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On some new congruences for ℓ-regular overpartitions

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Abstract. Andrews investigated the function $\overline{C_{k,j}}(n)$ which counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm j \pmod k$ may be overlined. Let $\overline{A_\ell}(n)$ denote the number of ℓ-regular overpartitions of n. Very recently, Mahadeva Naika and Gireesh discovered some congruences for $\overline{C_{3,1}}(n)$ modulo 2^i3^j for some values of i and j and modulo 2^4 for $\overline{A_5}(n)$. Furthermore, they conjectured that $\overline{C_{3,1}}(12n+11) \equiv 0 \pmod{144}$. In this paper, we confirm this conjecture. We also establish several congruences for $\overline{A_5}(n)$ and $\overline{A_{3^r}}(n)$, $r \geq 2$ modulo 2^i3^j for few values of i and j.

1 Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots,$ λ_r such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition. We shall set $p(0) = 1$ and for $n \geq 1$, let $p(n)$ denote the number of partitions of n. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1}.$$

Here and throughout this paper, we assume that $|q| < 1$ and for any positive integer k, f_k is defined by

$$f_k := \prod_{n=1}^{\infty}(1 - q^{kn}).$$

In 1919, Ramanujan [16] found nice congruence properties for $p(n)$ moduli $5, 7$ and 11. Namely, for any nonnegative integer n,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Motivated by the above congruences, many mathematicians discovered many congruence properties for different partition functions such as singular overpartitions, ℓ-regular partitions, broken k-diamond partitions and ℓ-regular overpartitions. Among these, arithmetic properties of ℓ-regular overpartitons has received a great deal of attention. For a positive integer $l \geq 2$, a partition is called ℓ-regular if none of its parts is divisible by ℓ. An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the final occurrence of a part may be overlined.

In [13], Lovejoy proved the following theorem in the theory of overpartitions.

Theorem 1.1. *([13]) If $\overline{B_\ell}(n)$ denote the number of overpartitions of n of the form $y_1 + y_2 + \cdots +$ y_s, where $y_j - y_{j+\ell-1} \geq 1$ if $y_{j+\ell-1}$ is overlined and $y_j - y_{j+\ell-1} \geq 2$ otherwise. Let $\overline{A_\ell}(0) = 1$*

and for $n \geq 1$, let $\overline{A_\ell}(n)$ denote number of overpartitions of n with no parts divisible by ℓ. Then $\overline{A_\ell}(n) = \overline{B_\ell}(n)$.

The generating function for $\overline{A_\ell}(n)$ is given by [18]

$$\sum_{n=0}^{\infty} \overline{A_\ell}(n)q^n = \frac{f_\ell^2 f_2}{f_1^2 f_{2\ell}}. \tag{1.1}$$

Setting $\ell = 3$ in (1.1), Shen [18] observed that $\overline{A_3}(n) = \overline{C_{3,1}}(n)$, where $\overline{C_{k,j}}(n)$ counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm j \pmod{k}$ may be overlined. This function was introduced and investigated by Andrews in [3]. As noted in [3], the generating function for $\overline{C_{k,j}}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C_{k,j}}(n)q^n = \frac{(q^k;q^k)_\infty(-q^j;q^k)_\infty(-q^{k-j};q^k)_\infty}{(q;q)_\infty}, \tag{1.2}$$

where $k \geq 3$ and $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$. Using generating function dissection techniques, Shen [18] established several interesting congruences modulo $2, 6, 24$ for $\overline{A_3}(n)$ and modulo $3, 24$ for $\overline{A_4}(n)$. For example

Theorem 1.2. *([18])For all non-negative integer n,*

$$\begin{aligned}
\overline{A_3}(9n+3) &\equiv 0 \pmod{6},\\
\overline{A_3}(9n+6) &\equiv 0 \pmod{24},\\
\overline{A_4}(12n+8) &\equiv 0 \pmod{3},\\
\overline{A_4}(12n+7) &\equiv 0 \pmod{24}.
\end{aligned}$$

In the same paper, Shen gave a combinatorial interpretation of first two congruences in the above theorem by introducing the rank of vector partitions. Very recently, Mahadeva Naika and Gireesh [14] employed dissection formulas of certain quotients of theta functions to establish several infinite families of congruences for $\overline{C_{k,j}}(n)$ for different values of k and j. They also considered the function $\overline{A_5}(n)$ and proved some congruences modulo 16. For example, they proved the following theorems:

Theorem 1.3. *([14]) For all integers $n \geq 0$, we have*

$$\begin{aligned}
\overline{C_{3,1}}(8n+7) &\equiv 0 \pmod{12},\\
\overline{C_{3,1}}(8n+6) &\equiv 0 \pmod{24},\\
\overline{C_{3,1}}(24n+14) &\equiv 0 \pmod{72}.
\end{aligned}$$

Theorem 1.4. *([14]) Let $p \geq 5$ be prime and $\left(\frac{-2}{p}\right) = -1$.Then for all integers $n \geq 0$, $\alpha \geq 1$ and $1 \leq j \leq p-1$, we have*

$$\overline{A_5}\Big(8p^{2\alpha}n + p^{2\alpha-1}(3p+8j)\Big) \equiv 0 \pmod{2^4}.$$

In the same paper, they also proposed the following conjecture for $\overline{C_{3,1}}(n)$.

Conjecture 1.5. [14] For all integer $n \geq 0$,

$$\overline{C_{3,1}}(12n+11) \equiv 0 \pmod{144}.$$

Alanazi, Munagi and Sellers [2] established several Ramanujan type congruences for ℓ-regular overpatitions. In particular, Alanazi et al. [2] discovered the following theorem.

Theorem 1.6. *([2]) For all $n \geq 0$, we have $\overline{A_9}(6n+5) \equiv 0 \pmod{3}$.*

The following theorem was proved by Alanazi et al. [2] using a congruence relation due to Munagi and Sellers [15].

Theorem 1.7. *([2]) For all $n \geq 0$ and all $j \geq 3$, we have* $\overline{A_{3^j}}(27n+18) \equiv 0 \pmod 3$.

The main aim of this paper is to show that Conjecture 1.5 is true and also to prove some new congruences for $\overline{A_5}(n)$ and $\overline{A_{3^r}}(n)$. The paper is organized as follows: In Section 2, we recall some notations, definitions and also we collect some lemmas and theorems which are useful to prove our main results. In Section 3, we give a simple proof of Conjecture 1.5 and also establish a p-dissection formula for f_1^5/f_2^2 which seems to be new. In Section 4, we derive some new congruences modulo 8 and 16 for $\overline{A_5}(n)$. In Section 5, we discover several infinite families of congruences modulo 6, 8 and 16 for $\overline{A_9}(n)$. We also deduce Theorem 1.6 as a special case of one of our theorems. In Section 6, we prove infinite families of congruences for $\overline{A_{3^r}}(n)$, $r \geq 2$ modulo 3, 4, 8 and 16. We also provide a short and simple proof of the Theorem 1.7.

2 Set of preliminary results

In this section, we present some identities which are useful to prove our main results.

