Corrigendum to “When the juxtaposition of two minimal ring extensions produces no new intermediate rings”

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The author recently received a message [7] from Gabriel Picavet and Martine Picavet-L’Hermite, who kindly informed him that [1, Theorem 2.8] is incorrect. At that time, they provided a counterexample to [1, Theorem 2.8]. The Example given below presents that counterexample and (the author’s rendering of) their proof of it. One should note that the first five paragraphs of the published “proof” of [1, Theorem 2.8] are correct. Contained therein is the proof of a special case where the statement of [1, Theorem 2.8] is correct. That partial result appears as the Proposition below. A closing Remark identifies where the published “proof” of [1, Theorem 2.8] was in error and announces some recent results.

Most of the following material refers to chains of (commutative unital) rings where the base ring is a special principal ideal ring (SPIR). Suitable background on SPIRs can be found in [8, page 245]. Our convention here is that no field can be an SPIR. As usual, if $A$ is a ring, then $[A, B]$ denotes the set of rings $C$ such that $A \subseteq C \subseteq B$.

Example. (G. Picavet and M. Picavet-L’Hermite) Let $(R, M)$ be an SPIR such that $M^2 = 0$. Fix $p \in R$ such that $M = Rp$ (so $p^2 = 0$). With $Y$ an indeterminate over $R$, set $T := \{R[Y]/(Y^2 - Y)\} \cup \{0\}$, for if $r_1, r_2 \in R$ with $r_1 + r_2 Y \in (Y^2 - Y)$, then $r_1 = 0 = r_2$. If $x \in R$, then $x = py \in R \cap Ry = \{0\}$, a contradiction.

Hence $x \notin R$, and so $R \subset S = R[x]$. Note also that $(R : T) \neq M$, since $px = x \notin R$.

Of course, $(R : T) \neq R$ since $R \neq T$. Thus, as the only ideals of the SPIR $R$ are $0$, $M$, and $R$, we have $(R : T) = 0$. In addition, as $x^2 = 0 \in M$ and $xM = ppyR = p^2 Ry = 0 \cdot Ry = \{0\} \subseteq M$, it follows that $R \cap S$ is ramified, necessarily with crucial maximal ideal $M$ (cf. [3, Theorem 2.2]). Also, the unique prime ideal of $S$ is $N := M + Rx$ (cf. [3, Theorem 2.3]).

If $T = S$, then there exist $a, b \in R$ such that $y = a + bx = a +spy$, so that $a = (1 - bp)y \in R \cap Ry = \{0\}$, whence $1 - bp \in (0 :_R y) \subseteq M$ and $1 \in M$, a contradiction. Hence $S \subset T = S[y]$. Also $yx = ppy^2 = ppy = x$. As $y^2 - y = 0 \in N$ and $yN = y(M + Rx) = yM + Ryx \subseteq Rx \subseteq N$, it follows that $S \subset T$ is decomposed, necessarily with crucial maximal ideal $N$ (cf. [3, Theorem 2.2]). Finally, $N \cap R = M$ since $\text{Spec}(R) = \{M\}$. Thus, the data satisfy the hypotheses of [1, Theorem 2.8].

Note that $N/R$ is a Noetherian ring of Krull dimension 0, i.e., an Artinian ring. As $T$ is a finitely generated $R$-module, it follows from [3, Theorem 4.2 (a)] that $R \subset T$ satisfies the FCP property; i.e., each chain in (the poset) $(R, T)$ is finite. Also, basic facts about ramified extensions and decomposed extensions (as in [3, Theorem 2.3]) give that the extension $R \subset T$ is infra-integral (for each $P \in \text{Spec}(T) = \text{Max}(T)$, the canonical map $R/(P \cap R) \rightarrow T/P$ is an isomorphism of fields). Furthermore, the above-cited basic facts about ramified extensions give $M \cap N \subseteq N^2 \subseteq M$ and $\dim_{R/M}(S/M) = 2$. Thus, as $0 \subset N/M \subset S/M$, the length of $N/M$ as an $R$-module is $L_R(N/M) = \dim_{R/M}(N/M) = 1$. Therefore, it follows from [3, Lemma
5.4] that each maximal chain in \([R, T]\) has length
\[
\mathbf{L}_R(N/M) + \text{Max}(T) - 1 = 1 + 2 - 1 = 2.
\]

