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On Gaussian Pell Polynomials and Their Some Properties

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Abstract In this study, we define firstly Gaussian Pell polynomials. Then, we give the generating functions and Binet formulas for this type polynomials. We also obtain some important identities involving the Gaussian Pell polynomials.

1 Introduction

The investigation of Gaussian numbers is a research topic of great interest. The set of these numbers is denoted by $\mathbb{Z}[i]$. One of the first studies on this subject belongs to Gauss. Gaussian numbers were investigated by Gauss in 1832. Then, in 1963, Horadam [8, 9] introduced the concept of Gaussian Fibonacci numbers, that is, complex Fibonacci numbers. And then J. H. Jordan [10] considered the two different sequences of Gaussian Fibonacci numbers and extented some relationships which are known about the common Fibonacci sequences. He gave many identities related with them. For example, $n \geq 2$, some of these identities are

$$GF_n = F_n + iF_{n-1}; \ GF_{-n} = iGF_n,$$

$$GF_{n+1}GF_{n-1} - GF_n^2 = (-1)^n(2-i),$$

$$GF_{n+1}^2 - GF_{n-1}^2 = F_{2n-1}(1+2i),$$

$$GF_n^2 + GF_{n+1}^2 = F_{2n}(1+2i); \ \Sigma_{k=0}^{n} GF_k = GF_{n+2} - 1.$$

In fact the above identities are known as the relationship between the usual Fibonacci sequence and the Gaussian Fibonacci, Lucas sequences in the literature. Also, Horadam studied also the complex Fibonacci polynomials. Fibonacci-like recursion relations are a special case of difference equations that can be solved by the combinatorics function technique method. Polynomials that can be defined by Fibonacci-like recursion relations are called Fibonacci Polynomials and they were studied in 1883 by Catalan and Jacobsthal. More mathematicians were involved in the study of Fibonacci polynomials such as Y. Yuan, W. Zhang [20], among others. G. Berzsenyi presented a natural manner of extension of the Fibonacci numbers into the complex plane [4]. In addition to this, Berzsenyi obtained some interesting identities for the classical Fibonacci numbers. And, he gave a closed form for Gaussian Fibonacci numbers by the Fibonacci Q matrix. In 1981, Harman gave the extension of Fibonacci numbers into the complex plane [6]. And, the author generalized the methods are given by Horadam and Berzsenyi. We notice that the generalized Gaussian Fibonacci and Lucas numbers are defined as similar to generalized Fibonacci and Lucas numbers. In [1], we investigated Gaussian Pell and Pell-Lucas numbers.

In this study, we define and study the Gaussian Pell polynomials. We give generating functions and Binet formulas of them. Also, we obtain some important identities involving the terms of these polynomials.

2 Gaussian Pell Polynomials and Their Some Properties

In this section, we introduce Gaussian Pell polynomials $GP_n(x)$ and present their some basic properties. Now we define Gaussian Pell polynomials $GP_n(x)$ as

$$GP_{n+1}(x) = 2xGP_n(x) + GP_{n-1}(x);\;\; GP_0(x) = i,\;\; GP_1(x) = 1.$$

That is, the Gaussian Pell polynomials sequence $\{GP_n(x)\}$ can be written as

$$\{GP_n(x)\} = \{i, 1, 2x + i, 4x^2 + 1 + 2xi, 8x^3 + 4x + 4x^2 i + i, \cdots\}.$$

It is note that we have an important relation between Gaussian Pell polynomials and usual Pell polynomials as follows.

$$GP_{n+1}(x) = P_n(x) + iP_{n-1}(x).$$

And, if we write $x = 1$ in the equation $GP_{n+1}(x) = P_n(x)+iP_{n-1}(x)$, then we obtain $GP_n(1) = GP_n$. When we pay attention to this the generalized Gaussian Fibonacci sequence can be written as

$$GU_{n+1} = pGU_n + qGU_{n-1};\;\; GU_0 = a,\;\; GU_1 = b. \tag{2.1}$$

Let us note that writing $p = q = 1$, $a = i, b = 1$ in the equation (1) we have the Gaussian Fibonacci sequence, that is

$$\{GF_n\} = \{i, 1, 1 + i, 2 + i, 3 + 2i, \cdots\}.$$

Similarly, the values of equation (1) at $p = 2$, $q = 1$, $a = i$, $b = 1$, are just the values of Gaussian Pell sequence, that is

$$\{GP_n\} = \{i, 1, 2 + i, 5 + 2i, 12 + 5i, \cdots\}.$$

Likewise, writing $p = 2$, $q = 1$, $a = 2-2i$, $b = 2+2i$ in the equation (1), we have the Gaussian Pell-Lucas sequence, that is

