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A NOTE ON SOME GROWTH PROPERTIES OF ENTIRE FUNCTIONS USING THEIR GENERALIZED RELATIVE L^*-ORDERS

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Abstract In this paper we investigate some growth properties of entire functions on the basis of generalized relative order (generalized relative lower order) as well as generalized relative L^*-order (generalized relative L^*-lower order).

1 Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. For entire $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$, the maximum modulus symbolized as $M_f(r)$ is defined as $\max\limits_{|z|=r} |f(z)|$. If f is non-constant entire then $M_f(r)$ is strictly increasing and continuous and therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$.Moreover for another entire function g, $M_g(r)$ is too defined and the ratio $\frac{M_f(r)}{M_g(r)}$ when $r \to \infty$ is called the comparative growth of f with respect to g in terms of their maximum moduli.

The order ρ_f of an entire function f which is classical in complex analysis is defined in the following way:

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} .$$

An entire function for which *order* and *lower order* are the same is said to be of *regular growth*. Functions which are not of *regular growth* are said to be of *irregular growth*.

In this connection let us recall that Sato [4] defined the *generalized order* and *generalized lower order* of an entire function f, respectively, as follows:

$$\rho_f^{[k]} = \limsup_{r\to\infty} \frac{\log^{[k]} M_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[k]} = \liminf_{r\to\infty} \frac{\log^{[k]} M_f(r)}{\log r} \right)$$

where k is any positive integer and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$. These definitions extended the order ρ_f and lower order λ_f of an entire function f since these correspond to the particular cases $\rho_f^{[2]} = \rho_f$ and $\lambda_f^{[2]} = \lambda_f$.

An entire function for which *generalized order* and *generalized lower order* are the same is said to be of *generalized regular growth*. Functions which are not of *generalized regular growthh* are said to be of *generalized irregular growth*.

Somasundaram and Thamizharasi [5] introduced the notions of *L-order* and *L-lower order* for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalized concept for *L-order* and *L-lower order* for entire function are L^**-order* and L^**-lower order* respectively. Their definitions are as follows:

Definition 1.1. [5] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

In the line of Sato [5] , Somasundaram and Thamizharasi [5] one can define the *generalized L^*-order* $\rho_f^{[k]L^*}$ and *generalized L^*-lower order*$\lambda_f^{[k]L^*}$ of an entire function f in the following way:

Definition 1.2. Let k be an integer ≥ 1. The generalized L^*-order $\rho_f^{[k]L^*}$ and generalized L^*-lower order $\lambda_f^{[k]L^*}$ of an entire function f are defined as

$$\rho_f^{[k]L^*} = \limsup_{r\to\infty} \frac{\log^{[k]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[k]L^*} = \liminf_{r\to\infty} \frac{\log^{[k]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

respectively.

An entire function for which *generalized L^* -order* and *generalized L^* -lower order* are the same is said to be of *generalized L^* -regular growth*. Functions which are not of *generalized L^* -regular growth* are said to be of *generalized L^*-irregular growth*.

For any two entire functions f and g, Bernal {[1], [2]} initiated the definition of relative order $\rho_g(f)$ of f with respect to g which keep away from comparing growth just with $\exp z$ to find out *order* of entire functions as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r},\end{aligned}$$

and of course this definition corresponds with the classical one [6] for $g = \exp z$.

Analogously, one may define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

In the line of Somasundaram and Thamizharasi [5] and Bernal {[1], [2]}, one can define the relative L^*-order and relative L^*-lower order of an entire function in the following way :

Definition 1.3. The relative L^*-order and relative L^*- lower order of an entire function f with respect to another entire function g , denoted respectively by $\rho_g^{L^*}(f)$ and $\lambda_g^{L^*}(f)$ are defined in the following way

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Lahiri and Banerjee [3] gave a more generalized concept of relative order in the following way:

Definition 1.4. [3] If $k \geq 1$ is a positive integer, then the k- th generalized relative order of f with respect to g, denoted by $\rho_g^{[k]}(f)$ is defined by

$$\begin{aligned}\rho_g^{[k]}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[k-1]} r^\mu\right) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[k]} M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

Clearly $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp z}^1(f) = \rho_f$.

