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SOME IDENTITIES OF THE APOSTOL TYPE POLYNOMIALS ARISING FROM UMBRAL CALCULUS

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Abstract The aim of this paper is to introduce and investigate several new identities related to the unified families of Apostol type polynomials. The results presented here are based upon the theory of the umbral calculus and the umbral algebra. We consider Apostol type polynomials related to associated sequences of polynomials and finally give some new and interesting identities of those polynomials arising from transfer formula for the associated sequences. Further, we derive several identities involving Apostol type polynomials arising from umbral calculus to have alternative ways. Some new and known identities are also derived for the families of Apostol type polynomials using umbral calculus as special cases.

1 Introduction

Umbral calculus provide powerful tool to deal with the properties of special polynomials and functions. It has been used in numerous problems of mathematics and its related field like combinotorics (for example, see [1, 3, 6, 7, 9]). Its techniques have been used in different areas of physics for example, it was used in group theory and quantum mechanics by Biedenharn et al. [5, 6]. Here we first recall the notations and definitions related to the umbral algebra and calculus [32, 33].

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left(f(t) = \Sigma_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k | \alpha_k \in \mathbb{C} \right).$$

Let us assume that $\mathbb{P}$ be the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. As a notation, the action of the linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$. The formal power series

$$f(t) = \Sigma_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k \in \mathcal{F},$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = \alpha_n, (n \geq 0). \tag{1.1}$$

From (1.1), we note that

$$\langle t^k|x^n\rangle = \ n!\delta_{n,k}, (n, k \geq 0), \tag{1.2}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [10, 14, 15, 32, 33]).

Let $f_L(t) = \Sigma_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$. Then, by (1.1), we get $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. The map $L \to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Hence forth, $\mathcal{F}$ thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [10, 14, 15, 31, 32, 33]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [10, 14, 15, 32, 33]). If $o(f(t)) = 1$, then $f(t)$ is called a *delta series*, and if $o(f(t)) = 0$, then $f(t)$ is called an *invertible series*. Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k|S_n(x)\rangle = n!\delta_{n,k}, (n,k \geq 0)$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the *associated sequence* for $f(t)$ (see [10, 14, 31, 32, 33]).

From (1.1), we note that

$$\langle e^{yt}|p(x)\rangle = p(y).$$

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \Sigma_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \quad p(x) = \Sigma_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k \tag{1.3}$$

and by (1.2), we get:

$$p^{(k)}(0) = \langle t^k|p(x)\rangle \qquad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{1.4}$$

Thus from (1.3), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^K p(x)}{dx^k}. \tag{1.5}$$

Also we note that

$$\langle f(t)(g(t)|p(x)\rangle = \langle (g(t)|f(t)p(x)\rangle.$$

Let $S_n(x) \sim (g(t), f(t))$. Then we see that

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \Sigma_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \qquad \forall \ \ y \in \mathbb{C}, \tag{1.6}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [32, 33]). Let $p_n(x) \sim (1, f(t)), q_n(x) \sim (1, g(t))$. Then, the transfer formula for the associated sequence is given by

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x). \tag{1.7}$$

Further, let $S_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$ then

$$S_n(x) = \Sigma_{k=0}^{n} C_{n,k} r_k(x), \tag{1.8}$$

where the connection constant $C_{n,k}$ are given by

$$C_{n,k} = \frac{1}{k!}\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k|x^n\rangle. \tag{1.9}$$

Equations (1.8) and (1.9) are called the alternative ways of Sheffer sequences.

The polynomials $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ are defined by the following generating functions [11, 34]:

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \ \ |t| < 2\pi, \tag{1.10}$$

$$\left(\frac{2}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \ \ |t| < \pi, \tag{1.11}$$

$$\left(\frac{2t}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!}, \ \ |t| < \pi. \tag{1.12}$$

It is easy to see that $B_n(x)$, $E_n(x)$ and $G_n(x)$ are given, respectively, by

$$B_n^{(1)}(x) = B_n(x);\ E_n^{(1)}(x) = E_n(x);\ G_n^{(1)}(x) = G_n(x),\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1.13}$$

