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The L^p-function over the product of the boundaries of the Hyperbolic spaces.

Fouzia El Wassouli

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Abstract. Let $B(\mathbb{F}^n)$ be the hyperbolic space over $\mathbb{F}$ ($\mathbb{F}$ being the field of real $\mathbb{R}$, or complex $\mathbb{C}$ or the quaternions $\mathbb{H}$) and $\partial B(\mathbb{F}^n)$ its boundary.
We give a necessary and sufficient conditions on the Poisson transform $P_\lambda f$ of an element $f \in A'(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$ for f to be in $L^p(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$, $2 \leq p < \infty$, where $A'(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$ is the space of all hyperfunctions on $\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n)$.

1 Introduction and statement of main result.

In classical harmonic function theory, it is well-known that the Poisson integral of complex-valued integrable function defined on the unit circle $S = \{z \in \mathbb{C}, \quad |z| = 1\}$ of the complex plane $\mathbb{C}$ determines an harmonic functions on the corresponding unit disk $D = \{z \in \mathbb{C}, \quad |z| < 1\}$. Namely, if $f(z)$ is a bounded harmonic function on D; then almost everywhere on the circle S it has radial boundary values

$$\lim_{r \longrightarrow 1} f(re^{i\alpha}) = \varphi(e^{i\alpha})$$

and the function f can be expressed in terms of φ with the help of the well-known Poisson transformation

$$f(re^{i\alpha}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2\cos(\alpha - \beta) + r^2} \varphi(e^{i\beta}) d\beta.$$

This transformation was generalized first to classical bounded domains and next to Riemannian symmetric spaces $X = G/K$, where G is a non-compact semi-simple Lie group, and K is its a maximal compact subgroup. Not only harmonic functions are considered, but also functions that are eigenfunctions of the algebra of G-invariant differential operators on $X = G/K$(see [3], [4], [5]).

Furthermore, in rank one symmetric spaces of non compact type, the Poisson transform appears naturally through the Fourier-Helgason transform in the L^2-Plancherel formula of the Laplace-Beltrami operator on $X = G/K$.
It is of great interest to look an analogue concrete a description of the range of the Poisson transform of L^p-functions on $X \times X, 1 < p < \infty$, and moreover on the product $E \times E$ of line bundle E over X
Below we have to deal the particular case of the unit ball $B(\mathbb{F}^n)$. Mainly, the aim of this paper is
⋆ to give the necessary and sufficient condition on the Poisson transform $P_\lambda f (\lambda \in \mathbb{R}^*)$ of an element f in the space $A'(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$ for f to be in $L^p(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n)), p \in [2, \infty[$.
⋆to extend in a unified manner the result in [2] to the classical hyperboplic spaces $B(\mathbb{F}^n)$.

The main result of this paper are the following theorems.

Theorem 1.1. *Let* $\lambda \in \mathbb{R}^*$. *Then,*
(i) *For every* $F = P_\lambda f$ *with* $f \in L^2(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$, *we have*

$$||F||^2_{\lambda,2} = \sup_{0 \leq r_1, r_2 < 1} (1 - r_1^2)^{-\frac{\sigma}{2}} (1 - r_2^2)^{-\frac{\sigma}{2}} \int_{\partial B(\mathbb{F}^n)} \int_{\partial B(\mathbb{F}^n)} \left| F(r_1\theta_1, r_2\theta_2) \right|^2 d\theta_1 d\theta_2 < \infty,$$

where $\sigma = \frac{d}{2}(n+1) - 1$ *and* $d = \dim_{\mathbb{R}} \mathbb{F}$.
(ii) *Let* $f \in A'(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$. *such that* $F = P_\lambda f$ *satisfies* $||F||_{\lambda,2} < \infty$. *Then* f *belongs to* $L^2(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$.
Moreover, there exist positive constants γ_1 *and* $\gamma_2(\lambda)$ *such that for every* $f \in L^2(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$ *we have the following estimates:*