Likewise one can define the generalized relative lower order of f with respect to g denoted by $\lambda_g^{[k]}(f)$ as

$$\lambda_g^{[k]}(f) = \liminf_{r\to\infty} \frac{\log^{[k]} M_g^{-1} M_f(r)}{\log r}.$$

An entire function for which *generalized relative order* and *generalized relative lower order* are the same is said to be of *generalized relative regular growth*. Functions which are not of *generalized relative regular growth* are said to be of *generalized relative irregular growth*.

Similarly in the line of Somasundaram and Thamizharasi [5], Lahiri and Banerjee [3], one can define the generalized relative L^*-order and generalized relative L^*-lower order of an entire function in the following way :

Definition 1.5. Let k be an integer ≥ 1. The generalized relative L^*-order and generalized relative L^*- lower order of an entire function f with respect to another entire function g , denoted respectively by $\rho_g^{[k]L^*}(f)$ and $\lambda_g^{[k]L^*}(f)$ are defined in the following way

$$\rho_g^{[k]L^*}(f) = \limsup_{r\to\infty} \frac{\log^{[k]} M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{[k]L^*}(f) = \liminf_{r\to\infty} \frac{\log^{[k]} M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} .$$

An entire function for which *generalized relative L^* -order* and *generalized relative L^* -lower order* are the same is said to be of *generalized relative L^* -regular growth*. Functions which are not of *generalized relative L^* -regular growth* are said to be of *generalized relative L^*-irregular growth*.

In this paper we have established some comparative growth properties of entire functions on the basis of generalized relative order (generalized relative lower order) as well as generalized relative L^*- order (generalized relative L^*-lower order). We do not explain the standard definitions and notations in the theory of entire function as those are available in [7].

2 Main Results

In this section we present the main results of the paper.

Theorem 2.1. *Let f, g and h be any three entire functions such that* $0 \leq \lambda_h^{[k]L^*}(f) \leq \rho_h^{[k]L^*}(f) < \infty$ *and* $0 \leq \lambda_h^{[k]}(g) \leq \rho_h^{[k]}(g) < \infty$ *where k is an integer* ≥ 1. *Then*

$$\begin{aligned} \frac{\lambda_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)} &\leq \lambda_g^{L^*}(f) \leq \min\left\{\frac{\lambda_h^{[k]L^*}(f)}{\lambda_h^{[k]}(g)}, \frac{\rho_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)}\right\} \\ &\leq \max\left\{\frac{\lambda_h^{[k]L^*}(f)}{\lambda_h^{[k]}(g)}, \frac{\rho_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)}\right\} \leq \rho_g^{L^*}(f) \leq \frac{\rho_h^{[k]L^*}(f)}{\lambda_h^{[k]}(g)} . \end{aligned}$$

Proof. From the definitions of $\rho_h^{[k]L^*}(f)$ and $\lambda_h^{[k]L^*}(f)$, we have for all sufficiently large values of r that

$$M_f(r) \leq M_h\left[\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}(f) + \varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right], \tag{2.1}$$

$$M_f(r) \geq M_h\left[\exp^{[k]}\left\{\left(\lambda_h^{[k]L^*}(f) - \varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right] \tag{2.2}$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq M_h\left[\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}(f) - \varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right], \tag{2.3}$$

$$M_f(r) \leq M_h\left[\exp^{[k]}\left\{\left(\lambda_h^{[k]L^*}(f) + \varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right]. \tag{2.4}$$

Similarly from the definitions of $\rho_h^{[k]}(g)$ and $\lambda_h^{[k]}(g)$, it follows for all sufficiently large values of r that

$$\begin{aligned} M_h^{-1}M_g(r) &\leq \exp^{[k]}\left\{\left(\rho_h^{[k]}(g) + \varepsilon\right)\log r\right\} \\ i.e.,\ M_g(r) &\leq M_h\left[\exp^{[k]}\left\{\left(\rho_h^{[k]}(g) + \varepsilon\right)\log r\right\}\right] \\ i.e.,\ M_h(r) &\geq M_g\left[\exp\left[\frac{\log^{[k]} r}{\left(\rho_h^{[k]}(g) + \varepsilon\right)}\right]\right], \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
M_h^{-1} M_g\left(r\right) &\geq \exp^{[k]}\left\{\left(\lambda_h^{[k]}\left(g\right)-\varepsilon\right)\log r\right\} \\
i.e.,\ M_h\left(r\right) &\leq M_g\left[\exp\left[\frac{\log^{[k]} r}{\left(\lambda_h^{[k]}\left(g\right)-\varepsilon\right)}\right]\right]
\end{aligned}
\tag{2.6}
$$

