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Corrigendum to “When the juxtaposition of two minimal ring extensions produces no new intermediate rings"

David E. Dobbs

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The author recently received a message [7] from Gabriel Picavet and Martine Picavet-L'Hermitte, who kindly informed him that [1, Theorem 2.8] is incorrect. At that time, they provided a counterexample to [1, Theorem 2.8]. The Example given below presents that counterexample and (the author's rendering of) their proof of it. One should note that the first five paragraphs of the published “proof" of [1, Theorem 2.8] are correct. Contained therein is the proof of a special case where the statement of [1, Theorem 2.8] is correct. That partial result appears as the Proposition below. A closing Remark identifies where the published “proof" of [1, Theorem 2.8] was in error and announces some recent results.

Most of the following material refers to chains of (commutative unital) rings where the base ring is a special principal ideal ring (SPIR). Suitable background on SPIRs can be found in [8, page 245]. Our convention here is that no field can be an SPIR. As usual, if $A \subseteq B$ are rings, then $[A, B]$ denotes the set of rings C such that $A \subseteq C \subseteq B$.

Example. (G. Picavet and M. Picavet-L'Hermitte) Let (R, M) be an SPIR such that $M^2 = 0$. Fix $p \in R$ such that $M = Rp$ (so $p^2 = 0$). With Y an indeterminate over R, set $T := R[Y]/(Y^2 - Y)$ and $y := Y + (Y^2 - Y)$ (so that $y^2 = y$ and $T = R[y]$). Set $x := py$ and $S := R[x]$. The canonical ring homomorphism $R \to T$ is an injection and so we can view $R \subseteq S \subseteq T$. Then $R \subset S$ is a ramified extension (necessarily with crucial maximal ideal M), $S \subset T$ is a decomposed extension (whose crucial maximal ideal necessarily lies over M), and $|[R, T]| = 3$.

Proof. The canonical map $R \to T$ is injective since $R \cap (Y^2 - Y) = \{0\}$. One sees by similar degree arguments that $x \neq 0$ since $pY \notin (Y^2 - Y)$; and $R \cap Ry = \{0\}$, for if $r_1, r_2 \in R$ with $r_1 + r_2 Y \in (Y^2 - Y)$, then $r_1 = 0 = r_2$. If $x \in R$, then $x = py \in R \cap Ry = \{0\}$, a contradiction. Hence $x \notin R$, and so $R \subset S = R[x]$. Note also that $(R : T) \neq M$, since $py = x \notin R$. Of course, $(R : T) \neq R$ since $R \neq T$. Thus, as the only ideals of the SPIR R are $0, M$ and R, we have $(R : T) = 0$. In addition, as $x^2 = 0 \in M$ and $xM = pypR = p^2Ry = 0 \cdot Ry = \{0\} \subseteq M$, it follows that $R \subset S$ is ramified, necessarily with crucial maximal ideal M (cf. [3, Theorem 2.2]). Also, the unique prime ideal of S is $N := M + Rx$ (cf. [3, Theorem 2.3]).

If $S = T$, then there exist $a, b \in R$ such that $y = a + bx = a + bpy$, so that $a = (1 - bp)y \in R \cap Ry = \{0\}$, whence $1 - bp \in (0 :_R y) \subseteq M$ and $1 \in M$, a contradiction. Hence $S \subset T = S[y]$. Also $yx = py^2 = py = x$. As $y^2 - y = 0 \in N$ and $yN = y(M + Rx) = yM + Ryx \subseteq Rx \subseteq N$, it follows that $S \subset T$ is decomposed, necessarily with crucial maximal ideal N (cf. [3, Theorem 2.2]). Finally, $N \cap R = M$ since $\text{Spec}(R) = \{M\}$. Thus, the data satisfy the hypotheses of [1, Theorem 2.8].

Note that $R/(R : T) = R/0 \cong R$ is a Noetherian ring of Krull dimension 0, i.e., an Artinian ring. As T is a finitely generated R-module, it follows from [3, Theorem 4.2 (a)] that $R \subset T$ satisfies the FCP property; i.e., each chain in (the poset) $[R, T]$ is finite. Also, basic facts about ramified extensions and decomposed extensions (as in [3, Theorem 2.2]) give that the extension $R \subset T$ is infra-integral (for each $P \in \text{Spec}(T) = \text{Max}(T)$, the canonical map $R/(P \cap R) \to T/P$ is an isomorphism of fields). Furthermore, the above-cited basic facts about ramified extensions give $MN \subseteq N^2 \subseteq M$ and $\dim_{R/M}(S/M) = 2$. Thus, as $0 \subset N/M \subset S/M$, the length of N/M as an R-module is $\text{L}_R(N/M) = \dim_{R/M}(N/M) = 1$. Therefore, it follows from [3, Lemma

