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CHARACTERIZATIONS OF ORDERED SEMIGROUPS BY THE PROPERTIES OF THEIR ORDERED (m, n) QUASI-IDEALS

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Abstract The aim of this paper is to study the concept of ordered (m, n) quasi-ideals in ordered semigroups that are studied analogously to the concept of (m, n) quasi-ideals in semigroups considered by Ansari, Khan and Kaushik in 2009. Regular ordered semigroups are characterized by their ordered (m, n) quasi-ideals and the fact that every ordered (m, n) quasi-ideal of a regular ordered semigroup has the ordered (m, n) intersection property, i.e., the intersection of an ordered m left ideal and an ordered n right ideal of an ordered semigroup.

1 Introduction and Preliminaries

The study of ordered semigroups began about 1950 by several authors, for example, Alimov [1], Chehata [5] and Vinogradov [22]. The theory of different types of ideals in semigroups and in ordered semigroups was studied by several researches such as: In 1956, Steinfeld [18] introduced the notion of quasi-ideals in semigroups. Steinfeld [19] gave some characterization of 0-minimal quasi-ideals in semigroups. In 1963, Saitô [17] gave a catalog of all possible types of subsemigroups generated by regular pairs of ordered semigroups. In 1998, Kehayopulu [8] gave some characterization of quasi-ideals and bi-ideals in completely regular ordered semigroups. Kehayopulu [7] gave some characterization of quasi-ideals in strongly regular ordered semigroups. In 2002, Cao [3] gave some characterization of quasi-ideals in regular ordered semigroups. Kehayopulu, Ponizovskii and Tsingelis [9] studied bi-ideals in ordered semigroups and ordered groups. In 2003, Kehayopulu, Ponizovskii and Tsingelis [10] proved that in commutative ordered semigroups with identity each maximal ideal is a prime ideal, the converse statement does not hold, in general. In 2006, Lee and Lee [15] gave some characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups. In 2008, Iampan [6] studied the concept of (0-)minimal and maximal ordered quasi-ideals in ordered semigroups. In 2009, Kim [14] introduced and characterized the notion of intuitionistic fuzzy semiprime ideals in ordered semigroups. Ansari, Khan and Kaushik [2] characterized the notion of (m, n) quasi-ideals in semigroups. In 2010, Khan, Khan and Hussain [12] characterized regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of intuitionistic fuzzy left (resp. right) ideals. Tang and Xie [21] characterized ordered semigroups in which the radical of every ideal (right ideal, bi-ideal) is an ordered subsemigroup (resp., ideal, right ideal, left ideal, bi-ideal, interior ideal) by using some binary relations on an ordered semigroup. Xie and Tang [23] introduced the concept of fuzzy generalized bi-ideals of ordered semigroups and characterized fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals in regular ordered semigroups. In 2011, Zeb and Khan [24] introduced the concept of anti-fuzzy quasi-ideals in ordered semigroups and investigate the quasi-ideals of ordered semigroups in terms of anti-fuzzy quasi-ideals and characterized left (resp. right) regular and completely regular ordered semigroups in terms of anti-fuzzy quasi-ideals and semiprime anti-fuzzy quasi-ideals. In 2012, Mohanraj, Krisnaswamy and Hema [16] introduced and characterized the notions of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals, $(\in, \in \vee q)$-

antifuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-antifuzzy bi-ideals of an ordered semigroup. Tang and Xie [20] characterized fuzzy quasi-ideals of ordered semigroups, and introduced the notion of completely semiprime fuzzy quasi-ideals of ordered semigroups and characterized strongly regular ordered semigroups in terms of completely semiprime fuzzy quasi-ideals. Khan, Sarmin, Khan and Faizullah [13] introduced and characterized the concept of $(\in, \in \vee q_k)$-fuzzy quasi-ideals in ordered semigroups. In 2013, Changphas [4] characterized 0-minimal (m, n)-ideals in ordered semigroups.