Let $p \geq 3$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo p and } p \nmid a, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo p and } p \nmid a, \\ 0 & \text{if } p \mid a. \end{cases}$$

For $|\, ab \,| < 1$, Ramanujan's general theta function $f(a,b)$ is defined by [1]

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

The following lemma is a consequence of Entry 25 of (i), (ii), (v) and (vi) in [1, pp. 35–36].

Lemma 2.1. *The following 2-dissection formulas are true:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8} \tag{2.1}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.2}$$

The following 2-dissection formula for $\frac{f_3^3}{f_1}$ was proved by Hirschhorn, Garvan and Borwein [9] and also by Xia and Yao [19].

Lemma 2.2. *The following 2-dissection formulas are true:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4} \tag{2.3}$$

and

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.4}$$

For a proof of (2.4), see [5] and [19].

From [8], we recall the following lemma.

Lemma 2.3. *The following 3-dissection formula holds:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}.$$

From [1, p.49], we recall the following p-dissection formula.

Lemma 2.4. *For any prime p, we have*

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_{2p^2}^5}{f_{p^2}^2 f_{4p^2}^2} + \sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}).$$

Theorem 2.5. *([7, Theorem 2.1]) For any odd prime p,*

$$\frac{f_2^2}{f_1} = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \frac{f_{2p^2}^2}{f_{p^2}}.$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$ *for* $0 \le m \le \frac{p-3}{2}$.

For all integers $n, k \ge 0$, let $t_k(n)$ (respectively $r_k(n)$) denote the number of representations of n as sum of k triangular (respectively square) numbers.

Theorem 2.6. *For $1 \le k \le 7$, we have*

$$r_k(8n+k) = 2^{k-1}\left\{2 + \binom{k}{4}\right\} t_k(n).$$

In [12], Hirschhorn and Sellers proved the following arithmetic identity for $a_3(n)$.

Theorem 2.7. *Let $p \equiv 2 \pmod{3}$. For all integers $n \ge 0$, we have*

$$a_3\left(p^{2\alpha} n + \frac{p^{2\alpha}-1}{3}\right) = a_3(n),$$

where $a_3(n)$ denote the number of 3-core partitions of n.

3 Proof of Conjecture 1.5 and a p-dissection formula

In this section, we give a simple proof of Conjecture 1.5 and also establish a p-dissection formula for f_1^5/f_2^2 which will be used to prove congruence properties for $\overline{A_5}(n)$ and $\overline{A_9}(n)$.

Theorem 3.1. *Conjecture 1.5 is true.*

Proof. On using Lemma 2.3, Yao [20] proved that

$$\sum_{n=0}^{\infty} \overline{C_{3,1}}(6n+5) q^n = 16 \frac{f_2^2 f_3^3 f_4^4}{f_1^9}. \tag{3.1}$$

By the binomial theorem, it is easy to check that, for all positive integers k and m,

$$f_k^{3m} \equiv f_k^{3m} \pmod{3}, \tag{3.2}$$

$$f_k^{9m} \equiv f_{3k}^{3m} \pmod{3^3}. \tag{3.3}$$

In view of congruence (3.3), we have

$$\sum_{n=0}^{\infty} \overline{C_{3,1}}(6n+5) q^n = 16 \frac{f_2^2 f_3^3 f_4^4}{f_1^9} \equiv 16 f_2^2 f_4^4 \pmod{144}. \tag{3.4}$$

Now, comparing the odd powers of q in (3.4), we obtain the required congruence. □

Theorem 3.2. *Let $p \ge 5$ be a prime. Then*

$$\frac{f_1^5}{f_2^2} = \sum_{\substack{k=-\frac{p-1}{2} \\ k \not\equiv \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{\frac{pn(3pn+6k+1)}{2}} \pm pq^{\frac{p^2-1}{24}} \frac{f_{p^2}^5}{f_{2p^2}^2}.$$

Furthermore, if $-\frac{p-1}{2} \le k \le \frac{p-1}{2}$, $k \ne \frac{\pm p-1}{6}$, *we have* $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

Proof. From [6, Corollary 1.3.21], we recall that

$$\frac{f_1^5}{f_2^2} = \sum_{n=-\infty}^{\infty} (6n+1) q^{\frac{3n^2+n}{2}}.$$

Dissecting the right side into p terms, we find that

$$\begin{aligned}
\frac{f_1^5}{f_2^2} &= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty} (6(pn+k)+1) q^{\frac{3(pn+k)^2+(pn+k)}{2}} \\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} q^{\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{\frac{pn(3pn+6k+1)}{2}} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{\frac{pn(3pn+6k+1)}{2}} \pm q^{\frac{p^2-1}{24}} \sum_{n=-\infty}^{\infty} p\,(6n+1) q^{\frac{p^2(3n^2+n)}{2}} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{\frac{pn(3pn+6k+1)}{2}} \pm p q^{\frac{p^2-1}{24}} \frac{f_{p^2}^5}{f_{2p^2}^2}.
\end{aligned}$$

If $\frac{3k^2+k}{2} \equiv \frac{p^2-1}{24} \pmod{p}$, which implies that $(6k+1)^2 \equiv 0 \pmod{p}$. This implies that $k = \frac{mp-1}{6}$ for some integer m. Since $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, we have $m = \pm 1$. Thus $k = \frac{\pm p-1}{6}$ which is a contradiction. □

4 Congruences for $\overline{A_5}(n)$ modulo powers of 2

In this section, we prove infinite families of congruences modulo 2^3 and 2^4 for $\overline{A_5}(n)$.

Theorem 4.1. *If $p \geq 5$ is a prime such that $\left(\frac{-5}{p}\right) = -1$ and $1 \leq j \leq p-1$, then for all non-negative integers n and α, we have*

$$\overline{A_5}\left(4p^{2\alpha+2}n + 4p^{2\alpha+1}j + p^{2\alpha+2}\right) \equiv 0 \pmod{2^3}, \tag{4.1}$$

$$\overline{A_5}\left(4 \cdot 5^{\alpha+1}n + 13 \cdot 5^{\alpha}\right) \equiv 0 \pmod{2^3}, \tag{4.2}$$

$$\overline{A_5}\left(4 \cdot 5^{\alpha+1}n + 17 \cdot 5^{\alpha}\right) \equiv 0 \pmod{2^3}. \tag{4.3}$$

Proof. In [14], Mahadeva Naika and Gireesh showed that

$$\sum_{n=0}^{\infty} \overline{A_5}(2n+1) q^n = 8q \frac{f_{10} f_4^2 f_8^4}{f_2^7} + 2 \frac{f_{10} f_4^{14}}{f_2^{11} f_8^4}. \tag{4.4}$$

Extracting the even powers of q in (4.4), we obtain

$$\sum_{n=0}^{\infty} \overline{A_5}(4n+1) q^n = 2 \frac{f_2^{14} f_5}{f_1^{11} f_4^4}. \tag{4.5}$$

By the binomial theorem, for any positive integers m and k, we have

$$f_{2k}^m \equiv f_k^{2m} \pmod{2}, \tag{4.6}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{2^2}. \tag{4.7}$$

