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ON SEMIGROUP IDEALS OF PRIME NEAR-RINGS WITH GENERALIZED SEMIDERIVATION

Öznur Gölbaşı and Emine Koç

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Abstract Let N be a $3-$prime near-ring with a nonzero generalized semiderivation F associated with a semiderivation d and an automorphism g associated with d and U be a nonzero semigroup ideal of N. In this paper, it is shown that N is commutative ring, if any one of the following conditions are satisfied: (i) $F(U) \subseteq Z$, (ii) $F([u,v]) = 0$, (iii) $F([u,v]) = \pm[u,v]$, (iv) $F([u,v]) = [F(u),v]$, (v) $[F(u),v] \in Z$, for all $u,v \in U$.

1 Introduction

Throughout this paper, N denotes a zero-symmetric left near-ring with multiplicative center Z. For any $x,y \in N$, as usual $[x,y] = xy - yx$ will denote the commutator product. A near-ring N is said to be 3-prime if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN \subseteq U$ (resp. $NU \subseteq U$) and if U is both a semigroup right ideal and a semigroup left ideal, it will be called a semigroup ideal. As for terminologies used here without mention, we refer to G. Pilz [9].

The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [1] and [2]. An additive mapping $d : N \longrightarrow N$ is said to be a derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x,y \in N$. An additive mapping $F : N \to N$ is called a generalized derivation if there exists a derivation $d : N \to N$ such that $F(xy) = F(x)y + xd(y)$, for all $x,y \in N$. Over the last sixteen years, many authors have proved commutativity theorems for prime or semiprime rings or near-rings admitting derivations or generalized derivations. In [4], J. Bergen has introduced the notion of semiderivation of a ring which extends the notion of derivation of a ring. This definition was given for near-rings as following: An additive mapping $d : N \to N$ is called a semiderivation if there exists an additive function $g : N \to N$ such that (i) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and (ii) $d(g(x)) = g(d(x))$ hold for all $x,y \in N$, or equivalently, as noted in [5], that (i) $d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y$ and (ii) $d(g(x)) = g(d(x))$ hold for all $x,y \in N$. In case g is an identity map of N, then all semiderivations associated with g are merely ordinary derivations. But the generalization is not trivial. For example $N = N_1 \oplus N_2$, where N_1 is a zero-symmetric near-ring and N_2 is a ring. Then the map $d : N \to N$ defined by $d(x,y) = (0,y)$ is a semiderivation associated with function $g : N \to N$ such that $g(x,y) = (x,0)$. However d is not a derivation on N. This example was given by Boua et al. in [7].

Recently, semiderivations have received significant attention. Many authors have studied commutativity of prime and semiprime near-rings with semiderivations. In [7], the definition of a generalized semiderivation was extended by A. Boua et al. as follows:

An additive mapping $F : N \to N$ is said to be a generalized semiderivation of N if there exists a semiderivation $d : N \to N$ associated with a map $g : N \to N$ such that (i) $F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y)$ and (ii) $F(g(x)) = g(F(x))$ hold for all $x,y \in N$. All semiderivations are generalized semiderivations. If g is the identity map on N, then all generalized semiderivations are merely generalized derivations, again the notion of generalized semiderivation generalizes that of generalized derivation. Morever, the generalization is not trivial as the following example which was given by Boua et al.

Let S be a 2−torsion free left near-ring and

$$N=\left\{\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix} \mid x,y,z\in S\right\}.$$

Define maps $F,d,g:N\rightarrow N$ by

$$F\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix}=\begin{pmatrix}0 & xy & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, d\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix}=\begin{pmatrix}0 & 0 & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix},$$

$$g\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix}=\begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It can be verified that N be a left near-ring and F is a generalized semiderivation with associated semiderivation d and a map g associated with d. However F is not a generalized derivation on N.