Some interesting analogues of the classical Bernoulli and Euler polynomials were first investigated by Apostol [2] and further studied by Srivastava [36]. Luo and Srivastava [27] introduced the Apostol Bernoulli polynomials of order $\alpha \in \mathbb{N}_0$, denoted by $\mathfrak{B}_n^{(\alpha)}(x;\lambda),\ \lambda \in \mathbb{C}$, which are defined by the generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ \ |t| < 2\pi,\ \text{when}\ \lambda = 1;\ |t| < |\log\lambda|,\ \text{when}\ \lambda \neq 1, \tag{1.14}$$

with

$$\mathfrak{B}_n^{(\alpha)}(x;1) = B_n^{(\alpha)}(x). \tag{1.15}$$

The Apostol Euler polynomials of order $\alpha \in \mathbb{N}_0$, denoted by $\mathfrak{E}_n^{(\alpha)}(x;\lambda),\ \lambda \in \mathbb{C}$ are introduced by Luo [21] and are defined by the generating function

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ |t| < |\log(-\lambda)|, \tag{1.16}$$

with

$$\mathfrak{E}_n^{(\alpha)}(x;1) = E_n^{(\alpha)}(x). \tag{1.17}$$

Further, Luo [24] introduced the Apostol Genocchi polynomials of order $\alpha \in \mathbb{N}_0$, denoted by $\mathcal{G}_n^{(\alpha)}(x;\lambda),\ \lambda \in \mathbb{C}$, which are defined by the generating function

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},\ |t| < |\log(-\lambda)|, \tag{1.18}$$

with

$$\mathcal{G}_n^{(\alpha)}(x;1) = G_n^{(\alpha)}(x). \tag{1.19}$$

Certain properties of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials such as asymptotic estimates, fourier expansions, multiplication formulas etc. are studied by several researchers, see for example [4, 17, 22, 23, 25, 30]. Recently, Luo and Srivastava [28] introduced the Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)$ $(\alpha \in \mathbb{N}, \lambda, \mu, \nu \in \mathbb{C})$ of order α, which are defined by means of the generating function

$$\left(\frac{2^{\mu}\, t^{\nu}}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)\frac{t^n}{n!},\ |t| < |\log(-\lambda)| \tag{1.20}$$

where

$$\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu) := \mathcal{F}_n^{(\alpha)}(0;\lambda;\mu,\nu), \tag{1.21}$$

denotes the Apostol type numbers of order α, defined by the generating function

$$\left(\frac{2^{\mu}\, t^{\nu}}{\lambda e^t + 1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu)\frac{t^n}{n!},\ |t| < |\log(-\lambda)|. \tag{1.22}$$

These polynomials are viewed as a unification and generalization of the polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$, $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ and $\mathcal{G}_n^{(\alpha)}(x;\lambda)$. In fact, from equations (1.14), (1.16), (1.18) and (1.20), we have

$$(-1)^{\alpha}\mathcal{F}_n^{(\alpha)}(x;-\lambda;0,1) = \mathfrak{B}_n^{(\alpha)}(x;\lambda), \tag{1.23}$$

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;1,0) = \mathfrak{E}_n^{(\alpha)}(x;\lambda), \tag{1.24}$$

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;1,1) = \mathcal{G}_n^{(\alpha)}(x;\lambda). \tag{1.25}$$

Recently, some results for the Apostol type polynomials and their generalized forms are established, see for example [8, 18, 19, 20, 26, 35].

The Hermite polynomials [11] are defined by the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\, \frac{t^n}{n!}. \tag{1.26}$$

where

$$H_n = H_n(0)$$

denotes the Hermite numbers.

The Touchard polynomials [29] are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n\geq 0} T_n(x)\, \frac{t^n}{n!}. \tag{1.27}$$

By comparing (1.27) with the generating function of the Bell numbers given by

$$e^{e^t-1} = \sum_{n\geq 0} Bell_n\, \frac{t^n}{n!}, \tag{1.28}$$

we get the following relationship between the Touchard polynomial and the Bell numbers

$$T_n(1) = Bell_n. \tag{1.29}$$

Furthermore, we note that the first Stirling number is given by

$$(x)_n = x(x-1)(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\, x^l, \qquad (see\ [14, 32, 33]) \tag{1.30}$$

and the second Stirling number is defined by the generating function

$$(e^t-1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}, \qquad (see\ [10, 32, 33]). \tag{1.31}$$

Motivated by the above mentioned work on Apostol type polynomials and due to the importance of the umbral calculus in this paper, we consider Apostol type polynomials related to associated sequences of polynomials by the use of umbral calculus. Finally, we give some new and interesting identities of those polynomials arising from transfer formula for the associated sequences. Further, we establish a connection between our polynomial and several known families of polynomials arising from umbral calculus to have alternative ways.