From (4.5) and (4.7), we find that

$$\sum_{n=0}^{\infty} \overline{A_5}(4n+1)q^n \equiv 2\frac{f_1^5 f_5^5}{f_2^2 f_{10}^2} \pmod{2^3}. \tag{4.8}$$

Define

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f_1^5}{f_2^2}\frac{f_5^5}{f_{10}^2}. \tag{4.9}$$

Then, in view of (4.8) and (4.9), we have

$$\overline{A_5}(4n+1) \equiv 2a(n) \pmod{2^3}. \tag{4.10}$$

Using Lemma 3.2, we can rewrite (4.9) as

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n = &\left[\sum_{\substack{j=-\frac{p-1}{2}\\ j\not\equiv\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{\frac{3j^2+j}{2}} \sum_{n=-\infty}^{\infty}(6pn+6j+1)q^{\frac{pn(3pn+6j+1)}{2}} \pm pq^{\frac{p^2-1}{24}}\frac{f_{p^2}^5}{f_{2p^2}^2}\right]\\ &\times\left[\sum_{\substack{m=-\frac{p-1}{2}\\ m\not\equiv\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{5\frac{3m^2+m}{2}} \sum_{n=-\infty}^{\infty}(6pn+6m+1)q^{5\frac{pn(3pn+6m+1)}{2}} \pm pq^{5\frac{p^2-1}{24}}\frac{f_{5p^2}^5}{f_{10p^2}^2}\right]\end{aligned} \tag{4.11}$$

Let $p \geq 5$ be a prime with $\left(\frac{-5}{p}\right) = -1$. For $-\frac{p-1}{2} \leq j, m \leq \frac{p-1}{2}$, consider the following congruence equation

$$\frac{3j^2+j}{2} + 5\frac{3m^2+m}{2} \equiv \frac{p^2-1}{4} \pmod{p}, \tag{4.12}$$

which is equivalent to

$$(6j+1)^2 + 5(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-5}{p}\right) = -1$, the above congruence holds if and only if $j = m = \frac{\pm p-1}{6}$. So. in (4.11), extracting the terms involving $q^{pn+\frac{p^2-1}{4}}$ and then replacing q^p by q in the resulting congruence, we obtain

$$\sum_{n=0}^{\infty} a\Big(pn + \frac{p^2-1}{4}\Big)q^n = (-1)^{\frac{\pm p-1}{6}} p^2 \frac{f_p^5}{f_{2p}^2}\frac{f_{5p}^5}{f_{10p}^2}.$$

This implies that, for $1 \leq t \leq p-1$,

$$a\Big(p(pn+t)n + \frac{p^2-1}{4}\Big) = 0 \tag{4.13}$$

and

$$\sum_{n=0}^{\infty} a\Big(p^2 n + \frac{p^2-1}{4}\Big)q^n = (-1)^{\frac{\pm p-1}{6}} p^2 \frac{f_1^5}{f_2^2}\frac{f_5^5}{f_{10}^2}.$$

From the above identity and (4.9), we find that

$$a\Big(p^2 n + \frac{p^2-1}{4}\Big) = (-1)^{\frac{\pm p-1}{6}} p^2 a(n),$$

and by induction on $\alpha \geq 0$, we deduce

$$a\Big(p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\Big) = (-1)^{\frac{\pm p-1}{6}\alpha} p^{2\alpha} a(n).$$

Replacing n by $p^2n + pt + \frac{p^2-1}{4}$ $(1 \leq t \leq p-1)$ in the above identity and then invoking (4.13), we deduce that for $\alpha \geq 0$ and $n \geq 0$,

$$a\Big(p^{2\alpha+2}n + p^{2\alpha+1}t + \frac{p^{2\alpha+1}-1}{4}\Big) = 0. \tag{4.14}$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}t + \frac{p^{2\alpha+1}-1}{4}$ in (4.10) and then using (4.14), we obtain (4.1).

From [1, pp.82], we recall that

$$f_1 = f_{25}\frac{f(-q^{15},-q^{10})}{f(-q^{20},-q^{5})} - q^2\frac{f(-q^{20},-q^{5})}{f(-q^{15},-q^{10})}f_{25} - qf_{25}. \tag{4.15}$$

In view of (4.8), (4.15) and by induction, we find that for all non-negative integers n and α

$$\sum_{n=0}^{\infty} \overline{A_5}(4\cdot 5^{\alpha}n + 5^{\alpha})q^n \equiv 2(-1)^{\alpha} f_1 f_5 \pmod{2^3}.$$

Substituting (4.15) into the above congruence and then equating the coefficients of q^{5n+3} and q^{5n+4} in the resulting congruence, we obtain the remaining two congruences of the above theorem. □

Theorem 4.2. *Let p be an odd prime and N be a positive integer with $p \nmid N$ such that $pN \equiv 3 \pmod{2^3}$. Let $\alpha \geq 0$ be an integer.*
(1) If $p \equiv -1 \pmod{2^4}$, *then* $\overline{A_5}(p^{4\alpha+3}N) \equiv 0 \pmod{2^4}$,
(2) If $p \equiv 3, 11 \pmod{2^4}$, *then* $\overline{A_5}(p^{16\alpha+15}N) \equiv 0 \pmod{2^4}$,
(3) If $p \equiv 1, 5, 9 \pmod{2^4}$, *then* $\overline{A_5}(p^{32\alpha+31}N) \equiv 0 \pmod{2^4}$,
(4) If $p \equiv 7 \pmod{2^4}$, *then* $\overline{A_5}(p^{8\alpha+7}N) \equiv 0 \pmod{2^4}$,
(5) If $p \equiv 13 \pmod{2^4}$, *then* $\overline{A_5}(p^{64\alpha+63}N) \equiv 0 \pmod{2^4}$.

Proof. Hirschhorn and Sellers [11] obtained the following 2−dissection formula:

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{4.16}$$

From (2.1), (4.4) and (4.16), we find that

$$\sum_{n=0}^{\infty} \overline{A_5}(4n+3)q^n \equiv 8\frac{f_4^4 f_8 f_{20}^2}{f_2^3 f_{40}} + 8q\frac{f_4^7 f_{10} f_{40}}{f_2^4 f_8 f_{20}} \pmod{2^4}. \tag{4.17}$$

Extracting the even powers of q in (4.17) and then using (4.6), we find that

$$\sum_{n=0}^{\infty} \overline{A_5}(8n+3)q^n \equiv 8\frac{f_2^6}{f_1^3} = 8\sum_{n=0}^{\infty} t_3(n)q^n \pmod{2^4}.$$

Equating the coefficients of q^n on both sides of the above congruence, we obtain

$$\overline{A_5}(8n+3) \equiv 8t_3(n) \pmod{2^4}.$$

Setting $k = 3$ in Theorem 2.6, we obtain $r_3(8n+3) = 8t_3(n)$. Hence

$$\overline{A_5}(8n+3) \equiv r_3(8n+3) \pmod{2^4}. \tag{4.18}$$

Hirschhorn and Sellers [10] proved that if $p \geq 3$ is a prime and n is a positive integer, then

$$r_3(p^{2\alpha}n) = \left(\frac{p^{\alpha+1}-1}{p-1} - \left(\frac{-n}{p}\right)\frac{p^{\alpha}-1}{p-1}\right) r_3(n) - p\,\frac{p^{\alpha}-1}{p-1}\,r_3(n/p^2), \quad \alpha \geq 0. \tag{4.19}$$

Here $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol and we take $r_3(n/p^2) = 0$ if $p^2 \nmid n$.

