SOME RESULTS ON $\lambda$ - SPIRALLIKE GENERALIZED SAKAGUCHI TYPE FUNCTIONS

Shilpa. N. and LATHA. S

Communicated by Ayman Badawi

MSC 2010 Classifications: 30C45.

Keywords and phrases: Univalent functions, $\lambda$-Spirallike functions, Convolution, Subordination.

Abstract. The aim of the present paper is to introduce a new subclass $\mathcal{L}_s(\alpha, \beta, \lambda, t)$ using Sakaguchi type functions and $\lambda$-Spirallike functions and to investigate Characterization and Subordination results for functions in this class. We discuss several consequences of our results.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{K}$ be the familiar class of functions that are convex in $\mathcal{U}$ and let $S^\lambda(\alpha)$ denote the class of $\lambda$-spirallike functions of order $\alpha$. A function $f(z) \in \mathcal{A}$ is said to be in the class $S^\lambda(\alpha)$ if

$$\Re \left\{ e^{i\lambda} \frac{zf''(z)}{f'(z)} \right\} > \alpha \cos \lambda, \quad (z \in \mathcal{U}, |\lambda| < \pi/2, 0 \leq \alpha < 1)$$

Note that $S^\lambda(0) = S^\lambda$ is the class $\lambda$-spirallike functions introduced by Spacek [10]. Further, a function $f(z) \in \mathcal{A}$ is said to be in the class $C^\lambda(\alpha)$ if

$$\Re \left\{ e^{i\lambda} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha \cos \lambda, \quad (z \in \mathcal{U}, |\lambda| < \pi/2, 0 \leq \alpha < 1)$$

Note that $C^\lambda(0) = C^\lambda$ is the class of functions for which $zf'(z)$ is $\lambda$-spirallike in $\mathcal{U}$ introduced by Robertson [7] and the class $C^\lambda(\alpha)$ was introduced and studied by Chichra [1]. A function $f(z) \in C^\lambda(\alpha)$ if and only if $zf'(z) \in S^\lambda(\alpha)$.

Now we introduce a new subclass $\mathcal{L}_s(\alpha, \beta, \lambda, t)$ defined using Sakaguchi type functions and $\lambda$-Spirallike functions as follows.

Definition 1.1. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{L}_s(\alpha, \beta, \lambda, t)$ if it satisfies

$$\Re \left\{ e^{i\lambda} \frac{(1-t)zf'(z) + \beta(1-t)z^2f''(z)}{(1-\beta)[f(z) - f(tz)] + \beta zf(z) - f(tz)} \right\} > \alpha \cos \lambda,$$

for some $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $t \neq 1$, $|t| \leq 1$, $|\lambda| < \pi/2$, and $z \in \mathcal{U}$.

By giving specific values to $\lambda, \beta, t$ we obtain the following subclasses studied by various researchers in earlier works

- For $\beta = 0$ and $\beta = 1$, we obtain the subclass $\mathcal{D}(\alpha, t)$ and $\mathcal{M}(\alpha, t)$ introduced and studied by Goyal and Goswami [3].
- For $\lambda = 0$, $\beta = 0$ and $\lambda = 0$, $\beta = 1$, we obtain the subclass $S(\alpha, t)$ and $T(\alpha, t)$ introduced and studied by Owa et. al. [5].
• For $\beta = 0$ and $t = 0$ we obtain the subclass $S^\beta_0(\lambda)$ studied in [6].
• For $\lambda = 0$ we obtain the subclass $L(\alpha, \beta, \gamma)$ studied in [9].

In our present investigation we need the following definitions and also a related result due to Wilf [11].

**Definition 1.2.** (Convolution) Given two functions $f$ and $g$ in the class $A$, where $f$ is given by (1.1) and $g$ is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product (or convolution) $f \ast g$ is defined by the power series 

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in U).$$

**Definition 1.3.** (Subordination Principle) For two functions $f$ and $g$ analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$ and write $f \prec g$, if there exists a Schwarz function $\omega$, which is analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, $z \in U$.