$$\{GQ_n\} = \{2 - 2i, 2 + 2i, 6 + 2i, 14 + 6i, 34 + 14i, \cdots\}.$$

Let $\alpha(x)$ and $\beta(x)$ denote the roots of the characteristic equation of the recurrence relation $GP_{n+1}(x) = 2xGP_n(x) + GP_{n-1}(x)$. Then these roots are

$$\alpha(x) = x + \sqrt{1 + x^2},\;\; \beta(x) = x - \sqrt{1 + x^2}.$$

It is clear that the sum and product of these roots are $\alpha(x) + \beta(x) = 2x$ and $\alpha(x)\beta(x) = -1$.

Binet's formulas are well known in the theory of Fibonacci numbers. These formulas can also carried out for the Gaussian Pell polynomials. We obtain the following Binet's formula for Gaussian Pell polynomials in the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$GP_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i\frac{\alpha(x)\beta^n(x) - \beta(x)\alpha^n(x)}{\alpha(x) - \beta(x)}.$$

Proof. From the theory of difference equations we know that the general term of the Gaussian Pell polynomials may be expressed in the form

$$GP_n(x) = c\alpha^n(x) + d\beta^n(x),$$

for some coefficients c and d. Using the values $n = 0$ and $n = 1$ for c and d we obtain

$$c = \frac{1 - i\beta(x)}{\alpha(x) - \beta(x)},\;\; d = \frac{-1 + i\alpha(x)}{\alpha(x) - \beta(x)}$$

Considering the values c, d and making needed calculations, we get the following equation.

$$GP_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i\frac{\alpha(x)\beta^n(x) - \beta(x)\alpha^n(x)}{\alpha(x) - \beta(x)}$$

which is desired. □

The generating function $g(x,t)$ of the sequence $\{GP_n(x)\}$ is defined by

$$g(x,t) = \Sigma_0^\infty GP_n(x)t^n$$

which is a formal power series. For detailed knowledge on generating functions we refer to the books [13, 19].

Theorem 2.2. *The generating function for the Gaussian Pell polynomials is*

$$g(x,t) = \Sigma_0^\infty GP_n(x)t^n = \frac{t + (1 - 2xt)i}{1 - 2xt - t^2}.$$

Proof. Let $g(x,t)$ be the generating function of the sequence $\{GP_n(x)\}$. Then, one can write

$$g(x,t) = \Sigma_0^\infty GP_n(x)t^n = GP_0(x) + GP_1(x)t + GP_2(x)t^2 + \cdots$$

Using the recursive relation of this sequence we have

$$g(x,t)(1 - 2xt - t^2) = GP_0(x) + (GP_1(x) - 2xGP_0(x))t$$

If we make our computations, then the result as follows.

$$g(x,t) = \frac{t + (1 - 2xt)i}{1 - 2xt - t^2}.$$

Thus, the proof is completed. □

And now we define some special matrices which we need to prove our theorems.

$$Q(x) = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$P(x) = \begin{pmatrix} 2x + i & 1 \\ 1 & i \end{pmatrix}$$

Theorem 2.3. *For $n \geq 1$, we have*

$$Q^n(x)P(x) = \begin{pmatrix} GP_{n+2}(x) & GP_{n+1}(x) \\ GP_{n+1}(x) & GP_n(x) \end{pmatrix}$$

where $GP_n(x)$ is the nth Gaussian Pell Polynomial.

Proof. We can prove the theorem by the induction method on n. For $n = 1$, we have

$$\begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2x + i & 1 \\ 1 & i \end{pmatrix} = \begin{pmatrix} 4x^2 + 1 + 2xi & 2xi \\ 2xi & 1 \end{pmatrix} = \begin{pmatrix} GP_3(x) & GP_2(x) \\ GP_2(x) & GP_1(x) \end{pmatrix}.$$

Now, assume that the theorem holds for $n = k$, that is

$$\begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}^k\begin{pmatrix} 2x + i & 1 \\ 1 & i \end{pmatrix} = \begin{pmatrix} 4x^2 + 1 + 2xi & 2xi \\ 2xi & 1 \end{pmatrix} = \begin{pmatrix} GP_{k+2}(x) & GP_{k+1}(x) \\ GP_{k+1}(x) & GP_k(x) \end{pmatrix}.$$

Then, for $n = k+1$, we have

$$\begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} 2x+i & 1 \\ 1 & i \end{pmatrix} = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} GP_{k+2}(x) & GP_{k+1}(x) \\ GP_{k+1}(x) & GP_k(x) \end{pmatrix},$$
$$= \begin{pmatrix} GP_{k+3}(x) & GP_{k+2}(x) \\ GP_{k+2}(x) & GP_{k+1}(x) \end{pmatrix}.$$

Thus, we obtain the desired result. □

It is well known that the Cassini identity is one of the oldest identities involving the Fibonacci numbers [7, 12, 16]. In the following theorem, we give the Cassini formula for the Gaussian Pell polynomials.