$$\gamma_1 |C(\lambda)|^2 |f||_{L^2} \leq ||P_\lambda f||_{\lambda,2} \leq \gamma_2(\lambda)||f||_{L^2}, \tag{1.1}$$

where

$$C(\lambda) = \frac{2^{\sigma - i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{i\lambda+\sigma}{2})\Gamma(\frac{i\lambda+\sigma+2-d}{2})} \tag{1.2}$$

is the Harish-Chandra c-function associated to $B(\mathbb{F}^n)$.
(iii) *Let* $F = P_\lambda f$ *with* $f \in L^2(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$. *Then its* L^2*-boundary value is given by following inversion formula*

$$\begin{aligned} f(w_1, w_2) &= |C(\lambda)|^{-4} \lim_{\substack{t_1 \longrightarrow \infty \\ t_2 \longrightarrow \infty}} \frac{1}{t_1 t_2} \\ &\times \int_0^{tht_1} \int_0^{tht_2} \Big(\int_{\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n)} F(r_1\theta_1, r_2\theta_2) \overline{P_\lambda(\lambda, r_1 w_1, \theta_1) P_\lambda(\lambda, r_2 w_2, \theta_2)} d\theta_1 d\theta_2 \Big) \\ &(1 - r_1^2)^{-\sigma-1}(1 - r_2^2)^{-\sigma-1}(r_1 r_2)^{dn-1} dr_1 dr_2, \quad in\ L^2(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n)). \end{aligned}$$

Theorem 1.2. *Let* $\lambda \in \mathbb{R}^*$ *and* $p \in [2, \infty[$. *Then,*
(i) *For every* $F = P_\lambda f$ *such that* $f \in L^p(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$, *we have*

$$||F||^p_{\lambda,p} = \sup_{0 \leq r_1, r_2 < 1} (1 - r_1^2)^{-\frac{\sigma}{2}} (1 - r_2^2)^{-\frac{\sigma}{2}} \int_{\partial B(\mathbb{F}^n)} \int_{\partial B(\mathbb{F}^n)} \Big| F(r_1\theta_1, r_2\theta_2) \Big|^p d\theta_1 d\theta_2 < \infty,$$

where $\sigma = \frac{d}{2}(n+1) - 1$ *and* $d = \dim_{\mathbb{R}} \mathbb{F}$.
(ii) *Let* $f \in A'(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$ *such that* $F = P_\lambda f$ *satisfies* $||F||_{\lambda,p} < \infty$. *Then* f *is belongs to* $L^p(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$.

Moreover, there exist positive constants γ_1 *and* $\gamma_2(\lambda, p)$ *such that for every* $f \in L^p(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$ *we have the following estimates:*

$$\gamma_1 |C(\lambda)|^2 ||f||_{L^p} \leq ||P_\lambda f||_{\lambda,p} \leq \gamma_2(\lambda, p)||f||_{L^p}, \tag{1.3}$$

where $C(\lambda)$ *is the Harish-Chandra c-function given by (1.2)*

The article is organized as follows. In Section 2, we recall some classical results from harmonic analysis on hyperbolic spaces $B(\mathbb{F}^n)$. In Section 3, we give the precise action of P_λ on $L^2(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$. Section 4 is devoted to the proof of Theorems 1.1 and 1.2.

2 Preliminary results.

In this section, we recall some known results of harmonic analysis on the hyperbolic space $B(\mathbb{F}^n) = U(n, 1; \mathbb{F})/U(n, \mathbb{F} \times U(1, \mathbb{F}))$. We refer the reader to [1] for more details on the subject.
Let $\mathbb{F}$ be one of the classical fields, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$. On $\mathbb{F}^{n+1}$ considered as a right vector space over $\mathbb{F}$, we consider the quadratic form

$$J(x_1, ..., x_{n+1}) = \sum_{j=1}^{n} |x_j|^2 - |x_{n+1}|^2,$$

where $|x|^2 = x\bar{x}$ and $x \longrightarrow \bar{x}$ is the standard involution of $\mathbb{F}$.
Let $G = U(n, 1; \mathbb{F})$ be the group of all $\mathbb{F}$-linear transformations g on $\mathbb{F}^{n+1}$ leaving the quadratic form J invariant, with the additional property that $\det g = 1$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then G is one of the

classical groups, $SO(n,1), SU(n,1)$ or $Sp(n,1)$ accordingly to $\mathbb{F}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$. Moreover, the group G acts on the unit ball $B(\mathbb{F}^n)=\{x\in\mathbb{F}^n; \quad |x|<1\}$ by fractional transforms:

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in G: x\longmapsto (Ax+B)(Cx+D)^{-1}$$

with $A\in\mathbb{F}^{n\times n}, B\in\mathbb{F}^{n\times 1}, C\in\mathbb{F}^{1\times n}$ and $D\in\mathbb{F}$. This action of G on $B(\mathbb{F}^n)$ is transitive so that $B(\mathbb{F}^n)$ can be seen as homogeneous space $B(\mathbb{F}^n)=G/K$ where K is the stabilizer of $0\in B(\mathbb{F}^n)$ in G.
The action of G mentioned above extends naturally to $\overline{B(\mathbb{F}^n)}$ and under this action, K acts transitively on the topological boundary $\partial B(\mathbb{F}^n)=\{w\in\mathbb{F}^n; \quad |w|=1\}$ of $B(\mathbb{F}^n)$. Moreover, for M being the stabilizer in K of $e=(1,0,...,0)$, we have $\partial B(\mathbb{F}^n)=K/M$.
Now, let $L^2(\partial B(\mathbb{F}^n))$ be the space of all square integrable $\mathbb{C}$-valued functions on $\partial B(\mathbb{F}^n)$, with respect to the normalized superficial measure of $\partial B(\mathbb{F}^n)$. Then the group K acts on $L^2(\partial B(\mathbb{F}^n))$ by composition $f\longmapsto f\circ k;\ k\in K$.
It is well known that under the action of K, the Peter-Weyl decomposition of $L^2(\partial B(\mathbb{F}^n))$ is given by $L^2(\partial B(\mathbb{F}^n))=\oplus_{p,q\in\hat{K}_0}V_{p,q}$, where $V_{p,q}$ is the finite linear span $\{\varphi_{p,q}\circ k, \quad k\in K\}$ and $\varphi_{p,q}$ the zonal spherical functions.
The parametrized set $\hat{K}_0$ consists of pairs (p,q) of integers satisfying:

$$\begin{aligned} i) &\qquad p\equiv q\ (mod 2),\\ ii) &\qquad p\geq 0 \text{ and } 0\leq q\leq 1 \text{ if } \mathbb{F}=\mathbb{R},\\ &\qquad p\geq |q| \text{ if } \mathbb{F}=\mathbb{C},\\ &\qquad p\geq q\geq 0 \text{ if } \mathbb{F}=\mathbb{H}. \end{aligned}$$

3 The Poisson transform P_λ on $A'(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$.

In this section, we give an explicit form of the Poisson transform P_λ defined for fixed $\lambda\in\mathbb{C}$ on the space $A'(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$ of all hyperfunctions on $\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)$ by

$$(P_\lambda F)(x_1,x_2)=\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)} P_\lambda(\lambda,x_1,w_1)P_\lambda(\lambda,x_2,w_2)F(w_1,w_2)dw_1dw_2$$

for every $(x_1,x_2)\in B(\mathbb{F}^n)\times B(\mathbb{F}^n)$, where

$$P_\lambda(\lambda,x_j,w_j)=\left[\frac{1-|x_j|^2}{\left|1-\left\langle x_j,w_j\right\rangle\right|^2}\right]^{\frac{i\lambda+\sigma}{2}},$$

with $\sigma=\frac{d}{2}(n+1)-1$ and $d=\dim_{\mathbb{R}}\mathbb{F}$.
The following generalized spherical function associated to the hyperbolic space $B(\mathbb{F}^n)$ are defined by