Replacing n by pN $(p \nmid N)$ in (4.19), we get

$$r_3(p^{2\alpha+1}N) = \left(\frac{p^{\alpha+1}-1}{p-1}\right) r_3(pN). \tag{4.20}$$

By (4.20), if $p \equiv -1 \pmod{2^4}$, then

$$r_3(p^{2\alpha+1}N) \equiv \begin{cases} 0 \pmod{16} & \text{if } \alpha \text{ is odd,} \\ r_3(pN) \pmod{16} & \text{if } \alpha \text{ is even.} \end{cases}$$

The above congruence implies that $r_3(p^{4\alpha+3}N) \equiv 0 \pmod{2^4}$. Setting $n = \frac{p^{4\alpha+3}N-3}{8}$ in (4.18), we obtain

$$\overline{A_5}(p^{4\alpha+3}N) \equiv r_3(p^{4\alpha+3}N) \equiv 0 \pmod{2^4}.$$

This completes the proof of (1).

Let $p \equiv 3, 11 \pmod{16}$. Replacing α by $8\alpha+7$ in (4.20) and using the fact that

$$\frac{p^{8\alpha+8}-1}{p-1} = 1 + p + \cdots + p^{8\alpha+7} \equiv 0 \pmod{2^4},$$

we obtain

$$r_3(p^{16\alpha+15}N) \equiv 0 \pmod{2^4}. \tag{4.21}$$

Putting $n = \frac{p^{8\alpha+7}N-3}{8}$ in (4.18) and then using the above congruence, we get (2). The other statements follow in a similar way. □

Theorem 4.3. *Let $p \geq 3$ be a prime and $N, \alpha \geq 1$ are integers.*
(1) If $p \equiv 7 \pmod{2^4}$, then $\overline{A_5}(p^{8\alpha}(8N+3)) \equiv \overline{A_5}(8N+3) \pmod{2^4}$,
(2) If $p \equiv 1, 5, 9 \pmod{2^4}$, then $\overline{A_5}(p^{32\alpha}(8N+3)) \equiv \overline{A_5}(8N+3) \pmod{2^4}$,
(3) If $p \equiv -1 \pmod{2^4}$, then $\overline{A_5}(p^{4\alpha}(8N+3)) \equiv \overline{A_5}(8N+3) \pmod{2^4}$,
(4) If $p \equiv 3, 11 \pmod{2^4}$, then $\overline{A_5}(p^{16\alpha}(8N+3)) \equiv \overline{A_5}(8N+3) \pmod{2^4}$,
(5) If $p \equiv 13 \pmod{2^4}$, then $\overline{A_5}(p^{64\alpha}(8N+3)) \equiv \overline{A_5}(8N+3) \pmod{2^4}$.

Proof. We give a proof of (1).The proof of other congruences follows similarly. Replacing n by $p^2(8N+3)$ and α by $4\alpha+3$ in (4.19), we obtain

$$r_3(p^{8\alpha+8}(8N+3)) = r_3(p^2(8N+3))\frac{p^{8\alpha+8}-1}{p-1} - r_3(8N+3)p\,\frac{p^{8\alpha+7}-1}{p-1} \quad (\alpha \geq 0). \tag{4.22}$$

If $p \equiv 7 \pmod{16}$, then we have

$$\frac{p^{8\alpha+8}-1}{p-1} = 1 + p + \cdots + p^{8\alpha+7} \equiv 0 \pmod{2^4}$$

and

$$p\,\frac{p^{8\alpha+7}-1}{p-1} = p + p^2 + \cdots + p^{8\alpha+6} \equiv -1 \pmod{2^4}.$$

Using above two congruences in (4.22), we get

$$r_3(p^{8\alpha+8}(8N+3)) \equiv r_3(8N+3) \pmod{2^4}. \tag{4.23}$$

Putting $n = \frac{p^{32\alpha+32}(8N+3)-3}{8}$ in (4.18) and then using (4.23) and (4.18), we get the required result. □

Theorem 4.4. *If $p \geq 3$ is a prime with $\left(\frac{-10}{p}\right) = -1$, then for all non-negative integers n and α,*

$$\overline{A_5}\left(p^{2\alpha}8n + 7p^{2\alpha}\right) \equiv 8f_2^3 f_5^3 \pmod{2^4}. \tag{4.24}$$

Moreover, for $1 \leq r \leq p-1$,

$$\overline{A_5}\left(p^{2\alpha+2}(8n+7) + 8p^{2\alpha+1}r\right) \equiv 0 \pmod{2^4}.$$

Proof. Extracting the terms involving q^{2n+1} in (4.17) and then using (4.6), we deduce that

$$\sum_{n=0}^{\infty} \overline{A_5}(8n+7)q^n = 8f_2^3 f_5^3 \pmod{2^4}. \tag{4.25}$$

Thus (4.24) is true for $\alpha = 0$. In view of Theorem 2.5 and (4.7), we have

$$f_1^3 \equiv \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} f_{p^2}^3 \pmod{2^2}. \tag{4.26}$$

Assume that (4.24) holds for $\alpha = j$. With the aid of (4.26), we can rewrite (4.24) with $\alpha = j$ as

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A_5}(p^{2j}8n + 7p^{2j})q^n =& 8\left[\sum_{m=0}^{\frac{p-3}{2}} q^{2\frac{m^2+m}{2}} f\left(q^{2\frac{p^2+(2m+1)p}{2}}, q^{2\frac{p^2-(2m+1)p}{2}}\right) + q^{2\frac{p^2-1}{8}} f_{2p^2}^3\right] \\ &\times \left[\sum_{k=0}^{\frac{p-3}{2}} q^{5\frac{k^2+k}{2}} f\left(q^{5\frac{p^2+(2k+1)p}{2}}, q^{5\frac{p^2-(2k+1)p}{2}}\right) + q^{5\frac{p^2-1}{8}} f_{5p^2}^3\right] \pmod{2^4}.\end{aligned} \tag{4.27}$$

Now consider the congruence equation,

$$m^2 + m + 5 \cdot \frac{k^2+k}{2} \equiv 7 \cdot \frac{p^2-1}{8} \pmod{p}.$$

where $0 \leq m, k \leq \frac{p-3}{2}$ and p is a prime such that $\left(\frac{-10}{p}\right) = -1$. We can rewrite the above congruence as follows:

$$(4m+2)^2 + 10(2k+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-10}{p}\right) = -1$, it implies that

$$4m + 2 = 2k + 1 \equiv 0 \pmod{p}.$$

Thus $m = k = \frac{p-1}{2}$. Using the above fact in (4.27), extracting the terms involving $q^{pn+7\frac{p^2-1}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A_5}\left(8p^{2j+1}n + 7p^{2j+2}\right)q^n \equiv 8f_{2p}^3 f_{5p}^3 \pmod{2^4}. \tag{4.28}$$

Again Extracting the terms involving q^p in the above congruence, we see that (4.24) is true for $\alpha = j+1$. Hence the proof of (4.24).