5.4] that each maximal chain in $[R, T]$ has length

$$\mathrm{L}_R(N/M) + \mathrm{Max}(T) - 1 = 1 + 2 - 1 = 2.$$

It suffices to get a contradiction from the supposed existence of some $S' \in [R, T] \setminus \{R, S, T\}$. By the above conclusion about length, $R \subset S'$ must be a (necessarily integral) minimal ring extension. We claim that $R \subset S'$ is not an inert extension. If this claim fails, M is a common maximal ideal of distinct members (namely R and S') of $[R, T]$, which is a contradiction to [3, Lemma 5.2] (which applies since $R \subset T$ is an integral infra-integral extension). This proves the claim. Since $R \subset S'$ is not inert, it must be either ramified or decomposed. Suppose, for the moment, that $R \subset S'$ is ramified. Then $R \subset S'$ is subintegral and so, by [3, Proposition 4.5 (b)], $S' \subseteq {}^+_R T$. Note that this seminormalization (of R in T) contains S (also by [3, Proposition 4.5 (b)]) but cannot be T. (Indeed, $R \subset T$ is not subintegral because the “decomposed" hypothesis ensures that two distinct prime ideals of T meet S in N and, hence, meet R in M.) Therefore, since $S \subset T$ is a minimal ring extension, ${}^+_R T = S$, and so $S' \subseteq S$. Since $R \subset S$ is a minimal extension, S' must be either R or S, a contradiction to the choice of S'. Therefore, $R \subset S'$ is not ramified. Hence, $R \subset S'$ is decomposed.

Since $R \subset S'$ is decomposed (necessarily with crucial maximal ideal M), it follows that $\mathrm{Spec}(S') = \mathrm{Max}(S') = \{N_1, N_2\}$ with $N_1 \neq N_2$ and $M = N_1 \cap N_2 = N_1 N_2$. Without loss of generality, we can take N_1 to be the crucial maximal ideal of $S' \subset T$. Then $N_1 = (S' : T)$ by [5, Théorème 2.2 (ii)]. Thus N_1 is an ideal of T, and so $N_1 T = N_1$. Hence $M = N_1 N_2 = (N_1 T) N_2 = N_1 (T N_2)$, which is a product of ideals of T. Thus M is an ideal of T, and so $MT = M \subseteq R$. Hence $M \subseteq (R : T) = 0$, and so $M = 0$, the desired contradiction. □

Proposition. Let k be a field, $k \subset S$ a ramified extension, and $S \subset T$ a decomposed extension. Then $|[k, T]| > 3$.

Proof. The assertion was established in the (valid) fourth paragraph of the published “proof" of [1, Theorem 2.8]. □

Remark. (a) There may be some question as to the validity of the process whereby the ideals I and J of $S(+)R/M$ were obtained in paragraphs 6-8 of the published “proof" of [1, Theorem 2.8]. However, it is certain that the published “proof" of [1, Theorem 2.8] was in error in its twelfth paragraph. Indeed, an error occurred on lines 3–5 of [1, page 39], where an appeal was made to a result [6, Theorem 25.1 (1), (2)] which is known to be false. That incorrect step purported to obtain certain descriptions of I and J. (Note that after this use of the incorrect result from [6], the remainder of the “proof" did make valid use of those improperly obtained descriptions.)

(b) In view of the above Example and Proposition, the statement of [1, Theorem 2.9] needs to be revised as follows. Relegate condition (ix) (which pertains to data such that $R \subset S$ is ramified with crucial maximal ideal M and $S \subset T$ is decomposed with crucial maximal ideal N with $N \cap R = M$), to part (c), rather than part (b), of the statement of [1, Theorem 2.9]. This correction needs to be taken into account in reading the final sentence of (c) below.

(c) Since receiving [7], the author has collaborated with Picavet and Picavet-L'Hermitte. This work has produced a manuscript [4] which gives necessary and sufficient conditions for the assertion of [1, Theorem 2.8] to be valid in case the base ring R is either a PID (but not a field) [4, Theorem 2.8] or an SPIR [4, Theorem 2.2]. When these results are taken in conjunction with those of [2] (which had been based in part on the result [1, Theorem 2.9] mentioned above in (b)), one obtains a characterization of the (commutative unital) rings with exactly two proper (unital) subrings.

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Author information

David E. Dobbs, Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, USA.
E-mail: dobbs@math.utk.edu, dedobbs@comporium.net

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