$$\begin{aligned}\Phi_{\lambda,pq}(|x|) &= \Big(\frac{i\lambda+\sigma}{2}\Big)_{\frac{p+q}{2}}\Big(\frac{i\lambda+\sigma+2-d}{2}\Big)_{\frac{p-q}{2}}\{(1)_{p+\frac{dn}{2}}\}^{-1}|x|^p(1-|x|^2)^{\frac{i\lambda+\sigma}{2}}\\ &\times\ F\Big(\frac{i\lambda+\sigma+p+q}{2},\frac{i\lambda+\sigma+2-d+p-q}{2},p+\frac{dn}{2};|x|^2\Big),\end{aligned}$$

where $(a)_k=a(a+1)(a+2)...(a+k-1)$ is the Pochammer symbol and $F(a,b,c;x)$ is the classical Gauss hypergeometric function.
We assert the following

Proposition 3.1. *Let* $A'(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$ *and* $f(w_1,w_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}f_{p_1q_1}(w_1)f_{p_2q_2}(w_2)$ *its K-type decomposition. Then,*

$$(P_\lambda f)(x_1,x_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}\Phi_{\lambda,p_1q_1}(|x_1|)\Phi_{\lambda,p_2q_2}(|x_2|)f_{p_1q_1}\Big(\frac{x_1}{|x_1|}\Big)f_{p_2q_2}\Big(\frac{x_2}{|x_2|}\Big).$$

Proof. According to definition of P_λ and the K-type decomposition of f, we have

$$(P_\lambda f)(x_1,x_2) = \int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)} P_\lambda(\lambda,x_1,w_1)P_\lambda(\lambda,x_2,w_2)f(w_1,w_2)dw_1dw_2$$

$$= \sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} \int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)} a_{p_1q_1,p_2q_2}P_\lambda(\lambda,x_1,w_1)P_\lambda(\lambda,x_2,w_2)f_{p_1q_1}(w_1)f_{p_2q_2}(w_2)dw_1dw_2.$$

Now, using the fact that [3]

$$\int_{\partial B(\mathbb{F}^n)} \left[\frac{1-|x|^2}{\left|1-\left\langle x,w\right\rangle\right|^2}\right]^{\frac{i\lambda+\sigma}{2}} \psi(w)dw = \Phi_{\lambda,pq}(|x|)\psi(\frac{x}{|x|}); \quad x\in B(\mathbb{F}^n).$$

For every $\psi\in V_{pq}$, it follows

$$\begin{aligned}(P_\lambda f)(x_1,x_2) &= \sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} \int_{\partial B(\mathbb{F}^n)} a_{p_1q_1,p_2q_2}\Phi_{\lambda,p_1q_1}(|x_1|)f_{p_1q_1}(\frac{x_1}{|x_1|})P_\lambda(\lambda,x_2,w_2)f_{p_2q_2}(w_2)dw_2\\ &= \sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}\Phi_{\lambda,p_1q_1}(|x_1|)\Phi_{\lambda,p_2q_2}(|x_2|)f_{p_1q_1}(\frac{x_1}{|x_1|})f_{p_2q_2}(\frac{x_2}{|x_2|}).\end{aligned}$$

4 Proof of Theorem 1.1 and Theorem 1.2

For prove our main results Theorems 1.1 and 1.2, we are need to the following technical lemmas:

Lemma 4.1. *[1] Let λ be a non zero real number. Then*

$$\sup_{p,q\in\hat{K}_0} |\Phi_{\lambda,pq}(r)| \leq \gamma(\lambda)(1-r^2)^{\frac{\sigma}{2}}$$

for some numerical positive constant γ.

Lemma 4.2. *[1] Let λ be a non zero real number. Then there exists a positive constant $\gamma>0$ such that we have:*

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\int_{B(0,t)} |\Phi_{\lambda,pq}(|x|)|^2(1-|x|^2)^{-\sigma-1}dm(x) = \gamma|C(\lambda)|^2,$$

for every $p,q\in\hat{K}_0$. Here $B(0,t)$ is the ball of radius t centered at 0 with respect to the $U(n,1;\mathbb{F})$-invariant metric on $B(\mathbb{F}^n)$.