The notion of quasi-ideals (some authors called an *ordered quasi-ideal*) play an important role in studying the structure of ordered semigroups. Now, we know that the notion of ordered (m, n) quasi-ideals is a generalization of ordered quasi-ideals in ordered semigroups. The main purpose of this paper is to investigate some properties of ordered (m, n) quasi-ideals of ordered semigroups which extends the results of Ansari, Khan and Kaushik [2].

Before going to prove the main results we need the following definitions that we use later.

An *ordered semigroup* (some authors called a *po-semigroup*) $(S, \cdot, \leq)$ is a poset $(S, \leq)$ at the same time a semigroup $(S, \cdot)$ such that: for any $a, b \in S$,

$$a \leq b \text{ implies } ac \leq bc \text{ and } ca \leq cb \text{ for all } c \in S.$$

If $(S, \cdot, \leq)$ is an ordered semigroup and A is a subsemigroup of S, then $(A, \cdot, \leq)$ is an ordered semigroup. For convenience, we simply write S instead of $(S, \cdot, \leq)$. Let now S be an ordered semigroup. For a subset H of S, we denote

$$(H] = \{s \in S \mid s \leq h \text{ for some } h \in H\}.$$

For nonempty subsets A and B of S, we denote

$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

Then, for nonempty subsets A, B and C of S. We have that (i) $A(B \cap C) \subseteq AB \cap AC$, and (ii) $A(B \cup C) = AB \cup AC$. A nonempty subset A of S is called an *ordered left ideal* of S if

(i) $SA \subseteq A$, and

(ii) $(A] \subseteq A$

an *ordered right ideal* of S if

(i) $AS \subseteq A$, and

(ii) $(A] \subseteq A$

an *ordered ideal* of S if A is both an ordered left ideal and an ordered right ideal of S. That is,

(i) $SA \subseteq A$ and $AS \subseteq A$, and

(ii) $(A] \subseteq A$.

A subsemigroup B of S is called an *ordered quasi-ideal* of S if

(i) $(SB] \cap (BS] \subseteq B$, and

(ii) $(B] \subseteq B$

an *ordered m left ideal* of S if

(i) $S^m B \subseteq B$, and

(ii) $(B] \subseteq B$

an *ordered n right ideal* of S if

(i) $BS^n \subseteq B$, and

(ii) $(B] \subseteq B$

an *ordered (m, n) quasi-ideal* of S if

(i) $(S^m B] \cap (BS^n] \subseteq B$, and

(ii) $(B] \subseteq B$.

We have the following lemma.

Lemma 1.1. *[11] Let S be an ordered semigroup, and A and B subsets of S. Then the following statements hold.*

(i) $A \subseteq (A]$.

(ii) $((A]] = (A]$.

(iii) If $A \subseteq B$, then $(A] \subseteq (B]$.

(iv) $(A \cap B] \subseteq (A] \cap (B]$.

(v) $(A \cup B] = (A] \cup (B]$.

(vi) $(A](B] \subseteq (AB]$.

(vii) $((A](B]] = (AB]$.

The following two lemmas are easy to verify, the proof will be omitted.

Lemma 1.2. *Let S be an ordered semigroup and $\{A_i \mid i \in I\}$ a nonempty family of subsemigroups of S. Then $\bigcap_{i \in I} A_i = \emptyset$ or $\bigcap_{i \in i} A_i$ is a subsemigroup of S.*

Lemma 1.3. *Let S be an ordered semigroup and A a subsemigroup of S. Then $A^n \subseteq A$ for all positive integer n.*

Proposition 1.4. *Let S be an ordered semigroup, Q an ordered (m, n) quasi-ideal of S and A a subsemigroup of S. Then $A \cap Q = \emptyset$ or $A \cap Q$ is an ordered (m, n) quasi-ideal of A.*

Proof. Suppose that $A \cap Q \neq \emptyset$. Since Q and A are subsemigroups of S, we have $A \cap Q$ is a subsemigroup of S. Since $A \cap Q \subseteq A$, we have $A \cap Q$ is a subsemigroup of A . Thus

$$\begin{aligned}(A^m(A \cap Q)] \cap ((A \cap Q)A^n] \cap A &\subseteq A \cap (A^m Q] \cap (QA^n] \\ &\subseteq A \cap (S^m Q] \cap (QS^n] \\ &\subseteq A \cap Q\end{aligned}$$

and

$$\begin{aligned}(A \cap Q] \cap A &\subseteq A \cap (A] \cap (Q] \\ &\subseteq A \cap (Q] \\ &= A \cap Q.\end{aligned}$$

