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Explicit expression for a first integral for a class of planar differential system

Rachid Boukoucha

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Abstract. In this paper we are interested in studying the existence of a First integral and to the curves which are formed by the trajectories of the 2-dimensional differential systems of the form

$$\begin{cases} x' = \sqrt{P(x,y)} + x\frac{R(x,y)}{S(x,y)}, \\ y' = \sqrt{Q(x,y)} + y\frac{R(x,y)}{S(x,y)}, \end{cases}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree n, n, m, b respectively. Concrete example exhibiting the applicability of our result is introduced.

1 Introduction

We consider two-dimensional autonomous systems of differential equations of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = F(x,y), \\ y' = \dfrac{dy}{dt} = G(x,y). \end{cases} \tag{1.1}$$

where $F(x,y)$ and $G(x,y)$ are reals functions. There exist three main open problems in the qualitative theory of real planar differential systems see [1],[8],[9] and [17], the distinction between a center and a focus, the determination of the number of limit cycles and their distribution see [2] and [3], and the determination of its integrability.

System (1.1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to R$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1.1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x}F(x,y) + \frac{\partial H(x,y)}{\partial y}G(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant.

The importance for searching first integrals of a given system was already noted by Poincaré see [18] in his discussion on a method to obtain polynomial or rational first integrals. One of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals. J. Giné and J. Llibre see [12] and [13] characterized a large classes of polynomial differential systems in terms of the existence of first integrals. Llibre and al see [15] and [16] Zhang see [19] studied the exact upper bound of algebraic limit cycles of polynomial differential systems with the help of Darboux theory of integrability. For more details about first integral see for instance see [4],[5],[7],[10],[11] and [14].

It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait see [6].

In this paper we are intersted in studying the existence of a First integral and to the curves which are formed by the trajectories of the 2-dimensional differential systems of the form

$$\begin{cases} x' = \sqrt{P(x,y)} + x\frac{R(x,y)}{S(x,y)}, \\ y' = \sqrt{Q(x,y)} + y\frac{R(x,y)}{S(x,y)}, \end{cases} \tag{1.2}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree n, n, m, b respectively

We define the trigonometric functions

$f_1(\theta) = \sqrt{P(\cos\theta, \sin\theta)}\cos\theta + \sqrt{Q(\cos\theta, \sin\theta)}\sin\theta$, $f_2(\theta) = \frac{R(\cos\theta, \sin\theta)}{S(\cos\theta, \sin\theta)}$, $f_3(\theta) = \sqrt{Q(\cos\theta, \sin\theta)}\cos\theta - P(\cos\theta, \sin\theta)\sin\theta$.

1.1 Main result

Our main result on the existence of a First integral and the curves which are formed by the trajectories of the 2-dimensional differential systems (1.2) is the following.

Theorem 1.1. *Consider a system (1.2), then the following statements hold.*

(a) If $f_3(\theta) \neq 0$, $\lambda \neq 0$, $P(\cos\theta, \sin\theta) \geq 0$, $Q(\cos\theta, \sin\theta) \geq 0$ and $S(\cos\theta, \sin\theta) \neq 0$, then system (1.2) has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{\lambda}{2}} \exp\left(-\lambda\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) - \\ &\quad \lambda\int^{\arctan\frac{y}{x}} \exp\left(-\lambda\int^{w} A(\omega)\,d\omega\right) B(w)\,dw, \end{aligned}$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$ and $\lambda = \frac{n}{2} - m + b - 1$.

Moreover the curves which are formed by the trajectories of the differential system (1.2), are written in Cartesian coordinates as

$$x^2+y^2 = \left(\begin{array}{c} h\exp\left(\lambda\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ \lambda\exp\left(\lambda\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right)\int^{\arctan\frac{y}{x}} \exp\left(-\lambda\int^{w} A(\omega)\,d\omega\right) B(w)\,dw \end{array}\right)^{\frac{2}{\lambda}}$$

where $h \in \mathbb{R}$.

(b) If $f_3(\theta) \neq 0$, $\lambda = 0$, $P(\cos\theta, \sin\theta) \geq 0$, $Q(\cos\theta, \sin\theta) \geq 0$ and $S(\cos\theta, \sin\theta) \neq 0$, then system (1.2) has the first integral

$$H(x,y) = \left(x^2+y^2\right)\exp\left(-\int^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right)d\omega\right).$$

Moreover the curves which are formed by the trajectories of the differential system (1.2), are written in Cartesian coordinates as

$$x^2+y^2 = h\exp\left(\int^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right)d\omega\right)$$

where $h \in \mathbb{R}$.

(c) If $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then system (1.2) has the first integral $H = \frac{y}{x}$.

Moreover the curves which are formed by the trajectories of the differential system (1.2), are written in Cartesian coordinates as $y = hx$ where $h \in \mathbb{R}$.

Proof. In order to prove our results we write the polynomial differential system (1.2) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then system (1.2) becomes

$$\begin{cases} r' = f_1(\theta)\,r^{\frac{n}{2}} + f_2(\theta)\,r^{m-b+1}, \\ \theta' = f_3(\theta)\,r^{\frac{n-2}{2}}, \end{cases} \tag{1.3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction, $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$

If $f_3(\theta) \neq 0$, $\lambda \neq 0$, $P(\cos\theta, \sin\theta) \geq 0$, $Q(\cos\theta, \sin\theta) \geq 0$ and $S(\cos\theta, \sin\theta) \neq 0$.