Next, comparing the coefficients of q^{pn+r} for $1 \leq r \leq p-1$ in (4.28), we obtain

$$\overline{A_5}\left(8p^{2j+1}(pn+r) + 7p^{2j+2}\right) = 0 \pmod{2^4}.$$

□

Theorem 4.5. *For all integers $n, \alpha \geq 0$, $j \in \{642, 842\}$ and $k \in \{242, 3242\}$, we have*

$$\overline{A_5}\Big(5^{2\alpha}\Big(10^3 n + j\Big) - 35\Big) \equiv 0 \pmod{2^4} \tag{4.29}$$

and

$$\overline{A_5}\Big(5^{2\alpha}\Big(5 \cdot 10^3 n + k\Big) - 35\Big) \equiv 0 \pmod{2^4}. \tag{4.30}$$

Proof. Setting $p = 5$ in (4.26), we obtain

$$f_1^3 \equiv f(q^{10}, q^{15}) + q f(q^5, q^{20}) + q^3 f_{25}^3 \pmod{2^2}. \tag{4.31}$$

Let $b(n)$ be defined by

$$\sum_{n=0}^{\infty} b(n) q^n = f_2^3 f_5^3. \tag{4.32}$$

Then from (4.25), we have

$$\overline{A_5}(8n+7) \equiv 8b(n) \pmod{2^4}. \tag{4.33}$$

In view of (4.31) and (4.32), we see that

$$\sum_{n=0}^{\infty} b(n) q^n \equiv f(q^{20}, q^{30}) f_5^3 + q^2 f(q^{10}, q^{40}) f_5^3 + q^6 f_{50}^3 f_5^3 \pmod{2^2}.$$

Equating the coefficients of q^{5n+3}, q^{5n+4} and q^{5n+1} in the above congruence, we find that

$$b(5n+3) \equiv b(5n+4) \equiv 0 \pmod{2^2}, \tag{4.34}$$

$$\sum_{n=0}^{\infty} b(5n+1) q^n \equiv q f_1^3 f_{10}^3 \pmod{2^2}.$$

Employing (4.31) in the above congruence and then equating the coefficients of q^{5n}, q^{5n+3} and q^{5n+4}, we obtain

$$b(25n+1) \equiv b(25n+16) \equiv 0 \pmod{2^2}, \tag{4.35}$$

$$\sum_{n=0}^{\infty} b(25n+21) q^n \equiv f_2^3 f_5^3 \pmod{2^2}. \tag{4.36}$$

In view of (4.32), (4.36) and by mathematical induction, we find that for $\alpha, n \geq 0$

$$b\Big(5^{2\alpha+2} n + 21 \cdot \frac{5^{2\alpha} - 1}{4}\Big) \equiv b(n) \pmod{2^2}. \tag{4.37}$$

Replacing n by $5n+3$ and $5n+4$ in (4.37) and then using (4.34), we obtain

$$b\Big(5^{2\alpha+2}(5n+3) + 21 \cdot \frac{5^{2\alpha} - 1}{4}\Big) \equiv b\Big(5^{2\alpha+2}(5n+4) + 21 \cdot \frac{5^{2\alpha} - 1}{4}\Big) \equiv 0 \pmod{2^2}. \tag{4.38}$$

From (4.33) and (4.38), we deduce that

$$\overline{A_5}\Big(5^{2\alpha}(10^3 n + 642) - 35\Big) \equiv \overline{A_5}\Big(5^{2\alpha}(10^3 n + 842) - 35\Big) \equiv 0 \pmod{2^4}.$$

This completes the proof of (4.29). In a similar way, remaining one follows from (4.33), (4.35) and (4.37). □

5 Congruences modulo powers of 2 and 6 for $\overline{A_9}(n)$

In this section, we prove several infinite families of congruences for $\overline{A_9}(n)$ modulo $2^2, 6, 2^3$ and 2^4. The following lemma gives the generating functions for $\overline{A_9}(4n+1)$ and $\overline{A_9}(4n+3)$.

Lemma 5.1. *We have*

$$\sum_{n=0}^{\infty} \overline{A_9}(4n+1)q^n = 2\frac{f_3^2 f_2^{14}}{f_1^{12} f_4^4} \tag{5.1}$$

and

$$\sum_{n=0}^{\infty} \overline{A_9}(4n+3)q^n = 8\frac{f_3^2 f_2^2 f_4^4}{f_1^8}. \tag{5.2}$$

Proof. Setting $l = 9$ in (1.1), we have

$$\sum_{n=0}^{\infty} \overline{A_9}(n)q^n = \frac{f_9^2 f_2}{f_1^2 f_{18}}. \tag{5.3}$$

Xia and Yao [19] found the following 2-dissection formula for $\frac{f_9}{f_1}$:

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}. \tag{5.4}$$

In view of (5.4), we have

$$\frac{f_2}{f_{18}}\frac{f_9^2}{f_1^2} = \frac{f_2}{f_{18}}\left(\frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}\right)^2 = \frac{f_{18} f_{12}^6}{f_2^3 f_6^2 f_{36}^2} + 2q\frac{f_{12}^2 f_4^2}{f_2^4} + q^2\frac{f_4^4 f_6^2 f_{36}^2}{f_2^5 f_{18} f_{12}^2}. \tag{5.5}$$

Combining (5.5) and (5.3) and then extracting the terms involving q^{2n+1} in the resulting identity, we obtain

$$\sum_{n=0}^{\infty} \overline{A_9}(2n+1)q^n = 2\frac{f_6^2 f_2^2}{f_1^4}. \tag{5.6}$$

With the help of (2.2), we can rewrite the above identity as follows:

$$\sum_{n=0}^{\infty} \overline{A_9}(2n+1)q^n = 8q\frac{f_6^2 f_4^2 f_8^4}{f_2^8} + 2\frac{f_6^2 f_4^{14}}{f_2^{12} f_8^4}. \tag{5.7}$$

Extracting the even powers of q and the odd powers of q in (5.7), we arrive at (5.1) and (5.2) respectively.