Lemma 4.3. *[1] Let λ be a non zero real number and $p\in]1,\infty[$. Then, there exist a constant $A(\lambda,p)>0$ such that*

$$\sup_{0\leq r<1} ||Q_r(\lambda)||_p \leq A(\lambda,p),$$

where $||||_p$ stands for the L^p-operatorial norm.

4.1 Proof of Theorem 1.1

The necessary condition: Assure that $F=P_\lambda f, f\in L^2(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$ and let $f(w_1,w_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}f_{p_1q_1}(w_1)f_{p_2q_2}(w_2)$ be its K-type decomposition. Then making use of Proposition 3.2, we get

$$F(r_1\theta_1,r_2\theta_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}\Phi_{\lambda,p_1q_1}(r_1)\Phi_{\lambda,p_2q_2}(r_2)f_{p_1q_1}(\theta_1)f_{p_2q_2}(\theta_2),\quad in\ C^\infty\Big([0,1[\times\partial B(\mathbb{F}^n)\Big)^2.$$

Therefore

$$\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}|F(r_1\theta_1,r_2\theta_2)|^2d\theta_1d\theta_2=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}}|a_{p_1q_1,p_2q_2}|^2|\Phi_{\lambda,p_1q_1}(r_1)|^2|\Phi_{\lambda,p_2q_2}(r_2)|^2.$$

Next, using the Lemma 4.1 we get the right hand side of the estimate (1.1) in Theorem 1.1

$$||P_\lambda f||_{\lambda,2}\leq\gamma^2(\lambda)||f||_{L^2}.$$

For the sufficiency condition: Assume that $F=P_\lambda f$ for some, $f\in A'(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$.
By writting K-type decomposition of f

$$f(w_1,w_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}f_{p_1q_1}(w_1)f_{p_2q_2}(w_2)$$

and next using Proposition 3.1, we get

$$F(r_1\theta_1,r_2\theta_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}\Phi_{\lambda,p_1q_1}(r_1)\Phi_{\lambda,p_2q_2}(r_2)f_{p_1q_1}(\theta_1)f_{p_2q_2}(\theta_2),\quad in\ C^\infty\Big([0,1[\times\partial B(\mathbb{F}^n)\Big)^2.$$

The growth condition on F, that is $||F||_{\lambda,2}<\infty$, implies

$$\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}}\frac{1}{t_1t_2}\int_0^{th_1}\int_0^{th_2}|a_{p_1q_1,p_2q_2}|^2\quad|\Phi_{\lambda,p_1q_1}(r_1)|^2|\quad\Phi_{\lambda,p_2q_2}(r_2)|^2(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2$$

$$\leq\qquad c||F||^2_{\lambda,2}<\infty,$$

for every $t_1,t_2>0$. Next, by means of Lemma 4.2 giving the uniform asymptotic behaviour of the function $\Phi_{\lambda,pq}$, we obtain:

$$\gamma^4|C(\lambda)|^4\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}}|a_{p_1q_1,p_2q_2}|^2<c||F||^2_{\lambda,2}<\infty.$$

This gives use to the left hand side of the estimate (1,1) in Theorem 1.1.
Now, to establish the L^2-inversion formula, let $F=P_\lambda f$ with $f\in L^2(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$. Application of Proposition 3.1 to f expanded into its K-type series, $f(w_1,w_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}f_{p_1q_1}(w_1)f_{p_2q_2}(w_2)$

gives use to

$$F(r_1\theta_1,r_2\theta_2)=\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}\Phi_{\lambda,p_1q_1}(r_1)\Phi_{\lambda,p_2q_2}(r_2)f_{p_1q_1}(\theta_1)f_{p_2q_2}(\theta_2),\quad in\ C^\infty\Big([0,1[\times\partial B(\mathbb{F}^n)\Big)^2.$$