and for a sequence of values of r tending to infinity, we obtain that

$$
\begin{aligned}
M_h^{-1} M_g\left(r\right) &\geq \exp^{[k]}\left\{\left(\rho_h^{[k]}\left(g\right)-\varepsilon\right)\log r\right\} \\
i.e.\ M_h\left(r\right) &\leq M_g\left[\exp\left[\frac{\log^{[k]} r}{\left(\rho_h^{[k]}\left(g\right)-\varepsilon\right)}\right]\right] ,
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
M_h^{-1} M_g\left(r\right) &\leq \exp^{[k]}\left\{\left(\lambda_h^{[k]}\left(g\right)+\varepsilon\right)\log r\right\} \\
i.e.,\ M_h\left(r\right) &\geq M_g\left[\exp\left[\frac{\log^{[k]} r}{\left(\lambda_h^{[k]}\left(g\right)+\varepsilon\right)}\right]\right] .
\end{aligned}
\tag{2.8}
$$

Now from (2.3) and in view of (2.5) , we get for a sequence of values of r tending to infinity that

$$
M_g^{-1} M_f\left(r\right) \geq M_g^{-1} M_h\left[\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}\left(f\right)-\varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right]
$$

$$
\begin{aligned}
i.e.,\ M_g^{-1} M_f\left(r\right) &\geq M_g^{-1} M_g\left[\exp\left[\frac{\log^{[k]}\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}\left(f\right)-\varepsilon\right)\log\left[re^{L(r)}\right]\right\}}{\left(\rho_h^{[k]}\left(g\right)+\varepsilon\right)}\right]\right] \\
i.e.,\ \log M_g^{-1} M_f\left(r\right) &\geq \frac{\left(\rho_h^{[k]L^*}\left(f\right)-\varepsilon\right)}{\left(\rho_h^{[k]}\left(g\right)+\varepsilon\right)}\log r
\end{aligned}
$$

$$
i.e.,\ \frac{\log M_g^{-1} M_f\left(r\right)}{\log r} \geq \frac{\left(\rho_h^{[k]L^*}\left(f\right)-\varepsilon\right)}{\left(\rho_h^{[k]}\left(g\right)+\varepsilon\right)} .
$$

As $\varepsilon > 0$ is arbitrary, it follows that

$$
\rho_g^{L^*}\left(f\right) \geq \frac{\rho_h^{[k]L^*}\left(f\right)}{\rho_h^{[k]}\left(g\right)} .
\tag{2.9}
$$

Analogously from (2.2) and in view of (2.8) , it follows for a sequence of values of r tending to infinity that

$$
M_g^{-1} M_f\left(r\right) \geq M_g^{-1} M_h\left[\exp^{[k]}\left\{\left(\lambda_h^{[k]L^*}\left(f\right)-\varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right]
$$

$$
\begin{aligned}
i.e.,\ M_g^{-1} M_f\left(r\right) &\geq M_g^{-1} M_g\left[\exp\left[\frac{\log^{[k]}\exp^{[k]}\left\{\left(\lambda_h^{[k]L^*}\left(f\right)-\varepsilon\right)\log\left[re^{L(r)}\right]\right\}}{\left(\lambda_h^{[k]}\left(g\right)+\varepsilon\right)}\right]\right] \\
i.e.,\ \log M_g^{-1} M_f\left(r\right) &\geq \frac{\left(\lambda_h^{[k]L^*}\left(f\right)-\varepsilon\right)}{\left(\lambda_h^{[k]}\left(g\right)+\varepsilon\right)}\log r
\end{aligned}
$$

$$
i.e.,\ \frac{\log M_g^{-1} M_f\left(r\right)}{\log r} \geq \frac{\left(\lambda_h^{[k]L^*}\left(f\right)-\varepsilon\right)}{\left(\lambda_h^{[k]}\left(g\right)+\varepsilon\right)} .
$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\rho_g^{L^*}(f) \geq \frac{\lambda_h^{[k]L^*}(f)}{\lambda_h^{[k]}(g)} . \tag{2.10}$$