Theorem 5.2. *If $p \geq 5$ is a prime with $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq p-1$, then for all non-negative integers n and α, we have*

$$\overline{A_9}\left(p^{2\alpha+2}(8n+3) + 8p^{2\alpha+1}j\right) \equiv 0 \pmod{2^4}.$$

Proof. Substituting (2.1) and (2.4) into (5.2), we get

$$\sum_{n=0}^{\infty} \overline{A_9}(4n+3)q^n = 8\frac{f_2^2}{f_4^4}\frac{f_3^2}{f_1^2}\frac{1}{f_1^6} \equiv 8\frac{f_4^8 f_6 f_{12}^2 f_8^{14}}{f_2^{18} f_{24} f_{16}^6} \pmod{2^4}. \tag{5.8}$$

Employing (4.6), we deduce that

$$\frac{f_2^8 f_3 f_6^2 f_4^{14}}{f_1^{18} f_{12} f_8^6} \equiv f_2^3\frac{f_3^5}{f_6^2} \pmod{2}.$$

Extracting the even powers of q in (5.8) and then using the above congruence, we find that

$$\sum_{n=0}^{\infty}\overline{A_9}(8n+3)q^n \equiv 8f_2^3\frac{f_3^5}{f_6^2} \pmod{2^4}. \tag{5.9}$$

Using Lemma 3.2 and (4.26), we can rewrite the above congruence as

$$\overline{A_9}(8n+3)q^n \equiv 8\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3\frac{3k^2+k}{2}}\sum_{n=-\infty}^{\infty}(6pn+6k+1)q^{3\frac{pn(3pn+6k+1)}{2}} \pm pq^{\frac{p^2-1}{8}}\frac{f_{3p^2}^5}{f_{6p^2}^2}\right]$$
$$\times\left[\sum_{m=0}^{\frac{p-3}{2}} q^{m^2+m}f\left(q^{2\frac{p^2+(2m+1)p}{2}},q^{2\frac{p^2-(2m+1)p}{2}}\right)+q^{\frac{p^2-1}{4}}f_{2p^2}^3\right] \pmod{2^4}. \tag{5.10}$$

Let $p\geq 5$ be prime with $\left(\frac{-2}{p}\right)=-1$. For $0\leq m\leq\frac{p-3}{2}$ and $-\frac{p-1}{2}\leq k\leq\frac{p-1}{2}$, we consider the congruence equation

$$m^2+m+3\cdot\frac{3k^2+k}{2}\equiv 3\cdot\frac{p^2-1}{8} \pmod{p}. \tag{5.11}$$

We can rewrite the above congruence as follows:

$$2(2m+1)^2+(6k+1)^2\equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right)=-1$, it implies that

$$2m+1=6k+1\equiv 0 \pmod{p}.$$

Thus, the congruence (5.11) holds if and only if $m=\frac{p-1}{2}$ and $k=\frac{p-1}{6}$. Using the above fact in (5.10), extracting the terms involving $q^{p^2n+3\frac{p^2-1}{8}}$ and then replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty}\overline{A_9}\left(8p^2n+3p^2\right)q^n \equiv 8f_2^3\frac{f_3^5}{f_6^2} \pmod{2^4}. \tag{5.12}$$

From (5.9), (5.12) and by mathematical induction, we find that for $\alpha\geq 0$ and $n\geq 0$

$$\sum_{n=0}^{\infty}\overline{A_9}\left(8p^{2\alpha}n+3p^{2\alpha}\right)q^n \equiv 8f_2^3\frac{f_3^5}{f_6^2} \pmod{2^4}. \tag{5.13}$$

Again employing Lemma 3.2 and (4.26) into (5.13), extracting the terms involving $q^{pn+3\frac{p^2-1}{8}}$ in the resulting congruence and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{A_9}\left(8p^{2\alpha}\left(pn+3\frac{p^2-1}{8}\right)+3p^{2\alpha}\right)q^n \equiv 8f_{2p}^3\frac{f_{3p}^5}{f_{6p}^2} \pmod{2^4}.$$

Equating the coefficients of q^{pn+j} for $1\leq j\leq p-1$, we obtain the required congruence. □

Remark 5.3. Equating the coefficients of odd powers of q in (5.8), we see that for $n\geq 0$

$$\overline{A_9}(8n+7)\equiv 0 \pmod{2^4}.$$

Theorem 5.4. *If $p\geq 5$ is a prime with $\left(\frac{-1}{p}\right)=-1$ and $1\leq j\leq p-1$, then for all non-negative integers n and α,*

$$\overline{A_9}\left(p^{2\alpha+2}(8n+5)+8p^{2\alpha+1}j\right)\equiv 0 \pmod{2^3}.$$

Proof. In view of (2.1), (2.4) and (5.1), modulo 4, we find that

$$\frac{f_3^2 f_2^{14}}{f_1^{12} f_4^4} \equiv 2q\left(\frac{f_4^2 f_6 f_{12}^2 f_8^{18}}{f_2^{16} f_{24} f_{16}^6} + \frac{f_6^2 f_8^{26} f_{24}}{f_2^{15} f_4^3 f_{12} f_{16}^{10}}\right) + \frac{f_6 f_{12}^2 f_8^{24}}{f_2^{16} f_{24} f_{16}^{10}}. \tag{5.14}$$

Combining (5.1) and (5.14), extracting the odd powers of q and then using (4.6), we deduce

$$\sum_{n=0}^{\infty} \overline{A_9}(8n+5)q^n \equiv 4f_4^3 \frac{f_3^5}{f_6^2} + 4f_1^3 \frac{f_{12}^5}{f_{24}^2} \pmod{2^3}. \tag{5.15}$$

Now, we consider the following two congruences:

$$3\frac{3j^2+j}{2} + 2m^2 + 2m \equiv 5\frac{p^2-1}{8} \pmod{p}, \tag{5.16}$$

$$18j^2 + 6j + \frac{m^2+m}{2} \equiv 5\frac{p^2-1}{8} \pmod{p}. \tag{5.17}$$

where $0 \le m \le \frac{p-3}{2}$, $-\frac{p-1}{2} \le j \le \frac{p-1}{2}$ and $p \ge 5$ is a prime such that $\left(\frac{-1}{p}\right) = -1$. We can rewrite above congruences as follows:

$$(6j+1)^2 + (4m+2)^2 \equiv 0 \pmod{p},$$

$$(12j+2)^2 + (2m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-1}{p}\right) = -1$, above two congruence implies that

$$6j+1 = 2m+1 \equiv 0 \pmod{p}.$$

Thus, the congruences (5.16) and (5.17) holds if and only if $m = \frac{p-1}{2}$ and $j = \frac{p-1}{6}$. Substituting Lemma 3.2 and (4.26) into (5.15), using the above fact in the resulting congruence and then extracting the terms involving $q^{p^2 n + 5\frac{p^2-1}{8}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{A_9}\left(8p^2 n + 5p^2\right)q^n \equiv 4f_4^3 \frac{f_3^5}{f_6^2} + 4f_1^3 \frac{f_{12}^5}{f_{24}^2} \pmod{2^3}. \tag{5.18}$$

From (5.15), (5.18) and by mathematical induction, we see that for $\alpha \ge 0$ and $n \ge 0$

$$\sum_{n=0}^{\infty} \overline{A_9}\left(8p^{2\alpha} n + 5p^{2\alpha}\right)q^n \equiv 4f_4^3 \frac{f_3^5}{f_6^2} + 4f_1^3 \frac{f_{12}^5}{f_{24}^2} \pmod{2^3}. \tag{5.19}$$