Taking as new independent variable the coordinate θ, this differential system (1.3) writes

$$\frac{dr}{d\theta} = A(\theta) r + B(\theta) r^{1-\lambda}, \tag{1.4}$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$ and $\lambda = \frac{n}{2} - m + b - 1$, which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{\lambda}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = \lambda\left(A(\theta)\rho + B(\theta)\right). \tag{1.5}$$

The general solution of linear equation (1.5) is

$$\rho(\theta) = \exp\left(\lambda \int^{\theta} A(\omega)\, d\omega\right)\left(\alpha + \lambda \int^{\theta} \exp\left(-\lambda \int^{w} A(\omega)\, d\omega\right) B(w)\, dw\right),$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{\lambda}{2}} \exp\left(-\lambda \int^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) - \\ &\quad \lambda \int^{\arctan\frac{y}{x}} \exp\left(-\lambda \int^{w} A(\omega)\, d\omega\right) B(w)\, dw. \end{aligned}$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (1.2), are written in Cartesian coordinates as

$$x^2 + y^2 = \left(\begin{array}{c} h \exp\left(\lambda \int^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) + \\ \lambda \exp\left(\lambda \int^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) \int^{\arctan\frac{y}{x}} \exp\left(-\lambda \int^{w} A(\omega)\, d\omega\right) B(w)\, dw \end{array}\right)^{\frac{2}{\lambda}}$$

where $h \in \mathbb{R}$.

Hence statement (a) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$, $\lambda = 0$, $P(\cos\theta, \sin\theta) \geq 0$, $Q(\cos\theta, \sin\theta) \geq 0$ and $S(\cos\theta, \sin\theta) \neq 0$.

Taking as new independent variable the coordinate θ, this differential system (1.3) writes

$$\frac{dr}{d\theta} = \left(A(\theta) + B(\theta)\right) r, \tag{1.6}$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$.

The general solution of equation (1.6) is

$$r(\theta) = \alpha \exp\left(\int^{\theta} \left(A(\omega) + B(\omega)\right) d\omega\right),$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int^{\arctan\frac{y}{x}} \left(A(\omega) + B(\omega)\right) d\omega\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (1.2), are written in Cartesian coordinates as

$$x^2 + y^2 = h \exp\left(\int^{\arctan\frac{y}{x}} \left(A(\omega) + B(\omega)\right) d\omega\right)$$

where $h \in \mathbb{R}$.

Hence statement (b) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then from (1.3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (1.2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system.

The curves $H = h$ with $h \in \mathbb{R}$

, which are formed by trajectories of the differential system (1.2), are written in Cartesian coordinates as $y = hx$ where $h \in \mathbb{R}$

This completes the proof of statement (c) of Theorem 1. □

2 Examples

The following example are given to illustrate our result

Example 1 If we take $P(x, y) = x^4y^4 - 4x^5y^3 + 4x^6y^2$, $Q(x, y) = 4x^4y^4 + 4x^5y^3 + x^6y^2$, $R(x, y) = x^2 - xy$ and $S(x, y) = x + y$, then system (1.2) reads

$$\begin{cases} x' = \sqrt{x^4y^4 - 4x^5y^3 + 4x^6y^2} + x\frac{x^2-xy}{x+y}, \\ y' = \sqrt{4x^4y^4 + 4x^5y^3 + x^6y^2} + y\frac{x^2-xy}{x+y}, \end{cases} \tag{2.1}$$

the planar system (2.1) in Polar coordinates (r, θ) becomes

$$\begin{cases} r' = (\sin 2\theta)\, r^3 + \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} r, \\ \theta' = \frac{1}{2}(\sin 2\theta)\, r^2, \end{cases}$$

here $f_1(\theta) = \sin 2\theta$, $f_2(\theta) = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$ and $f_3(\theta) = \frac{1}{2}\sin 2\theta$, then the planar system (2.1) has the first integral

$$\begin{aligned} H(x, y) &= \left(x^2 + y^2\right)\exp\left(-4\arctan\frac{y}{x}\right) - \\ &\int^{\arctan\frac{y}{x}} \exp(-4w)\left(\frac{4\cos w - 4\sin w}{(\cos w + \sin w)\sin 2w}\right) dw. \end{aligned}$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2.1), in Cartesian coordinates are written as

$$\begin{aligned} x^2 + y^2 &= h\exp\left(4\arctan\frac{y}{x}\right) + \\ &\exp\left(4\arctan\frac{y}{x}\right)\int^{\arctan\frac{y}{x}} \exp(-4w)\left(\frac{4\cos w - 4\sin w}{(\cos w + \sin w)\sin 2w}\right) dw, \end{aligned}$$

where $h \in \mathbb{R}$

3 Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar differential systems of ODEs in order to obtain explicit expression for a first integral and characterizes its trajectories, this is a one of the classical tools in the classification of all trajectories of dynamical systems. In short our methods can be used to obtain rich integrable planar differential systems, and to get the explicit expression for a first integral and characterizes its trajectories of these systems.

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Author information

Rachid Boukoucha, Department of Technology, University of Bejaia, 06000 Bejaia, Algeria.
E-mail: `rachid_boukecha@yahoo.fr`

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