**Definition 1.4.** (Subordinating factor sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f$ of the form (1.1) is analytic, univalent and convex in $U$, we have the subordination given by

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \quad (z \in U, a_1 = 1).$$

**Lemma 1.5.** [11] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\Re\left\{1 + 2 \sum_{n=1}^{\infty} b_n z^n\right\} > 0, \quad (z \in U).$$

2 Main Results

In this section first we prove the Characterization results for the functions in the class $L(\alpha, \beta, \lambda, t)$.

**Theorem 2.1.** A function $f(z)$ of the form (1.1) is in the class $L(\alpha, \beta, \lambda, t)$ if

$$\left| \frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)|f(z) - f(tz)| + \beta z|f(z) - f(tz)|} - 1 \right| < 1 - \gamma$$

for some $0 \leq \gamma < 1, 0 \leq \beta < 1, |t| \leq 1, t \neq 1$, provided that

$$|\lambda| \leq \cos^{-1}\left(\frac{1-\gamma}{1-\alpha}\right)$$

where $0 \leq \gamma < 1, 0 \leq \beta < 1, |t| \leq 1, t \neq 1$.

**Proof.** Suppose that

$$\left| \frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)|f(z) - f(tz)| + \beta z|f(z) - f(tz)|} - 1 \right| < (1-\gamma)|\omega(z)|,$$

$|\omega(z)| < 1$ for all $z \in U$. Now

$$\Re\left\{e^{i\lambda} \left( \frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)|f(z) - f(tz)| + \beta z|f(z) - f(tz)|} \right)\right\} = \cos\lambda + (1-\gamma)\Re\{e^{i\lambda}\omega(z)\}$$

$$\geq \cos\lambda - (1-\gamma)|e^{i\lambda}\omega(z)|$$

$$\geq \cos\lambda - (1-\gamma) \geq a\cos\lambda$$

provided that $|\lambda| \leq \cos^{-1}\left(\frac{1-\gamma}{1-\alpha}\right)$.

This completes the proof. \(\square\)
Theorem 2.2. If

$$\left| \frac{(1 - t)z f'(z) + \beta(1 - t)z^2 f''(z)}{(1 - \beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]} - 1 \right| < (1 - \alpha)\cos \lambda$$  \hfill (2.3)

for some $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $t \neq 1$, $|t| \leq 1$, $|\lambda| < \pi /2$, and $z \in \mathcal{U}$. Then $f(z)$ belongs to the class $\mathcal{L}_s(\alpha, \beta, \lambda, t)$.

**Proof.** Set $\gamma = 1 - (1 - \alpha)\cos \lambda$, in the above Theorem. $\square$

Theorem 2.3. If the function $f(z) \in \mathcal{A}$, satisfies the inequality

$$\sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| \sec \lambda + (1 - \alpha) \left| u_n \right| \left| a_n \right| \right| \leq (1 - \alpha),$$  \hfill (2.4)

where $u_n = \sum_{k=0}^{n-1} t^k$, $(t \neq 1$, $|t| \leq 1$, $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $|\lambda| < \pi /2)$, then $f(z) \in \mathcal{L}_s(\alpha, \beta, \lambda, t)$.

**Proof.** By Theorem (2.2) it suffices to show that

$$\left| \frac{(1 - t)z f'(z) + \beta(1 - t)z^2 f''(z)}{(1 - \beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]} - 1 \right| < (1 - \alpha)\cos \lambda$$

Since

$$\left| \frac{(1 - t)z f'(z) + \beta(1 - t)z^2 f''(z)}{(1 - \beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]} - 1 \right| = \sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| a_n \right| z^n$$

$$= \sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| a_n \right| \left| z \right|^n$$

$$< \sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| a_n \right| \left| z \right|^{-n}$$

$$\sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| a_n \right|$$

$$< \sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| a_n \right|$$

The last expression is bounded above by $(1 - \alpha)\cos \lambda$, if

$$\sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| a_n \right| < (1 - \alpha)\cos \lambda$$

which is equivalent to

$$\sum_{n=2}^{\infty} \left| 1 + (n - 1)\beta \right| \left| n - u_n \right| \left| a_n \right| < (1 - \alpha)$$

$\square$

Remark: Suitable choices of $\lambda, \beta, t$ yield the characterization results derived in [3], [5], [6] and [9].
3 Subordination results

In this section we prove the Subordination results for the functions in the class \( \mathcal{L}_s(\alpha, \beta, \lambda, t) \).