Therefore, $A \cap Q$ is an ordered (m, n) quasi-ideal of A. □

Proposition 1.5. *Let S be an ordered semigroup and $\{Q_i \mid i \in I\}$ a nonempty family of ordered (m, n) quasi-ideals of S. Then $\bigcap_{i \in I} Q_i = \emptyset$ or $\bigcap_{i \in i} Q_i$ is an ordered (m, n) quasi-ideal of S.*

Proof. Suppose that $\bigcap_{i \in I} Q_i \neq \emptyset$. By Lemma 1.2, we have $\bigcap_{i \in I} Q_i$ is a subsemigroup of S. For all $i \in I$, we have

$$(S^m(\textstyle\bigcap_{i \in I} Q_i)] \cap ((\textstyle\bigcap_{i \in I} Q_i)S^n] \subseteq (s^m Q_i] \cap (Q_i S^n] \subseteq Q_i.$$

Thus $(S^m(\bigcap_{i \in I} Q_i)] \cap ((\bigcap_{i \in I} Q_i)S^n] \subseteq \bigcap_{i \in I} Q_i$ and $(\bigcap_{i \in I} Q_i] \subseteq \bigcap_{i \in I}(Q_i] = \bigcap_{i \in I} Q_i$. Therefore, $\bigcap_{i \in I} Q_i$ is an ordered (m, n) quasi-ideal of S. □

2 Ordered (m,n) Quasi-Ideals and Ordered (m,n) Intersection Property

In this section, we characterize ordered m left ideals and ordered n right ideals in ordered semigroups and investigate the ordered (m,n) intersection property of ordered (m,n) quasi-ideals in ordered semigroups.

Theorem 2.1. *Let S be an ordered semigroup. Then the following statements hold.*

(i) If $\{A_i \mid i \in I\}$ is a nonempty family of ordered m left ideals of S, then $\bigcap_{i\in I} A_i = \emptyset$ or $\bigcap_{i\in I} A_i$ is an ordered m left ideal of S.

(ii) If $\{B_i \mid i \in I\}$ is a nonempty family of ordered n right ideals of S, then $\bigcap_{i\in I} B_i = \emptyset$ or $\bigcap_{i\in I} B_i$ is an ordered n right ideal of S.

Proof. (i) Assume that $\{A_i \mid i \in I\}$ is a nonempty family of ordered m left ideals of S and let $\bigcap_{i\in I} A_i \neq \emptyset$. By Lemma 1.2, we have $\bigcap_{i\in I} A_i$ is a subsemigroup of S. For all $i \in I$, we have $S^m(\bigcap_{i\in I} A_i) \subseteq S^m A_i \subseteq A_i$. Thus $S^m(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} A_i$ and $(\bigcap_{i\in I} A_i] \subseteq \bigcap_{i\in I}(A_i] = \bigcap_{i\in I} A_i$. Therefore, $\bigcap_{i\in I} A_i$ is an ordered m left ideal of S.
(ii) In a similar way, we can prove that $\bigcap_{i\in I} B_i$ is an ordered n right ideal of S. □

Lemma 2.2. *Let S be an ordered semigroup and Q a nonempty subset of S. Then the following statements hold.*

(i) $(S^m Q]$ is an ordered m left ideal of S.

(ii) $(QS^n]$ is an ordered n right ideal of S.