Therefore, the $\mathbb{C}$-valued function on $\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)$ given by

$$\begin{aligned}g_{t_1,t_2}(w_1,w_2)&=|C(\lambda)|^{-4}\frac{1}{t_1t_2}\\ &\times\int_0^{tht_1}\int_0^{tht_2}\Big(\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}F(r_1\theta_1,r_2\theta_2)\overline{P_\lambda(\lambda,r_1w_1,\theta_1)P_\lambda(\lambda,r_2w_2,\theta_2)}d\theta_1d\theta_2\Big)\\ &\quad(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2,\quad in\ L^2(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)).\end{aligned}$$

Then, replacing F by its above series expansion, the function g_{t_1,t_2} can be rewritten as:

$$g_{t_1,t_2}(w_1,w_2) = |C(\lambda)|^{-4}\frac{1}{t_1t_2}\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}} a_{p_1q_1,p_2q_2}\int_0^{tht_1}\int_0^{tht_2}\Phi_{\lambda,p_1q_1}(r_1)\Phi_{\lambda,p_2q_2}(r_2)$$

$$\times \Big(\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)} f_{p_1q_1}(\theta_1)f_{p_2q_2}(\theta_2)\overline{P_\lambda(\lambda,r_1w_1,\theta_1)P_\lambda(\lambda,r_2w_2,\theta_2)}d\theta_1 d\theta_2\Big)$$

$$(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2$$

$$= |C(\lambda)|^{-4}\frac{1}{t_1t_2}\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}}\Big[a_{p_1q_1,p_2q_2}$$

$$\int_0^{tht_1}\int_0^{tht_2}|\Phi_{\lambda,p_1q_1}(r_1)|^2|\Phi_{\lambda,p_2q_2}(r_2)|^2(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2\Big]f_{p_1q_1}(w_1)f_{p_2q_2}(w_2$$

Hence the $L^2(\partial B(\mathbb{F}^n))$-norm of the function g_{t_1,t_2} is given by:

$$||g_{t_1,t_2}||^2_{L^2} = \Big||C(\lambda)|^{-4}\frac{1}{t_1t_2}\Big|^2$$

$$\sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}}\Big[a_{p_1q_1,p_2q_2}\int_0^{tht_1}\int_0^{tht_2}|\Phi_{\lambda,p_1q_1}(r_1)|^2|\Phi_{\lambda,p_2q_2}(r_2)|^2(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2\Big]^2.$$

Furthermore, we have

$$||g_{t_1,t_2}-f||^2_{L^2} = \sum_{\substack{p_1,q_1\in\hat{K}_0\\ p_2,q_2\in\hat{K}_0}}\Big[\frac{|C(\lambda)|^{-4}}{t_1t_2}\int_0^{tht_1}\int_0^{tht_2}|\Phi_{\lambda,p_1q_1}(r_1)|^2|\Phi_{\lambda,p_2q_2}(r_2)|^2(1-r_1^2)^{-\sigma-1}$$

$$\times\ (1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2 - 1\Big]^2|a_{p_1q_1,p_2q_2}|^2.$$

Finally using the asymptotic behaviour of the generalized spherical function $\Phi_{\lambda,pq}$ given Lemma 4.2 we see that

$$\lim_{\substack{t_1\longrightarrow\infty\\ t_2\longrightarrow\infty}}\Big|g_{t_1,t_2}-f\Big|^2_{L^2} = 0$$

which gives the desired result.