Theorem 2.4. *For $n \geq 1$, we have*

$$GP_{n+1}(x)GP_{n-1}(x) - GP_n^2(x) = (-1)^n 2(1-ix).$$

Proof. This theorem can be proved by matrix method. It is clear that determinants of matrices $Q^{n-1}(x)$ and $P(x)$

$$det(Q^{n-1}(x)) = det \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix} = (-1)^{n-1},$$

$$det(P(x)) = det \begin{pmatrix} 2x+i & 1 \\ 1 & i \end{pmatrix} = -2(1-ix).$$

And from the previous theorem

$$Q^{n-1}(x)P(x) = \begin{pmatrix} GP_{n+1}(x) & GP_n(x) \\ GP_n(x) & GP_{n-1}(x) \end{pmatrix}$$

can be written. Thus, from these facts

$$GP_{n+1}(x)GP_{n-1}(x) - GP_n^2(x) = det(Q^{n-1}(x)P(x)) = (-1)^{n-1}(-2)(1-ix) = (-1)^n 2(1-ix).$$

can be written. □

Theorem 2.5. *For the Gaussian Pell Polynomials, $n \geq 1$, we have the following sum formula.*

$$\sum_{k=1}^{n} GP_k(x) = \frac{1}{2x}[GP_{n+1}(x) + GP_n(x) - (1+i)]$$

Proof. From the recursive relation,

$$GP_n(x) = \frac{1}{2x}[GP_{n+1}(x) - GP_{n-1}(x)],$$

and

$$GP_1(x) = \frac{1}{2x}[GP_2(x) - GP_0(x)],$$

$$GP_2(x) = \frac{1}{2x}[GP_3(x) - GP_1(x)],$$

$$GP_3(x) = \frac{1}{2x}[GP_4(x) - GP_2(x)],$$

$$\vdots$$

$$GP_{n-1}(x) = \frac{1}{2x}[GP_n(x) - GP_{n-2}(x)],$$

$$GP_n(x) = \frac{1}{2x}[GP_{n+1}(x) - GP_{n-1}(x)]$$

can be written. Hence, we get

$$\sum_{k=1}^{n} GP_k(x) = \frac{1}{2x}[GP_{n+1}(x) + GP_n(x) - (GP_1(x) + GP_0(x))]$$

$$= \frac{1}{2x}[GP_{n+1}(x) + GP_n(x) - (1+i)].$$

This completes the proof. □

Moreover, by the aid of the last theorem we can give the following corollary.

Corollary 2.6. *For $n \geq 1$, we have*

$$i)\ \sum_{k=0}^{n} GP_{2k}(x) = \frac{1}{2x}[GP_{2n+1}(x) - (1-2i)],$$

and

$$ii)\ \sum_{k=0}^{n} GP_{2k-1}(x) = \frac{1}{2x}[GP_{2n}(x) - i].$$

Proof. This corollary is easily obtained by proceeding as in the proof of Theorem 5. So, we can omit the detail. □

Theorem 2.7. *(Catalan Formulas) Let n, k be a nonzero positive integers. Then*

$$GP_{n+k}(x)GP_{n-k}(x) - GP_n^2(x) = (-1)^n(1-ix)[1 + \frac{(-1)^{k+1}[\alpha^k(x) + \beta^k(x)]^2}{4}]$$

Proof. This theorem can be easily proved by using the Binet formula. □

Finally, we can give the following identity without proof.

Theorem 2.8. *(d'Ocagne's Identity) For all $m, n \in \mathbb{Z}$, we have*

$$GP_{m+1}(x)GP_n(x) - GP_m(x)GP_{n+1}(x) = (-1)^{n+1}2(1-ix)P_{m-n}(x).$$

Conclusions. This study proposes introduce of the Gaussian Pell polynomials. In this study, we give the generating functions and the Binet formulas of Gaussian Pell polynomials. We also obtain some important identities involving the Gaussian Pell polynomials. In future we shall further develop some results concerning the terms of these polynomials.

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