Again employing Lemma 3.2 and (4.26) into (5.19), extracting the terms involving $q^{pn+5\frac{p^2-1}{8}}$ in the resulting congruence and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A_9}\left(8p^{2\alpha}\left(pn + 5\frac{p^2-1}{8}\right) + 5p^{2\alpha}\right)q^n \equiv 4f_{4p}^3 \frac{f_{3p}^5}{f_{6p}^2} + 4f_p^3 \frac{f_{12p}^5}{f_{24p}^2} \pmod{2^3}.$$

Equating the coefficients of q^{pn+j} for $1 \le j \le p-1$ in the above congruence, we obtain the required result. □

Theorem 5.5. *If $p \ge 5$ is a prime with $\left(\frac{-2}{p}\right) = -1$ and $1 \le j \le p-1$, then for all non-negative integers n and α,*

$$\overline{A_9}\left(p^{2\alpha+2}(8n+1) + 8p^{2\alpha+1} j\right) \equiv 0 \pmod{2^3}.$$

Proof. Combining (5.1) and (5.14), extracting the even powers of q and then using (4.7), we see that

$$\sum_{n=0}^{\infty} \overline{A_9}(8n+1)q^n \equiv 2\frac{f_3^5}{f_6^2}\frac{f_6^2}{f_{12}} \pmod{2^3}. \tag{5.20}$$

Using Lemma 2.4 with q replaced by $-q$ and Lemma 3.2 in (5.20), we have

$$\sum_{n=0}^{\infty} \overline{A_9}\left(p^{2j}8n+p^{2j}\right)q^n \equiv 2\left[\sum_{\substack{k=\frac{-p-1}{2}\\ k\not\equiv\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty}(6pn+6k+1)q^{3\frac{pn(3pn+6k+1)}{2}} \pm pq^{\frac{p^2-1}{8}}\frac{f_{p^2}^5}{f_{2p^2}^2}\right]$$
$$\times\left[\frac{f_{6p^2}^2}{f_{12p^2}}+\sum_{r=1}^{p-1}(-1)^r q^{6r^2} f(-q^{6p(p-2r)},-q^{6p(p+2r)})\right] \pmod{2^3}. \tag{5.21}$$

Let $p\geq 5$ be a prime with $\left(\frac{-2}{p}\right)=-1$. For $-\frac{p-1}{2}\leq k\leq\frac{p-1}{2}$ and $1\leq r\leq p-1$, consider the congruence equation

$$3\frac{3k^2+k}{2}+r^2\equiv\frac{p^2-1}{8} \pmod{p}, \tag{5.22}$$

which is equivalent to

$$(6k+1)^2+2(2r)^2\equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right)=-1$, the only solution of the congruence (5.22) is $k=\frac{\pm p-1}{6}$ and $r=0$. Using the above fact in (5.21), extracting the terms involving $q^{p^2n+\frac{p^2-1}{8}}$ and then replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A_9}\left(8p^2n+p^2\right)q^n \equiv \pm 2p\frac{f_3^5}{f_6^2}\frac{f_6^2}{f_{12}} \pmod{2^3}. \tag{5.23}$$

From (5.20), (5.23) and by induction, we find that for $n\geq 0$ and $\alpha\geq 0$,

$$\sum_{n=0}^{\infty} \overline{A_9}\left(8p^{2\alpha}n+p^{2\alpha}\right)q^n \equiv 2\left(\pm p\right)^{\alpha}\frac{f_3^5}{f_6^2}\frac{f_6^2}{f_{12}} \pmod{2^3}. \tag{5.24}$$

Substituting Lemma 2.4 with q replaced by $-q$ and Lemma 3.2 into (5.24), extracting the terms invloving $q^{pn+\frac{p^2-1}{8}}$ in the resulting congruence, we deduce that

$$\sum_{n=0}^{\infty} \overline{A_9}\left(8p^{2\alpha}\left(pn+\frac{p^2-1}{8}\right)+p^{2\alpha}\right)q^n \equiv 2\left(\pm p\right)^{\alpha+1}\frac{f_{3p}^5}{f_{6p}^2}\frac{f_{6p}^2}{f_{12p}} \pmod{2^3}. \tag{5.25}$$

Equating the coefficients of q^{pk+j} for $1\leq j\leq p-1$ in (5.25), we obtain

$$\overline{A_9}\left(p^{2\alpha+1}8(pn+j)+p^{2\alpha+2}\right)\equiv 0 \pmod{2^3}.$$

Hence the proof. □

Theorem 5.6. *If p is a odd prime such that $\left(\frac{-3}{p}\right)=-1$ and $1\leq k\leq p-1$, then for all integers $n\geq 0$ and $\alpha\geq 0$*

$$\overline{A_9}\left(2p^{2\alpha+2}n+2p^{2\alpha+1}k+p^{2\alpha+2}\right)\equiv 0 \pmod{6}, \tag{5.26}$$

$$\overline{A_9}\left(3^{\alpha}(2n+1)\right)\equiv\overline{A_9}(2n+1) \pmod{6}, \tag{5.27}$$

$$\overline{A_9}\left(3^{\alpha}(6n+5)\right)\equiv 0 \pmod{6}. \tag{5.28}$$

Proof. It follows from (3.2) and (5.6) that

$$\sum_{n=0}^{\infty} \overline{A_9}(2n+1)q^n \equiv 2\frac{f_6^2}{f_3}\frac{f_2^2}{f_1} \pmod{6}. \tag{5.29}$$

Let p be odd prime such that $\left(\frac{-3}{p}\right) = -1$ and for $0 \le m, j \le \frac{p-3}{2}$, the following relation

$$3 \cdot \frac{m^2+m}{2} + \frac{j^2+j}{2} \equiv \frac{p^2-1}{2} \pmod{p}$$

holds if and only if $m = j = \frac{p-1}{2}$. From Theorem 2.5, (5.29) and by induction α, we find that for all integer $n \ge 0$

$$\sum_{n=0}^{\infty} \overline{A_9}\left(2p^{2\alpha}n + p^{2\alpha}\right)q^n \equiv 2\frac{f_6^2}{f_3}\frac{f_2^2}{f_1} \pmod{6}.$$

Now, substituting Theorem 2.5 into the above congruence and then extracting the terms involving $q^{pn+\frac{p^2-1}{2}}$, we deduce

$$\sum_{n=0}^{\infty} \overline{A_9}\left(2p^{2\alpha}\left(pn + \frac{p^2-1}{2}\right) + p^{2\alpha}\right)q^n \equiv 2\frac{f_{6p}^2}{f_{3p}}\frac{f_{2p}^2}{f_p} \pmod{6}.$$

Equating the coefficients of q^{pn+k} for $1 \le k \le p-1$ in the above congruence, we arrive at (5.26).

Form [1, pp.49], we recall that

$$\frac{f_2^2}{f_1} = f(q^3, q^6) + q\frac{f_{18}^2}{f_9}. \tag{5.30}$$

In view of (5.30), (5.29) and by induction, we arrive at (5.27) and (5.28). □

Remark 5.7. Setting $\alpha = 0$ in (5.28), we obtain Theorem 1.6.

6 Congruences modulo powers of 2 and 3 for $\overline{A_{3^r}}(n)$

In this section, by employing (2.3) and Lemma 2.3, we find several congruences modulo $2^2, 2^3$, 2^4 and 3 for $\overline{A_{3^r}}(n)$, $r \ge 2$.