Proof. (i) By Lemma 1.3, we have that

$$\begin{aligned}(S^mQ](S^mQ] &\subseteq ((S^mQ)(S^mQ)]\\ &\subseteq ((S^mS)S^mQ]\\ &\subseteq (S(SS^{m-1}Q)]\\ &= ((SS)(S^{m-1}Q)]\\ &\subseteq (S(S^{m-1}Q)]\\ &= ((SS^{m-1})Q]\\ &= (S^mQ].\end{aligned}$$

Thus $(S^mQ]$ is a subsemigroup of S. We see that

$$\begin{aligned}S^m(S^mQ] &\subseteq S(SS^{m-1}Q]\\ &= (S](SS^{m-1}Q]\\ &\subseteq (S(SS^{m-1}Q)]\\ &= ((SS)(S^{m-1}Q)]\\ &\subseteq (S(S^{m-1}Q)]\\ &= ((SS^{m-1})Q]\\ &= (S^mQ]\end{aligned}$$

and $((S^mQ]] = (S^mQ]$. Therefore, $(S^mQ]$ is an ordered m left ideal of S.
(ii) In a similar way, we can prove that $(QS^n]$ is an ordered n right ideal of S. □

Lemma 2.3. *Let S be an ordered semigroup. Then the following statements hold.*

(i) Every ordered m left ideal is an ordered (m,n) quasi-ideal of S for all positive integer n.

(ii) Every ordered n right ideal is an ordered (m,n) quasi-ideal of S for all positive integer m.

Proof. (i) Suppose that A is an ordered m left ideal of S and let n be a positive integer. Then A is a subsemigroup of S. Thus $(S^mA] \cap (AS^n] \subseteq (S^mA] \subseteq (A] \subseteq A$ and $(A] \subseteq A$. Therefore, A is an ordered (m,n) quasi-ideal of S for all positive integer n.
(ii) In a similar way, we can prove that every ordered n right ideal is an ordered (m,n) quasi-ideal of S for all positive integer m. □

Theorem 2.4. *Let S be an ordered semigroup, and A an ordered m left ideal and B an ordered n right ideal of S. Then $A \cap B = \emptyset$ or $A \cap B$ is an ordered (m, n) quasi-ideal of S.*

Proof. Suppose that $A \cap B \neq \emptyset$. Then $A \cap B$ is a subsemigroup of S. Thus

$$\begin{aligned} (S^m(A \cap B)] \cap ((A \cap B)S^n] &\subseteq (S^m A] \cap (BS^n] \\ &\subseteq (A] \cap (B] \\ &= A \cap B \end{aligned}$$

and $(A \cap B] \subseteq (A] \cap (B] = A \cap B$. Hence, $A \cap B$ is an ordered (m, n) quasi-ideal of S. □

Definition 2.5. A subsemigroup Q of an ordered semigroup S has the *ordered* (m, n) *intersection property* if Q is the intersection of an ordered m left ideal and an ordered n right ideal of S.

Theorem 2.6. *Let S be an ordered semigroup and Q an ordered (m, n) quasi-ideal of S. Then the following statements are equivalent.*

(i) Q *has the ordered* (m, n) *intersection property.*

(ii) $(Q \cup S^m Q] \cap (Q \cup QS^n] = Q$.

(iii) $(S^m Q] \cap (Q \cup QS^n] \subseteq Q$.

(iv) $(Q \cup S^m Q] \cap (QS^n] \subseteq Q$.

Proof. (i)⇒(ii) Assume that Q has the ordered (m, n) intersection property. Since $Q \subseteq Q \cup (S^m Q] = (Q] \cup (S^m Q] = (Q \cup S^m Q]$ and $Q \subseteq Q \cup (QS^n] = (Q] \cup (QS^n] = (Q \cup QS^n]$, we have $Q \subseteq (Q \cup S^m Q] \cap (Q \cup QS^n]$. Since Q has the ordered (m, n) intersection property, there exist an ordered m left ideal A and an ordered n right ideal B of S such that $Q = A \cap B$. Thus $Q \subseteq A$ and $Q \subseteq B$, so $(S^m Q] \subseteq (S^m A] \subseteq (A] = A$ and $(QS^n] \subseteq (BS^n] \subseteq (B] = B$. Thus $(Q \cup S^m Q] = (Q] \cup (S^m Q] = Q \cup (S^m Q] \subseteq A$ and $(Q \cup QS^n] = (Q] \cup (QS^n] = Q \cup (QS^n] \subseteq B$. Hence, $(Q \cup S^m Q] \cap (Q \cup QS^n] \subseteq A \cap B = Q$. Therefore, $(Q \cup S^m Q] \cap (Q \cup QS^n] = Q$.
(ii)⇒(i) Assume that $(Q \cup S^m Q] \cap (Q \cup QS^n] = Q$. We shall show that $(Q \cup S^m Q]$ is an ordered m left ideal and $(Q \cup QS^n]$ an ordered n right ideal of S. By Lemma 2.2, we have $(S^m Q]$ is an ordered m left ideal and $(QS^n]$ an ordered n right ideal of S and so $(S^m Q]$ and $(QS^n]$ are subsemigroups of S. We see that