In [8], M. N. Daif and H. E. Bell showed that the ideal I of a semiprime ring R is contined in the center of R if any one of the following conditions

$$d([x,y])=[x,y] \text{ and } d([x,y])=-[x,y] \text{ for all } x,y\in I,$$

are satisfied. Several authors investigated this result for prime or semiprime ring admitting derivation or generalized derivation. Motived by these works, we will prove this result and extend some commutativity theorems in the setting of semigroup ideal of a 3−prime near-ring involving generalized semiderivation.

2 Results

Lemma 2.1. *[3, Lemma 1.2] Let N be a 3−prime near-ring. If $z\in Z\backslash\{0\}$ and $x\in N$ such that $xz\in Z$ or $zx\in Z$, then $x\in Z$.*

Lemma 2.2. *[3, Lemma 1.3] Let N be a 3−prime near-ring, U a nonzero semigroup ideal of N and $x\in N$.*
i) If $Ux=\{0\}$ or $xU=\{0\}$, then $x=0$.
ii) If $[U,x]=\{0\}$, then $x\in Z$

Lemma 2.3. *[3, Lemma 1.4] Let N be a 3−prime near-ring, U a nonzero semigroup ideal of N and $a,b\in N$. If $aUb=\{0\}$, then $a=0$ or $b=0$.*

Lemma 2.4. *[6, Theorem 1] Let N be a 3−prime near-ring, U a nonzero semigroup ideal of N. If N admits a nonzero semiderivation d such that $d(U)\subseteq Z$, then N is commutative ring.*

Lemma 2.5. *[7, Lemma 5] Let N be a 3−prime near-ring admitting a nonzero generalized semiderivation F associated with a semiderivation d. If g is the map associated with d such that $g(xy)=g(x)g(y)$ for all $x,y\in N$, then N satisfies the following partial distrubitive laws:*

$$(F(x)y+g(x)d(y))z=F(x)yz+g(x)d(y)z$$
$$(d(x)y+g(x)d(y))z=d(x)yz+g(x)d(y)z, \text{ for all } x,y,z\in N.$$

Lemma 2.6. *Let N be a near-ring admitting a nonzero generalized semiderivation F associated with a semiderivation d. Then the following condition is satisfy*

$$F(xy)=g(x)d(y)+F(x)y=xF(y)+d(x)g(y), \text{ for all } x,y\in N.$$

Proof. For any $x, y \in N$, we get

$$F(x(y+y)) = F(x)(y+y) + g(x)d(y+y)$$
$$= F(x)y + F(x)y + g(x)d(y) + g(x)d(y)$$

and

$$F(xy+xy) = F(x)y + g(x)d(y) + F(x)y + g(x)d(y).$$

Comparing these two expressions, one can obtain

$$F(xy) = g(x)d(y) + F(x)y, \text{ for all } x, y \in N.$$

Similarly, we can prove $F(xy) = xF(y) + d(x)g(y)$, for all $x, y \in N$. □

Lemma 2.7. *Let N be a 3−prime near-ring admitting a nonzero generalized semiderivation F associated with a semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $F(U) = \{0\}$, then $d = 0$.*

Proof. Replacing u by $ux, x \in N$ in the hypothesis and using this, we have

$$0 = F(ux) = F(u)x + g(u)d(x)$$

and so

$$g(u)d(x) = 0, \text{ for all } u \in U, x \in N.$$

This equation shows that

$$Id(N) = 0.$$

where $I = g(U)$. It is clear that I is a nonzero semigroup ideal of N. Hence we get $d = 0$ by Lemma 2.2 (i). □

Lemma 2.8. *Let N be a 3−prime near-ring admitting a semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $ad(U) = \{0\}$ (or $d(U)a = \{0\}$), then $a = 0$ or $d = 0$.*

Proof. Writing u by $ux, x \in N$ in the hypothesis and using this, we have

$$0 = ad(ux) = ad(u)g(x) + aud(x)$$

and so

$$aUd(N) = 0.$$

By Lemma 2.3, we arrive at $a = 0$ or $d = 0$.
$d(U)a = \{0\}$ can be proved by the the expression of $d(xu)a = 0$. □

Remark 2.9. Let N be a 3−prime near-ring admitting a nonzero semiderivation d. If g is an automorphism on N, then $d(Z) \subset Z$.