4.2 Proof of Theorem 1.2

Proof of (i): Let f in $L^p(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$. Then, we have

$$(P_\lambda f)(r_1\theta_1,r_2\theta_2) = \int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)} P_\lambda(\lambda,r_1\theta_1,w_1)P_\lambda(\lambda,r_2\theta_2,w_2)f(w_1,w_2)dw_1dw_2$$

$$= (1-r_1^2)^{\frac{i\lambda+\sigma}{2}}\int_{\partial B(\mathbb{F}^n)} P_\lambda(\lambda,r_2\theta_2,w_2)[Q_{r_1}(\lambda)f_{w_1}](\theta_1)dw_2$$

with $f_{w_2}(w_1) = f(w_1,w_2)$. Putting $g(w_2) = [Q_{r_1}(\lambda)f_{w_2}](\theta_2)$. Then

$$P_\lambda f)(r_1\theta_1,r_2\theta_2) = (1-r_1^2)^{\frac{i\lambda+\sigma}{2}}(1-r_2^2)^{\frac{i\lambda+\sigma}{2}}[Q_{r_2}(\lambda)g](\theta_2).$$

Thus, from Lemma 4.3, we get

$$||P_\lambda f||_{\lambda,p} = \sup_{0\le r_1,r_2<1}(1-r_1^2)^{-\frac{\sigma}{2}}(1-r_2^2)^{-\frac{\sigma}{2}}\Big[\int_{\partial B(\mathbb{F}^n)}\int_{\partial B(\mathbb{F}^n)}\Big|P_\lambda f(r_1\theta_1,r_2\theta_2)\Big|^p d\theta_1 d\theta_2\Big]^{\frac{1}{p}}$$

$$= \sup_{0\le r_1,r_2<1}\Big[\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}\Big|[Q_{r_1}(\lambda)[Q_{r_2}(\lambda)(g)]\theta_2]\theta_1\Big|^p d\theta_1 d\theta_2\Big]^{\frac{1}{p}}$$

$$\le A(\lambda,p)||Q_{r_2}(\lambda)(g)(\theta_2)||_{L^p} \le A^2(\lambda,p)||f||_{L^p}.$$

This end the proof of (i).

Proof of (ii): Let F be a $\mathbb{C}$ valued function on $\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)$ such that $||F||_{\lambda,p}<\infty$. Using the fact that $||F||_{\lambda,2}\leq||F||_{\lambda,p}$ for every $p\in[2,\infty[$, there exists from Theorem 1.1 a function $g_{t_1,t_2}\in L^2(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$ such that $P_\lambda f=F$ and $f(w_1,w_2)=\lim_{\substack{t_1\longrightarrow\infty\\ t_2\longrightarrow\infty}} g_{t_1,t_2}(w_1,w_2),\quad in\ L^2(\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n))$. where More precisely, we have

$$\begin{aligned} g_{t_1,t_2}(w_1,w_2) &= \frac{|C(\lambda)|^{-4}}{t_1t_2}\\ &\times \int_0^{tht_1}\int_0^{tht_2}\Big(\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}F(r_1\theta_1,r_2\theta_2)\overline{P_\lambda(\lambda,r_1w_1,\theta_1)P_\lambda(\lambda,r_2w_2,\theta_2)}d\theta_1d\theta_2\Big)\\ &\quad(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2. \end{aligned}$$

Let Φ_i, $i\in\{1,2\}$ be continuous functions on $\partial B(\mathbb{F}^n)$. Then we have

$$\lim_{\substack{t_1\longrightarrow\infty\\ t_2\longrightarrow\infty}}\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}g_{t_1,t_2}(w_1,w_2)\overline{\Phi_1(w_1)\Phi_2(w_2)}dw_1dw_2=\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}f(w_1,w_2)\overline{\Phi_1(w_1)\Phi_2(w_2)}dw_1dw_2$$

However,

$$\begin{aligned} &\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}g_{t_1,t_2}(w_1,w_2)\overline{\Phi_1(w_1)\Phi_2(w_2)}dw_1dw_2=\frac{|C(\lambda)|^{-4}}{t_1t_2}\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}\Big[\\ &\int_0^{tht_1}\int_0^{tht_2}\Big(\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}F(r_1\theta_1,r_2\theta_2)\overline{P_\lambda(\lambda,r_1w_1,\theta_1)P_\lambda(\lambda,r_2w_2,\theta_2)}d\theta_1d\theta_2\Big)\\ &(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2\Big]\overline{\Phi_1(w_1)\Phi_2(w_2)}dw_1dw_2.\\ =\ &\frac{|C(\lambda)|^{-4}}{t_1t_2}\int_0^{tht_1}\int_0^{tht_2}\Big(\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}F(r_1\theta_1,r_2\theta_2)\overline{P_\lambda\Phi_1(r_1\theta_1)P_\lambda\Phi_2(r_2\theta_2)}d\theta_1d\theta_2\Big)\\ &(1-r_1^2)^{-\sigma-1}(1-r_2^2)^{-\sigma-1}(r_1r_2)^{dn-1}dr_1dr_2. \end{aligned}$$