$$\begin{aligned} (Q \cup S^m Q](Q \cup S^m Q] &= (Q \cup (S^m Q])(Q \cup (S^m Q]) \\ &= QQ \cup (S^m Q]Q \cup Q(S^m Q] \cup (S^m Q](S^m Q] \\ &= QQ \cup (S^m Q](Q] \cup (S](S^m Q] \cup (S^m Q](S^m Q] \\ &\subseteq QQ \cup (S^m QQ] \cup (SS^m Q] \cup (S^m QS^m Q] \\ &\subseteq Q \cup (S^m Q] \cup (S^m Q] \cup (S^m Q] \\ &= Q \cup (S^m Q] \\ &= (Q \cup S^m Q]. \end{aligned}$$

Thus $(Q \cup S^m Q]$ is a subsemigroup of S. Now,

$$\begin{aligned} S^m(Q \cup S^m Q] &= S^m(Q \cup (S^m Q]) \\ &= S^m Q \cup S^m(S^m Q] \\ &\subseteq S^m Q \cup (S^m Q] \quad \text{(by Lemma 2.2)} \\ &= (S^m Q] \\ &\subseteq (Q] \cup (S^m Q] \\ &= (Q \cup S^m Q] \end{aligned}$$

and $((Q \cup S^m Q]] = (Q \cup S^m Q]$. Hence, $(Q \cup S^m Q]$ is an ordered m left ideal of S. In a similar way, we can prove that $(Q \cup QS^n]$ is an ordered n right ideal of S. Therefore, Q has the ordered (m, n) intersection property.

(ii)⇒(iii) Assume that $(Q\cup S^mQ]\cap(Q\cup QS^n]=Q$. Since $(S^mQ]\subseteq(Q]\cup(S^mQ]=(Q\cup S^mQ]$, we have $(S^mQ]\cap(Q\cup QS^n]\subseteq(Q\cup S^mQ]\cap(Q\cup QS^n]=Q$. Hence, $(S^mQ]\cap(Q\cup QS^n]\subseteq Q$.
(iii)⇒(ii) Assume that $(S^mQ]\cap(Q\cup QS^n]\subseteq Q$. Since $Q\subseteq Q\cup(S^mQ]=(Q\cup S^mQ]$ and $Q\subseteq Q\cup(QS^n]=(Q\cup S^mQ]$, we have $Q\subseteq(Q\cup S^mQ]\cap(Q\cap QS^n]$. Now,

$$\begin{aligned}(Q\cup S^mQ]\cap(Q\cup QS^n] &= (Q\cup(S^mQ])\cap(Q\cup(QS^n])\\ &= (Q\cap(Q\cup(QS^n]))\cup((S^mQ]\cap(Q\cup QS^n])\\ &\subseteq Q\cup Q\\ &= Q.\end{aligned}$$

Therefore, $(Q\cup S^mQ]\cap(Q\cup QS^n]=Q$
(ii)⇒(iv) The proof is almost similar to the proof of (ii)⇒(iii).
(iv)⇒(ii) The proof is almost similar to the proof of (iii)⇒(ii). □

Lemma 2.7. *Every ordered m left ideal and ordered n right ideal of an ordered semigroup have the ordered (m,n) intersection property.*