Proof. For all $x \in N, z \in Z$, we get

$$zx = xz$$
$$d(zx) = d(xz)$$
$$g(z)d(x) + d(z)x = d(x)g(z) + xd(z)$$

Using $g(z) \in Z$, we have

$$d(z)x = xd(z), \text{ for all } x \in N$$

and so, $d(z) \in Z$. □

Lemma 2.10. *Let N be a 3−prime near-ring admitting a nonzero semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $d(U)d(U) \subset Z$, then N is commutative ring.*

Proof. Assume that
$$d(u)d(v) \in Z, \text{ for all } u,v \in U.$$
Replacing v by vw in this equation, we have
$$d(u)d(v)w + d(u)g(v)d(w) \in Z.$$
Commuting this equation with w and using Lemma 2.5, it can be rewritten as
$$wd(u)d(v)w + wd(u)g(v)d(w) = d(u)d(v)w^2 + d(u)g(v)d(w)w.$$
Since $d(u)d(v) \in Z$, the equation reduces to
$$wd(u)g(v)d(w) = d(u)g(v)d(w)w, \text{ for all } u,v,w \in U.$$
This equation shows that
$$wd(u)jd(w) = d(u)jd(w)w, \text{ for all } u,w \in U, j \in I$$
where $I = g(U)$. Writing $d(v)j$ instead of j in the last equation and using $d(u)d(v) \in Z$, we have
$$wd(u)d(v)jd(w) = d(u)d(v)jd(w)w$$
$$d(u)d(v)(wjd(w) - jd(w)w) = 0.$$
Since $d(u)d(v) \in Z$, the latter expression implies that
$$d(u)d(v)N(wjd(w) - jd(w)w) = 0,$$
and so
$$d(u)d(v) = 0 \text{ or } wjd(w) = jd(w)w, \text{ for all } u,v,w \in U, j \in I.$$
If $d(u)d(v) = 0$, for all $u,v \in U$, then we find that $d = 0$ by Lemma 2.8, a contradiction. So we must have
$$wjd(w) = jd(w)w, \text{ for all } w \in U, j \in I.$$
Substituting $jd(v)$ for j in this equation and using $d(v)d(w) \in Z$, we get
$$wjd(v)d(w) = jd(v)d(w)w$$
$$d(v)d(w)[w,j] = 0.$$
Again using $d(v)d(w) \in Z$, we find that
$$d(v)d(w) = 0 \text{ or } [w,j] = 0, \text{ for each } w \in U. \tag{2.1}$$
By Lemma 2.4, there exists $w_0 \in U$ such that $d(w_0) \neq 0$ and by Lemma 2.8 there exists $v_0 \in U$ such that $d(v_0)d(w_0) \neq 0$. Hence, we get $w_0 \in Z$ by (2.1), and so $d(w_0) \in Z$. Since $d(v)d(w_0) \in Z$ and $d(w_0) \in Z \backslash \{0\}$, we arrive at $d(v) \in Z$, for all $v \in U$ by Lemma 2.1. That is $d(U) \subset Z$ and N is commutative ring by Lemma 2.4. □

Theorem 2.11. *Let N be a $2-$torsion free $3-$prime near-ring admitting a nonzero generalized semiderivation F associated with a nonzero semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $F(U) \subseteq Z$, then N is commutative ring.*

Proof. Let assume $d(Z) \neq \{0\}$. There exists a $z \in Z$ such that $d(z) \neq 0$. Since $z \in Z$, we get $d(z) \in Z$. By the hypothesis $F(u) \in Z$, for all $u \in U$, and so
$$F(uz)x = xF(uz), \text{ for all } u \in U, x \in N.$$
By Lemma 2.5, we get
$$F(u)zx + g(u)d(z)x = xF(u)z + xg(u)d(z)$$

and so

$$g(u)d(z)x = xg(u)d(z).$$

Using $0 \neq d(z) \in Z$, we arrive at

$$d(z)[g(u), x] = 0, \text{ for all } u \in U, x \in N,$$

and so, we obtain that $g(u) \in Z$, for all $u \in U$. That is $d(g(U)) \subseteq Z$. Hence we get $d(I) \subseteq Z$ where $I = g(U)$. It is clear that I is a nonzero semigroup ideal of N. Therefore, N is commutative ring by Lemma 2.4.