Again in view of (2.6), we have from (2.1) for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \leq M_g^{-1}M_h\left[\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}(f)+\varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f(r) &\leq M_g^{-1}M_g\left[\exp\left[\frac{\log^{[k]}\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}(f)+\varepsilon\right)\log\left[re^{L(r)}\right]\right\}}{\left(\lambda_h^{[k]}(g)-\varepsilon\right)}\right]\right] \\ i.e.,\ \log M_g^{-1}M_f(r) &\leq \frac{\left(\rho_h^{[k]L^*}(f)+\varepsilon\right)}{\left(\lambda_h^{[k]}(g)-\varepsilon\right)}\log r \end{aligned}$$

$$i.e.,\ \frac{\log M_g^{-1}M_f(r)}{\log r} \leq \frac{\left(\rho_h^{[k]L^*}(f)+\varepsilon\right)}{\left(\lambda_h^{[k]}(g)-\varepsilon\right)} .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\rho_g^{L^*}(f) \leq \frac{\rho_h^{[k]L^*}(f)}{\lambda_h^{[k]}(g)} . \tag{2.11}$$

Again from (2.2) and in view of (2.5), it follows for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \geq M_g^{-1}M_h\left[\exp^{[k]}\left\{\left(\lambda_h^{[k]L^*}(f)-\varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f(r) &\geq M_g^{-1}M_g\left[\exp\left[\frac{\log^{[k]}\exp^{[k]}\left\{\left(\lambda_h^{[k]L^*}(f)-\varepsilon\right)\log\left[re^{L(r)}\right]\right\}}{\left(\rho_h^{[k]}(g)+\varepsilon\right)}\right]\right] \\ i.e.,\ \log M_g^{-1}M_f(r) &\geq \frac{\left(\lambda_h^{[k]L^*}(f)-\varepsilon\right)}{\left(\rho_h^{[k]}(g)+\varepsilon\right)}\log r \end{aligned}$$

$$i.e.,\ \frac{\log M_g^{-1}M_f(r)}{\log r} \geq \frac{\left(\lambda_h^{[k]L^*}(f)-\varepsilon\right)}{\left(\rho_h^{[k]}(g)+\varepsilon\right)} .$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\lambda_g^{L^*}(f) \geq \frac{\lambda_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)} . \tag{2.12}$$

Also in view of (2.7), we get from (2.1) for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \leq M_g^{-1}M_h\left[\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}(f)+\varepsilon\right)\log\left[re^{L(r)}\right]\right\}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f(r) &\leq M_g^{-1}M_g\left[\exp\left[\frac{\log^{[k]}\exp^{[k]}\left\{\left(\rho_h^{[k]L^*}(f)+\varepsilon\right)\log\left[re^{L(r)}\right]\right\}}{\left(\rho_h^{[k]}(g)-\varepsilon\right)}\right]\right] \\ i.e.,\ \log M_g^{-1}M_f(r) &\leq \frac{\left(\rho_h^{[k]L^*}+\varepsilon\right)}{\left(\rho_h^{[k]}(g)-\varepsilon\right)}\log r \end{aligned}$$

$$i.e., \ \frac{\log M_g^{-1} M_f(r)}{\log r} \leq \frac{\left(\rho_h^{[k]L^*} + \varepsilon\right)}{\left(\rho_h^{[k]}(g) - \varepsilon\right)} .$$

Since $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\lambda_g^{L^*}(f) \leq \frac{\rho_h^{[k]L^*}}{\rho_h^{[k]}(g)} . \tag{2.13}$$

Similarly from (2.4) and in view of (2.6), it follows for a sequence of values of r tending to infinity that

$$M_g^{-1} M_f(r) \leq M_g^{-1} M_h \left[\exp^{[k]} \left\{ \left(\lambda_h^{[k]L^*}(f) + \varepsilon\right) \log \left[r e^{L(r)}\right] \right\} \right]$$

$$i.e., \ M_g^{-1} M_f(r) \leq M_g^{-1} M_g \left[\exp\left[\frac{\log^{[k]} \exp^{[k]} \left\{ \left(\lambda_h^{[k]L^*}(f) + \varepsilon\right) \log \left[r e^{L(r)}\right] \right\}}{\left(\lambda_h^{[k]}(g) - \varepsilon\right)}\right]\right]$$

$$i.e., \ \log M_g^{-1} M_f(r) \leq \frac{\left(\lambda_h^{[k]L^*} + \varepsilon\right)}{\left(\lambda_h^{[k]}(g) - \varepsilon\right)} \log r$$

$$i.e., \ \frac{\log M_g^{-1} M_f(r)}{\log r} \leq \frac{\left(\lambda_h^{[k]L^*} + \varepsilon\right)}{\left(\lambda_h^{[k]}(g) - \varepsilon\right)} .$$