Proof. Let A be an ordered m left ideal and B an ordered n right ideal of an ordered semigroup S. By Lemma 2.3, we have that A is an ordered (m,n) quasi-ideal of S. Now,

$$\begin{aligned}(S^mA]\cap(A\cup AS^n] &= (S^mA]\cap(A\cup(AS^n])\\ &= ((S^mA]\cap A)\cup((S^mA]\cap(AS^n])\\ &\subseteq A\cup A\\ &= A.\end{aligned}$$

By Theorem 2.6, we have that A has the ordered (m,n) intersection property. Similarly, we can prove that B has the ordered (m,n) intersection property. □

Proposition 2.8. *Let S be an ordered semigroup and Q an ordered (m,n) quasi-ideal of S. If $S^mQ\subseteq QS^n$ or $QS^n\subseteq S^mQ$, then Q has the ordered (m,n) intersection property.*

Proof. Assume that $S^mQ\subseteq QS^n$. Then $(S^mQ]\subseteq(QS^n]$. Since Q is an ordered (m,n) quasi-ideal of S, we have $S^mQ\subseteq(S^mQ]=(S^mQ]\cap(QS^n]\subseteq Q$. Thus Q is an ordered m left ideal of S. By Lemma 2.7, we have that Q has the ordered (m,n) intersection property. Similarly, we can prove that Q has the ordered (m,n) intersection property. □

3 Ordered (m,n) Quasi-Ideals in Regular Ordered Semigroups

We have investigated in the previous section that every ordered m left ideal and ordered n right ideal of an ordered semigroup have the ordered (m,n) intersection property, but not for ordered (m,n) quasi-ideals in ordered semigroups. In this section, we will prove that every ordered (m,n) quasi-ideal of a regular ordered semigroup has the ordered (m,n) intersection property.

Definition 3.1. An ordered semigroup S is called *regular* if for any $x\in S$ there exists $y\in S$ such that $x\leq xyx$.

Lemma 3.2. *Let S be a regular ordered semigroup and A a nonempty subset of S. Then the following statements hold.*

(i) $A\subseteq(S^mA]$ for all positive integer m.

(ii) $A\subseteq(AS^n]$ for all positive integer n.

Proof. (i) Let $x\in A$. Since S is regular, there exists $y\in S$ such that $x\leq xyx$. Since $xy\in S$, we have $x\leq xyx=(xy)x\in SA$ and so $A\subseteq(SA]$. Let m be a positive integer such that $A\subseteq(S^mA]$. Then $SA\subseteq S(S^mA]=(S](S^mA]\subseteq(S(S^mA)]=(S^{m+1}A]$. Therefore, $A\subseteq(S^mA]$ for all positive integer m.
(ii) In a similar way, we can prove that $A\subseteq(AS^n]$ for all positive integer n. □

Theorem 3.3. *Every ordered (m, n) quasi-ideal of a regular ordered semigroup has the ordered (m, n) intersection property.*

Proof. Let Q be an ordered (m, n) quasi-ideal of a regular ordered semigroup S. By Lemma 3.2, we have $Q \subseteq (QS^n]$ and so $(Q \cup QS^n] = Q \cup (QS^n] = (QS^n]$. Thus $(S^m Q] \cap (Q \cup QS^n] = (S^m Q] \cap (QS^n] \subseteq Q$. By Theorem 2.6, we have that Q has the ordered (m, n) intersection property. □

Theorem 3.4. *Let S be a regular ordered semigroup and A a nonempty subset of S. Then A is an ordered (m, n) quasi-ideal of S if and only if $A = (S^m A] \cap (AS^n]$.*

Proof. Assume that A is an ordered (m, n) quasi-ideal of S. Then $(S^m A] \cap (AS^n] \subseteq A$. By Lemma 3.2, we have $A \subseteq (S^m A]$ and $A \subseteq (AS^n]$ and so $A \subseteq (S^m A] \cap (AS^n]$. Therefore, $A = (S^m A] \cap (AS^n]$.

Conversely, assume that $A = (S^m A] \cap (AS^n]$. By Lemma 2.2, we have $(S^m A]$ is an ordered m left ideal and $(AS^n]$ an ordered n right ideal of S. By Theorem 2.4, we have that A is an ordered (m, n) quasi-ideal of S. □

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