Now we assume that $d(Z) = \{0\}$. Since $F(ur) \in Z$, for all $u \in U, r \in N$, $d(F(ur)) = 0$. This implies that

$$\begin{aligned} 0 &= d(F(ur)) = d(F(u)r + g(u)d(r)) \\ &= d(F(u))g(r) + F(u)d(r) + g(u)d^2(r) + d(g(u))g(d(r)) \end{aligned}$$

and so

$$F(u)d(r) + g(u)d^2(r) + d(g(u))g(d(r)) = 0, \text{ for all } u \in U, r \in N.$$

On the other hand, $F(ud(r)) \in Z$. That is

$$F(ud(r)) = F(u)d(r) + g(u)d^2(r).$$

Using the last two equations together, we arrive at

$$\begin{aligned} F(ud(r)) &= -d(g(u))d(g(r)) \\ &= d(g(u))(-d(g(r))) \\ &= d(g(u))(d(g(-r))). \end{aligned}$$

and so

$$d(g(u))d(g(-r)) \in Z, \text{ for all } u \in U, r \in N.$$

Replacing r by $-r$ in this equation, we have

$$d(g(u))d(g(r)) \in Z, \text{ for all } u \in U, r \in N.$$

In particular, we get

$$d(g(u))d(g(v)) \in Z, \text{ for all } u, v \in U.$$

We can write the last equation $d(i)d(j) \in Z$, for all $i, j \in I$, where $I = g(U)$. Hence we find that

$$d(I)d(I) \subseteq Z.$$

Therefore we conclude that N is commutative ring by Lemma 2.10. □

Corollary 2.12. *Let N be a $2-$torsion free $3-$prime near-ring admitting a nonzero semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $d(U) \subseteq Z$, then N is commutative ring.*

Theorem 2.13. *Let N be a $3-$prime near-ring admitting a nonzero generalized semiderivation F associated with a semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $F([u, v]) = 0$, for all $u, v \in U$, then N is commutative ring.*

Proof. Replacing v by uv in the hypothesis and using this, we have

$$d(u)g([u, v]) = 0, \text{ for all } u, v \in U.$$

Since g is an automorphism, we get

$$d(u)g(u)g(v) = d(u)g(v)g(u), \text{ for all } u, v \in U.$$

This equation shows that

$$d(u)g(u)j = d(u)jg(u), \text{ for all } u \in U, j \in I$$

where $I = g(U)$. Writing $jg(x), x \in N$ instead of j in the last equation and using this, we have

$$d(u)j[g(u), g(x)] = 0, \text{ for all } u \in U, j \in I, x \in N.$$

By Lemma 2.3, this implies that $d(u) = 0$ or $[g(u), g(x)] = 0$. Since g is an automorphism, we can write $[u, x] = 0$. If $[u, x] = 0$, then $u \in Z$ by Lemma 2.2 (ii), and so $d(u) \in Z$. In both cases, we arrive at $d(U) \subseteq Z$, and so Lemma 2.4 assures N is commutative ring. □

Theorem 2.14. *Let N be a $3-$prime near-ring admitting a nonzero generalized semiderivation F associated with a semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $F([u, v]) = \pm[u, v]$, for all $u, v \in U$, then N is commutative ring.*

Proof. Assume that $F([u, v]) = [u, v]$, for all $u, v \in U$. Taking uv instead of v in this equation and using Lemma 2.6, we have

$$\begin{aligned} F([u, uv]) &= [u, uv] \\ F(u[u, v]) &= u[u, v] \\ uF([u, v]) + d(u)g([u, v]) &= u[u, v]. \end{aligned}$$

By the hypothesis, we get

$$d(u)g([u, v]) = 0, \text{ for all } u, v \in U.$$

Arguing in the similar manner as in the proof of Theorem 2.13, we get the result.

