n+1}, n+1) = n$.

Proof. **Case 1:** $V_1 = \{1, 3\}$. Then V_2 can be $Y \setminus \{2\}$ or $Y \setminus \{2, 3\}$. Since $N(1) \cap N(3) = \{2\},\ S = V_1 \cup V_2$, is not a G.B.D.S. The case $V_1 = \{2n-1, 2n+1\}$ is similar.

Case 2: $V_1 = \{3, 5\}$. Then V_2 can be $Y \setminus \{4\}$ or $Y \setminus \{4, 5\}$. Since $N(3) \cap N(5) = \{4\},\ S = V_1 \cup V_2$, is not a G.B.D.S. The case $V_1 = \{2n-3, 2n-1\}$ is similar.

Case 3: $V_1 = \{i, i+2\}, i \notin \{1, 3, 2n-3, 2n-1\}$. Then V_2 can be $Y \setminus \{i-1, i+1, i+3\}$ or $Y \setminus \{i-1, i+1\}$ or $Y \setminus \{i+1, i+3\}$. Since $N(i) \cap N(i+2) = \{i+1\},\ S = V_1 \cup V_2$, is not a G.B.D.S.

In cases 1 and 2 we have four dominating sets of order n and $n+1$. In case 3 there are $n-4$ dominating sets of order $n-1$ and $n+1$ and $2(n-4)$ dominating sets of order n. Thus the result follows. □

Theorem 2.22. *For* $|V_2| = 2$, *we have*

(*i*) $d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = n-1$.

(*ii*) $d(P_{2n+1}, n+1) - d_{gb}(P_{2n+1}, n+1) = 2n-2$.

(*iii*) $d(P_{2n+1}, n+2) - d_{gb}(P_{2n+1}, n+2) = n-1$.

Proof. Let $V_2 = \{i, i+2\},\ i \in Y \Rightarrow N(i) \cap N(i+2) = \{i+1\}$. Then V_1 can be $X \setminus \{i-1, i+1, i+3\}$ or $X \setminus \{i-1, i+1\}$ or $X \setminus \{i+1, i+3\}$. Since V_2 can be chosen in $n-1$ ways , we have $n-1$ dominating sets of order n and $2(n-1)$ dominating sets of order$n+1$ and $n-1$ dominating sets of order $n+2$. Since $N(i) \cap N(i+2) = \{i+1\},\ S = V_1 \cup V_2$ is not a G.B.D.S. of P_{2n+1}. This proves the result. □

Theorem 2.23. *For* $n \geq 6$,

$$\mathcal{D}(P_{2n+1}, x) - \mathcal{D}_{gb}(P_{2n+1}, x) = (2n-7)x^{n-1} + (6n-7)x^n + (5n-2)x^{n+1} + (n-1)x^{n+2}.$$

Proof. It follows from Theorems 2.19, 2.20, 2.21 and 2.22. □

Theorem 2.24. *[1] For every* $n \geq 4$,
$\mathcal{D}(C_n, x) = x[\mathcal{D}(C_{n-1}, x) + \mathcal{D}(C_{n-2}, x) + \mathcal{D}(C_{n-3}, x)$, *with initial values* $\mathcal{D}(C_1, x) = x$, $\mathcal{D}(C_2, x) = x^2 + 2x$, $\mathcal{D}(C_3, x) = x^3 + 3x^2 + 3x$.

Next, we find $\mathcal{D}(C_{2n}, x) - \mathcal{D}_{gb}(C_{2n}, x)$.
To prove theorems 2.25 to 2.29, we take $X = \{1, 3, 5, \ldots, 2n-1\}$ and $Y = \{2, 4, 6, \ldots, 2n\}$ be the bipartition of C_{2n} and $S = V_1 \cup V_2$ where $V_1 \subseteq X$ and $V_2 \subseteq Y$ be a dominating set of C_{2n}.

Theorem 2.25. *For* $|V_1| = 1$, *we have*

(i) $d(C_{2n}, n-1) - d_{gb}(C_{2n+1}, n-1) = n.$

(ii) $d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n.$

Proof. Let $V_1 = \{i\}$, $i \in X$. Then $N(i) = \{i-1, i+1\}$ (if $i = 1$, then we take $i - 1 = 2n$.) Then V_2 can be $Y \setminus \{i-1, i+1\}$ or $X \setminus \{i-1\}$ or $X \setminus \{i+1\}$. Since i can be chosen in n ways , we have n dominating sets of order $n-1$ and $2n$ dominating sets of order$n$. Since $N(i) = \{i-1, i+1\}$, $S = V_1 \cup V_2$ is not a G.B.D.S. of C_{2n}. Hence the result follows. □

Theorem 2.26. *For* $|V_2| = 1$, *we have*

(i) $d(C_{2n}, n-1) - d_{gb}(C_{2n+1}, n-1) = n.$

(ii) $d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n.$

Proof. The proof is exactly similar to Theorem 2.25. □

Theorem 2.27. *For* $|V_1| = 2$, *we have*

(i) $d(C_{2n}, n-1) - d_{gb}(C_{2n}, n-1) = n-1.$

(ii) $d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n-1).$

(iii) $d(C_{2n}, n+1) - d_{gb}(C_{2n}, n+1) = n-1.$

Proof. Let $V_1 = \{i, i+2\}$, $i \in X$. Then $N(i) \cap N(i+2) = \{i+1\}$(if $i = 2n-1$, then we take $i + 2 = 1$ and $i+3 = 2$.) Then V_2 can be $Y \setminus \{i-1, i+1, i+3\}$ or $Y \setminus \{i-1, i+1\}$ or $Y \setminus \{i+1, i+3\}$ or $Y \setminus \{i+1\}$. Since V_1 can be chosen in $n-1$ ways , we have $(n-1)$ dominating sets of order $n-1$,$2(n-1)$ dominating sets of order$n$ and $n-1$ dominating sets of order $n+1$. Since $N(i) \cap N(i+2) = \{i+1\}$, $S = V_1 \cup V_2$ is not a G.B.D.S. of C_{2n}. Hence the result follows. □

Theorem 2.28. *For* $|V_2| = 2$, *we have*

(i) $d(C_{2n}, n-1) - d_{gb}(C_{2n}, n-1) = n-1.$

(ii) $d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n-1).$

(iii) $d(C_{2n}, n+1) - d_{gb}(C_{2n}, n+1) = n-1.$

Proof. The proof is exactly similar to Theorem 2.27. □

Theorem 2.29. *For* $n \geq 6$,

$$\mathcal{D}(C_{2n}, x) - \mathcal{D}_{gb}(C_{2n}, x) = (4n-2)x^{n-1} + (8n-4)x^n + (2n-2)x^{n+1}.$$

Proof. It follows from Theorems 2.25, 2.26, 2.27 and 2.28. □

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Author information

Latheeshkumar A. R., Department of Mathematics,St. Mary’s College, Sulthan bathery, Wayanad, Kerala, 673 592, India.
E-mail: `latheesby@gmail.com`

Anil Kumar. V., Department of Mathematics,University of Calicut, Malappuram, Kerala, 673 635, India.
E-mail: `anil@uoc.ac.in`

Received: July 22, 2016.

Accepted: February 27, 2017.