Lemma 6.1. *We have*

$$\overline{A_{3^r}}(9n+3) \equiv 8a_3(n) \pmod{2^4}, \tag{6.1}$$

$$\overline{A_{3^r}}(6n+2) \equiv 4a_3(n) \pmod{2^3}, \tag{6.2}$$

$$\overline{A_{3^r}}(3n+1) \equiv 2a_3(n) \pmod{2^2}, \tag{6.3}$$

where $a_3(n)$ denote the number of 3−cores of n.

Proof. Setting $l = 3^r$ $(r \ge 2)$ in (1.1) and then employing Lemma 2.3, we find that

$$\sum_{n=0}^{\infty} \overline{A_{3^r}}(n)q^n = \frac{f_{3^r}^2 f_6^4 f_9^6}{f_{2\cdot 3^r} f_3^8 f_{18}^3} + 2q\frac{f_{3^r}^2 f_6^3 f_9^3}{f_{2\cdot 3^r} f_3^7} + 4q^2\frac{f_{3^r}^2 f_6^2 f_{18}^3}{f_{2\cdot 3^r} f_3^6}. \tag{6.4}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in (6.4), we obtain

$$\sum_{n=0}^{\infty} \overline{A_{3^r}}(3n)q^n = \frac{f_{3^{r-1}}^2 f_2^4 f_3^6}{f_{2\cdot 3^{r-1}} f_1^8 f_6^3}, \tag{6.5}$$

$$\sum_{n=0}^{\infty} \overline{A_{3^r}}(3n+1)q^n = 2\frac{f_{3^{r-1}}^2 f_2^3 f_3^3}{f_{2\cdot 3^{r-1}} f_1^7} \tag{6.6}$$

and

$$\sum_{n=0}^{\infty}\overline{A_{3^r}}(3n+2)q^n = 4\frac{f_{3^{r-1}}^2 f_2^2 f_6^3}{f_{2\cdot 3^{r-1}} f_1^6}. \tag{6.7}$$

In view of Lemma 2.3, modulo 16, we find that

$$\begin{aligned}\frac{f_{3^{r-1}}^2 f_3^6}{f_{2\cdot 3^{r-1}} f_6^3}\frac{f_2^4}{f_1^8} &\equiv \frac{f_{3^{r-1}}^2 f_3^6}{f_{2\cdot 3^{r-1}} f_6^3}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^4 \\ &\equiv \frac{f_{3^{r-1}}^2 f_6^{13} f_9^{24}}{f_3^{26} f_{2\cdot 3^{r-1}} f_{18}^{12}} + 8q\frac{f_{3^{r-1}}^2 f_6^{12} f_9^{21}}{f_3^{25} f_{2\cdot 3^{r-1}} f_{18}^9} + 8q^2\frac{f_{3^{r-1}}^2 f_6^{11} f_9^{18}}{f_3^{24} f_{2\cdot 3^{r-1}} f_{18}^6}. \end{aligned} \tag{6.8}$$

Combining (6.5) and (6.8), extracting the terms of the form q^{3n+1} and then using (4.6), we obtain

$$\sum_{n=0}^{\infty}\overline{A_{3^r}}(9n+3)q^n \equiv 8\frac{f_3^3}{f_1} = \sum_{n=0}^{\infty} a_3(n)q^n \pmod{2^4}.$$

Equating the coefficients of q^n on both sides of the above congruence, we arrive at (6.1).

Employing (4.6) in (6.7), we see that

$$\sum_{n=0}^{\infty}\overline{A_{3^r}}(3n+2)q^n \equiv 4\frac{f_6^3}{f_2} = \sum_{n=0}^{\infty} a_3(n)q^{2n} \pmod{2^3}. \tag{6.9}$$

Extracting even powers of q in (6.9), we obtain (6.2).

In view of (6.6) and (4.6), we deduce (6.3). □

Remark 6.2. Equating the odd powers of q in (6.9), we find that

$$\overline{A_{3^r}}(6n+5) \equiv 0 \pmod{2^3}, \qquad n \geq 0.$$

Utilizing (2.3), we can easily derive the following corollary.

Corollary 6.3. *For all non-negative integers n, α and $1 \leq j \leq 3$, we have*

$$a_3(4n+1) = 0, \tag{6.10}$$

$$a_3(8n+2j) \equiv 0 \pmod 2 \tag{6.11}$$

and

$$a_3(8n) \equiv \begin{cases} 1 \pmod 2, \text{if } n = k(3k-1)/2 \text{ for some integer } k, \\ 0 \pmod 2, \text{otherwise.} \end{cases}$$

Theorem 6.4. *If $p \equiv 2 \pmod 3$ and $j \in \{1,2,3\}$, then for all non-negative integers n and α, we have*

$$\overline{A_{3^r}}\left(p^{2\alpha}(9n+3)\right) \equiv \overline{A_{3^r}}(9n+3) \pmod{2^4}, \tag{6.12}$$

$$\overline{A_{3^r}}\left(p^{2\alpha}(36n+30)\right) \equiv 0 \pmod{2^4}, \tag{6.13}$$

$$\overline{A_{3^r}}\left(p^{2\alpha}(72n+18j+3)\right) \equiv 0 \pmod{2^4} \tag{6.14}$$

and

$$\overline{A_{3^r}}\left(p^{2\alpha}(72n+3)\right) \equiv \begin{cases} 2^3 \pmod{2^4}, \text{if } n = k(3k-1)/2 \text{ for some integer } k, \\ 0 \pmod{2^4}, \text{otherwise.} \end{cases}$$

Proof. Proof follows from Corollary 2.7, Corollary 6.3 and (6.1). □

Remark 6.5. Employing Corollary 2.7 and Corollary 6.3 in (6.2) and (6.2), we can also find infinite families of congruences modulo 8 and 4 for $\overline{A_{3^r}}(n)$ which are similar to congruences in Theorem 6.4.

Next, we present a short and simple proof of the Theorem 1.7.

Theorem 6.6. *For all non-negative integers $r \geq 3$ and n, we have*

$$\overline{A_{3^r}}(27n+18) \equiv 0 \pmod{3}.$$

Proof. From (3.2) and (6.5), it follows that

$$\sum_{n=0}^{\infty} \overline{A_{3^r}}(3n)q^n \equiv \frac{f_{3^{r-1}}^2 f_3^4}{f_{2\cdot 3^{r-1}} f_6^2} \frac{f_2}{f_1^2} \pmod{3}.$$

In view of above congruence, Lemma 2.3 and (3.2), we find that

$$\sum_{n=0}^{\infty} \overline{A_{3^r}}(9n)q^n \equiv \frac{f_{3^{r-2}}^2 f_3^5}{f_{2\cdot 3^{r-2}} f_6^2} \frac{f_2^2}{f_1} \pmod{3}. \tag{6.15}$$

Substituting (5.30) into (6.15) and then equating the coefficients of q^{3n+2}, we obtain the required congruence. Hence the proof. □

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