$F([u, v]) = -[u, v]$ can be proved by following the same lines as above with necessary variations. □

Theorem 2.15. *Let N be a $3-$prime near-ring admitting a nonzero generalized semiderivation F associated with a semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $F([u, v]) = [F(u), v]$, for all $u, v \in U$, then N is commutative ring.*

Proof. Replacing v instead of uv in the hypothesis and using Lemma 2.6, we get

$$\begin{aligned} F([u, uv]) &= [F(u), uv] \\ F(u[u, v]) &= F(u)uv - uvF(u) \\ uF([u, v]) + d(u)g([u, v]) &= F(u)uv - uvF(u). \end{aligned}$$

On the other hand, we can write $F(u)u = uF(u)$, for all $u \in U$ by thehypothesiss. Then the last equation gives that

$$\begin{aligned} uF([u, v]) + d(u)g([u, v]) &= uF(u)v - uvF(u) \\ uF([u, v]) + d(u)g([u, v]) &= u[F(u), v]. \end{aligned}$$

Using the hyphotesis, we get

$$d(u)g([u, v]) = 0, \text{ for all } u, v \in U.$$

Applying the same arguments as used in the proof of Theorem 2.13, we get the required result. □

Theorem 2.16. *Let N be a $3-$prime near-ring admitting a nonzero generalized semiderivation F associated with a nonzero semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $F([u, v]) = [u, F(v)]$, for all $u, v \in U$, then N is commutative ring.*

Proof. By the hypothesis

$$F([u,v]) = [u, F(v)], \text{ for all } u, v \in U.$$

Taking vu instead of u in the hypothesis and using this, we have

$$\begin{aligned} F([vu,v]) &= [vu, F(v)] \\ F(v[u,v]) &= vuF(v) - F(v)vu \\ vF([u,v]) + d(v)g([u,v]) &= vuF(v) - F(v)vu. \end{aligned}$$

Noting that $F(u)u = uF(u)$, for all $u \in U$ by the hyphotesis. Then the last equation gives that

$$\begin{aligned} vF([u,v]) + d(v)g([u,v]) &= vuF(v) - vF(v)u \\ vF([u,v]) + d(v)g([u,v]) &= v[u, F(v)]. \end{aligned}$$

Using the hyphotesis, we arrive at

$$d(v)g([u,v]) = 0, \text{ for all } u, v \in U.$$

Arguing in the similar manner as in the proof of Theorem 2.13, we get the result. □

Theorem 2.17. *Let N be a $2-$torsion free $3-$prime near-ring admitting a nonzero generalized semiderivation F associated with a semiderivation d and U be a nonzero semigroup ideal of N. If g is an automorphism on N and $[F(u), v] \in Z$, for all $u, v \in U$, then N is commutative ring.*

Proof. Replacing $F(u)v$ instead of v in the hypothesiss, we have

$$F(u)[F(u), v] \in Z, \text{ for all } u, v \in Z.$$

Commuting this term with $v \in U$ and using $[F(u), v] \in Z$, we get

$$\begin{aligned} F(u)[F(u), v]v &= vF(u)[F(u), v] \\ [F(u), v]F(u)v &= [F(u), v]vF(u) \end{aligned}$$

and so

$$[F(u), v]^2 = 0, \text{ for all } u, v \in U.$$

Since N be a $3-$prime near ring and $[F(u), v] \in Z$, we find that

$$[F(u), v] = 0, \text{ for all } u, v \in U.$$

By Lemma 2.2 (ii), we get $F(U) \subseteq Z$, and so N is commutative ring by Theorem 2.11. □

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Author information

Öznur Gölbaşı and Emine Koç, Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140, Sivas-Turkey,.
E-mail: ogolbasi@cumhuriyet.edu.tr, eminekoc@cumhuriyet.edu.tr

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