It suffices to get a contradiction from the supposed existence of some \(S' \in [R, T] \setminus \{R, S, T\}\). By the above conclusion about length, \(R \subset S'\) must be a (necessarily integral) minimal ring extension. We claim that \(R \subset S'\) is not an inert extension. If this claim fails, then \(M\) is a common maximal ideal of distinct members (namely \(R\) and \(S'\)) of \([R, T]\), which is a contradiction to \([3, \text{Lemma 5.2}]\) (which applies since \(R \subset T\) is an integral infra-integral extension). This proves the claim. Since \(R \subset S'\) is not inert, it must be either ramified or decomposed. Suppose, for the moment, that \(R \subset S'\) is ramified. Then \(R \subset S'\) is subintegral and so, by \([3, \text{Proposition 4.5 (b)}]\), \(S' \subseteq \frac{R}{T}\). Note that this seminormalization (of \(R\) in \(T\)) contains \(S\) (also by \([3, \text{Proposition 4.5 (b)}]\)) but cannot be \(T\). (Indeed, \(R \subset T\) is not subintegral because the “decomposed” hypothesis ensures that two distinct prime ideals of \(T\) meet \(S\) in \(N\) and, hence, meet \(R\) in \(M\).) Therefore, since \(S \subset T\) is a minimal ring extension, \(\frac{R}{T} = S\), and so \(S' \subseteq S\). Since \(R \subset S\) is a minimal extension, \(S'\) must be either \(R\) or \(S\), a contradiction to the choice of \(S'\). Therefore, \(R \subset S'\) is not ramified. Hence, \(R \subset S'\) is decomposed.

Since \(R \subset S'\) is decomposed (necessarily with crucial maximal ideal \(M\)), it follows that \(\text{Spec}(S') = \text{Max}(S') = \{N_1, N_2\}\) with \(N_1 \neq N_2\) and \(M = N_1 \cap N_2 = N_1 N_2\). Without loss of generality, we can take \(N_1\) to be the crucial maximal ideal of \(S' \subset T\). Then \(N_1 = (S' : T)\) by \([5, \text{Théorème 2.2 (i)}]\). Thus \(N_1\) is an ideal of \(T\), and so \(N_1 T = N_1\). Hence \(M = N_1 N_2 = (N_1 T) N_2 = N_1 (T N_2)\), which is a product of ideals of \(T\). Thus \(M\) is an ideal of \(T\), and so \(MT = M \subseteq R\). Hence \(M \subseteq (R : T) = 0\), and so \(M = 0\), the desired contradiction. 

\textbf{Proposition.} Let \(k\) be a field, \(k \subset S\) a ramified extension, and \(S \subset T\) a decomposed extension. Then \(|k, T| > 3\).

\textbf{Proof.} The assertion was established in the (valid) fourth paragraph of the published “proof” of \([1, \text{Theorem 2.8}]\). 

\textbf{Remark.} (a) There may be some question as to the validity of the process whereby the ideals \(I\) and \(J\) of \(S(+) R / M\) were obtained in paragraphs 6-8 of the published “proof” of \([1, \text{Theorem 2.8}]\). However, it is certain that the published “proof” of \([1, \text{Theorem 2.8}]\) was in error in its twelfth paragraph. Indeed, an error occurred on lines 3-5 of \([1, \text{page 39}]\), where an appeal was made to a result \([6, \text{Theorem 25.1 (1), (2)}]\) which is known to be false. That incorrect step purported to obtain certain descriptions of \(I\) and \(J\). (Note that after this use of the incorrect result from \([6]\), the remainder of the “proof” did make valid use of those improperly obtained descriptions.)

(b) In view of the above Example and Proposition, the statement of \([1, \text{Theorem 2.9}]\) needs to be revised as follows. Relocate condition (ix) (which pertains to data such that \(R \subset S\) is ramified with crucial maximal ideal \(M\) and \(S \subset T\) is decomposed with crucial maximal ideal \(N\) with \(N \cap R = M\), to part (c), rather than part (b), of the statement of \([1, \text{Theorem 2.9}]\). This correction needs to be taken into account in reading the final sentence of (c) below.

(c) Since receiving \([7]\), the author has collaborated with Picaret and Picavet-L’Hermitte. This work has produced a manuscript \([4]\) which gives necessary and sufficient conditions for the assertion of \([1, \text{Theorem 2.8}]\) to be valid in case the base ring \(R\) is either a PID (but not a field) \([4, \text{Theorem 2.8}]\) or an SPIR (\([4, \text{Theorem 2.2}]\). When these results are taken in conjunction with those of \([2]\) (which had been based in part on the result \([1, \text{Theorem 2.9}]\) mentioned above in (b)), one obtains a characterization of the (commutative unital) rings with exactly two proper (unital) subrings.

\textbf{References}


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