**Theorem 3.1.** Let \( f \in \mathcal{A} \) satisfies the inequality (2.4) and suppose that \( g \in \mathcal{K} \). Then

\[
(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]
2[(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)](f \ast g)(z) < g(z)
\]  
(3.1)

\( z \in \mathcal{U}, \ |t| \leq 1, t \neq 1, 0 \leq \beta < 1, 0 \leq \alpha < 1, |\lambda| < \pi/2 \) and

\[
\Re\{f(z)\} > \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]}{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)}(z \in \mathcal{U}).
\]  
(3.2)

The constant factor 
\[
(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]
2[(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)]
\]  

in the subordination result (3.1) cannot be replaced by any larger one.

**Proof.** Let \( f \in \mathcal{A} \) satisfy the inequality (2.4) and suppose that

\( g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K} \). Then we have

\[
(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]
2[(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)](f \ast g)(z)
\]  
(3.3)

\[
= \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]}{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)}(z + \sum_{n=2}^{\infty} a_n c_n z^n)
\]

By definition (1.4) the subordination result (3.1) holds true if the sequence

\[
\left\{ \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]}{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)} a_n \right\}_{n=1}^{\infty}
\]  
(3.4)

is a subordinating factor sequence with \( a_1 = 1 \).

In view of lemma (1.5) it is enough to prove the inequality:

\[
\Re\left\{ 1 + \sum_{n=1}^{\infty} \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]}{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)} a_n z^n \right\} > 0, \quad (z \in \mathcal{U}).
\]  
(3.5)

Now,

\[
\Re\left\{ 1 + \sum_{n=1}^{\infty} \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]}{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)} a_n z^n \right\}
= \Re\left\{ 1 + \sum_{n=1}^{\infty} \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]}{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)} a_n z^n \right\}
\]

\[
\sum_{n=2}^{\infty} (1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] a_n z^n
\]
when \( |z| = r, (0 < r < 1) \),

\[
\geq 1 - \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|]}{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)} r^n
\]

\[
- \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)}{\sum_{n=2}^{\infty} (1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] a_n z^n}
\]

\[
> 1 - \frac{(1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] + (1 - \alpha)}{\sum_{n=2}^{\infty} (1 + \beta)[|1 - t|\sec\lambda + (1 - \alpha)|1 + t|] a_n z^n}
\]

\[
> 0
\]
Then (3.5) holds in \( \mathcal{U} \). This proves the inequality (3.1). The inequality (3.2) follows from (3.1), by taking the convex function \( g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \). To prove the sharpness of the constant
\[
(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t] \\
2[(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t] + (1 - \alpha)]
\]
we consider the function \( f_0(z) \in \mathcal{L}(\alpha, \beta, \lambda, t) \) given by
\[
f_0(z) = z - \frac{(1 - \alpha)\sec \lambda}{[1 + \beta][1 - t|\sec \lambda + (1 - \alpha)|1 + t]} z^2 \tag{3.6}
\]
From (3.1),
\[
\frac{(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t]}{2[(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t] + (1 - \alpha)]} f_0(z) \sim \frac{z}{1-z}, \quad (z \in \mathcal{U}) \tag{3.7}
\]
For the function \( f_0 \), it is easy to verify that
\[
\min \left\{ \left\{ \frac{(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t]}{2[(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t] + (1 - \alpha)]} f_0(z) \right\} \right\} = - \frac{1}{2}. \quad (|z| \leq 1)
\]
This shows that the constant \( \frac{(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t]}{2[(1 + \beta)[1 - t|\sec \lambda + (1 - \alpha)|1 + t] + (1 - \alpha)]} \) is the best possible, which completes the proof. \( \square \)

**Remark:** Suitable choices of \( \lambda, \beta, t \) yield the subordination results derived in [2], [3], [6] and [9].

### References


### Author Information

Shilpa. N., PG Department of Mathematics, JSS College of Arts Commerce and Science, Ooty Road, Mysuru - 570025, India. 
E-mail: drshilpamaths@gmail.com

LATHA. S, Department of Mathematics, Yuvanja’s College, University of Mysore, Mysuru-570005, India. 
E-mail: drlatha@gmail.com

Received: June 13, 2016.

Accepted: January 7, 2017.