Thus by means of Holder inequality, we obtain

$$\begin{aligned} &\Big|\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}F(r_1\theta_1,r_2\theta_2)\overline{P_\lambda\Phi_1(r_1\theta_1)P_\lambda\Phi_2(r_2\theta_2)}d\theta_1d\theta_2\Big|\\ \leq\ &\Big(\int_{\partial B(\mathbb{F}^n)}|P_\lambda\Phi_1(r_1\theta_1)|^qd\theta_1\Big)^{\frac{1}{q}}\int_{\partial B(\mathbb{F}^n)}|P_\lambda\Phi_2(r_2\theta_2)|\Big[\int_{\partial B(\mathbb{F}^n)}|F(r_1\theta_1,r_2\theta_2)|^pd\theta_1\Big]^{\frac{1}{p}}d\theta_2\\ \leq\ &\Big(\int_{\partial B(\mathbb{F}^n)}|P_\lambda\Phi_1(r_1\theta_1)|^qd\theta_1\Big)^{\frac{1}{q}}\Big(\int_{\partial B(\mathbb{F}^n)}|P_\lambda\Phi_2(r_2\theta_2)|^qd\theta_2\Big)^{\frac{1}{q}}|\Big[\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}|F(r_1\theta_1,r_2\theta_2)|^pd\theta_1d\theta_2\Big] \end{aligned}$$

where q is such that $\frac{1}{p}+\frac{1}{q}=1$. Next, Lemma 4.3 shows that, for every $q>1$, the following estimate

$$\Big[\int_{\partial B(\mathbb{F}^n)}|P_\lambda\Phi_i(r_i\theta_i)|^qd\theta_i\Big]^{\frac{1}{q}}\leq(1-r_i^2)^{\frac{\sigma}{2}},A(\lambda,q)||\Phi_i||_{L^q}\quad i\in\{1,2\},$$

holds. Hence,

$$\begin{aligned} &\Big|\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}g_{t_1,t_2}(w_1,w_2)\overline{\Phi_1(w_1)\Phi_2(w_2)}dw_1dw_2\Big|\\ \leq\ &|C(\lambda)|^{-4}A^2(\lambda,q)||\Phi_1||_{L^q}||\Phi_2||_{L^q}||F||_{\lambda,p}. \end{aligned}$$

Thus

$$\Big|\int_{\partial B(\mathbb{F}^n)\times\partial B(\mathbb{F}^n)}f(w_1,w_2)\overline{\Phi_1(w_1)\Phi_2(w_2)}dw_1dw_2\Big|\leq|C(\lambda)|^{-4}A^2(\lambda,q)||\Phi_1||_{L^q}||\Phi_2||_{L^q}||F||_{\lambda,p}.$$

Taking the supremum over all continuous functions $\Phi_i, \quad i \in \{1,2\}$ with $||\Phi_i||_{L^q} \leq 1$, we deduce that $f \in L^p(\partial B(\mathbb{F}^n) \times \partial B(\mathbb{F}^n))$ with $|C(\lambda)|^2||f||_{L^p} \leq A^2(\lambda,p)||F||_{\lambda,p}$. This finishes the proof of Theorem 1.2.

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Author information

Fouzia El Wassouli, Department of Mathematics, Faculty of Sciences- Ain Chock, University Hassan II, Casablanca, Morocco.
E-mail: elwassouli@gmail.com

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