As $\varepsilon (> 0)$ is arbitrary, we obtain from above that

$$\lambda_g^{L^*}(f) \leq \frac{\lambda_h^{[k]L^*}}{\lambda_h^{[k]}(g)} . \tag{2.14}$$

The theorem follows from (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14). □

In view of Theorem 2.1, one can easily deduce the following corollaries:

Corollary 2.2. *Let f, g and h be any three entire functions such that* $0 \leq \lambda_h^{[k]L^*}(f) = \rho_h^{[k]L^*}(f) < \infty$ *and* $0 \leq \lambda_h^{[k]}(g) \leq \rho_h^{[k]}(g) < \infty$ *where k is an integer* ≥ 1. *Then*

$$\lambda_g^{L^*}(f) = \frac{\rho_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)} \quad and \quad \rho_g^{L^*}(f) = \frac{\rho_h^{[k]L^*}(f)}{\lambda_h^{[k]}(g)} .$$

Corollary 2.3. *Let f, g and h be any three entire functions such that* $0 \leq \lambda_h^{[k]L^*}(f) \leq \rho_h^{[k]L^*}(f) < \infty$ *and* $0 \leq \lambda_h^{[k]}(g) = \rho_h^{[k]}(g) < \infty$ *where k is an integer* ≥ 1. *Then*

$$\lambda_g^{L^*}(f) = \frac{\lambda_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)} \quad and \quad \rho_g^{L^*}(f) = \frac{\rho_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)} .$$

Corollary 2.4. *Let f, g and h be any three entire functions such that* $0 \leq \lambda_h^{[k]L^*}(f) = \rho_h^{[k]L^*}(f) < \infty$ *and* $0 \leq \lambda_h^{[k]}(g) = \rho_h^{[k]}(g) < \infty$ *where k is an integer* ≥ 1. *Then*

$$\lambda_g^{L^*}(f) = \rho_g^{L^*}(f) = \frac{\rho_h^{[k]L^*}(f)}{\rho_h^{[k]}(g)} .$$

Corollary 2.5. *Let f, g and h be any three entire functions such that* $0 \leq \lambda_h^{[k]L^*}(f) = \rho_h^{[k]L^*}(f) < \infty$ *and* $0 \leq \lambda_h^{[k]}(g) = \rho_h^{[k]}(g) < \infty$ *where k is an integer ≥ 1.Also suppose that* $\rho_h^{[k]L^*}(f) = \rho_h^{[k]}(g)$. *Then*

$$\lambda_g^{L^*}(f) = \rho_g^{L^*}(f) = 1 \ .$$

Corollary 2.6. *Let f and h be any two entire functions such that* $0 \leq \lambda_h^{[k]L^*}(f) \leq \rho_h^{[k]L^*}(f) < \infty$. *Then for any entire function g,*

$$\begin{aligned} (i)\ \lambda_g^{L^*}(f) &= \infty \text{ when } \rho_h^{[k]}(g) = 0\ , \\ (ii)\ \rho_g^{L^*}(f) &= \infty \text{ when } \lambda_h^{[k]}(g) = 0\ , \\ (iii)\ \lambda_g^{L^*}(f) &= 0 \text{ when } \rho_h^{[k]}(g) = \infty \end{aligned}$$

and

$$(iv)\ \rho_g^{L^*}(f) = \infty \text{ when } \lambda_h^{[k]}(g) = \infty,$$

where k is an integer ≥ 1.

Corollary 2.7. *Let g and h be any two entire functions such that* $0 \leq \lambda_h^{[k]}(g) \leq \rho_h^{[k]}(g) < \infty$. *Then for any entire function f,*

$$\begin{aligned} (i)\ \rho_g^{L^*}(f) &= 0 \text{ when } \rho_h^{[k]L^*}(f) = 0\ , \\ (ii)\ \lambda_g^{L^*}(f) &= 0 \text{ when } \lambda_h^{[k]L^*}(f) = 0\ , \\ (iii)\ \rho_g^{L^*}(f) &= \infty \text{ when } \rho_h^{[k]L^*}(f) = \infty \end{aligned}$$

and

$$(iv)\ \lambda_g^{L^*}(f) = \infty \text{ when } \lambda_h^{[k]L^*}(f) = \infty\ ,$$

where k is an integer ≥ 1.

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