2 Umbral calculus and Apostol type polynomials

From (1.6), we note that

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) \sim \left(\left(\frac{\lambda e^z+1}{2^\mu\, t^\nu}\right)^\alpha, t\right). \tag{2.1}$$

Thus, we get

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) \;=\; \left(\frac{2^\mu\, z^\nu}{\lambda e^z+1}\right)^\alpha x^n. \tag{2.2}$$

Let us assume that

$$p_n(x) \sim (1, z(\lambda e^z+1)), \quad q_n(x) \sim \left(1, \left(\frac{\lambda e^z+1}{2^\mu\, z^\nu}\right)^\alpha z\right), \quad (\alpha \neq 0). \tag{2.3}$$

From $x^n \sim (1,z)$, (1.7) and (2.3), we note that

$$p_n(x) \;=\; x\left(\frac{1}{\lambda e^z+1}\right)^n x^{n-1} \;=\; \frac{x}{(2^\mu\, z^\nu)^n} \mathcal{F}_{n-1}^{(n)}(x;\lambda;\mu,\nu) \tag{2.4}$$

and

$$q_n(x) = x\Big(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\Big)^{\alpha n}\ x^{n-1}\ =\ x\mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda;\mu,\nu). \tag{2.5}$$

In view of the fact (1.7), (2.3), (2.4) and (2.5), we can derive

$$\frac{x}{(2^\mu\ z^\nu)^n}\mathcal{F}_{n-1}^{(n)}(x;\lambda;\mu,\nu) = x\Bigg(\frac{\Big(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\Big)^\alpha}{\lambda e^z+1}\Bigg)^n\ x^{-1}x\ \mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda;\mu,\nu)$$

$$= \frac{x}{(2^\mu\ z^\nu)^{\alpha n}}\sum_{l=0}^{(\alpha-1)n}\binom{(\alpha-1)n}{l}\lambda^l\mathcal{F}_{n-1}^{(\alpha n)}(x+l;\lambda;\mu,\nu), \tag{2.6}$$

where $\alpha, n \in \mathbb{N}$. Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. For $\alpha, n \in \mathbb{N}$, we have :

$$\mathcal{F}_{n-1}^{(n)}(x;\lambda;\mu,\nu) = \frac{1}{(2^\mu\ z^\nu)^{(\alpha-1)n}}\sum_{l=0}^{(\alpha-1)n}\binom{(\alpha-1)n}{l}\lambda^l\mathcal{F}_{n-1}^{(\alpha n)}(x+l;\lambda;\mu,\nu). \tag{2.7}$$

Further in view of relation (1.23), we deduce the following result from (2.7)

Corollary 2.2. For $\alpha, n \in \mathbb{N}$, we have :

$$\mathcal{B}_{n-1}^{(n)}(x;\lambda) = \frac{1}{(z)^{(\alpha-1)n}}\sum_{l=0}^{(\alpha-1)n}\binom{(\alpha-1)n}{l}(-\lambda)^l\mathcal{B}_{n-1}^{(\alpha n)}(x+l;\lambda), \tag{2.8}$$

Also in view of relation (1.24), we deduce the following result from (2.7)

Corollary 2.3. For $\alpha, n \in \mathbb{N}$, we have :

$$\mathcal{E}_{n-1}^{(n)}(x;\lambda) = \frac{1}{(2)^{(\alpha-1)n}}\sum_{l=0}^{(\alpha-1)n}\binom{(\alpha-1)n}{l}\lambda^l\mathcal{E}_{n-1}^{(\alpha n)}(x+l;\lambda), \tag{2.9}$$

Furthermore, in view of relation (1.25), we deduce the following result from (2.7)

Corollary 2.4. For $\alpha, n \in \mathbb{N}$, we have :

$$\mathcal{G}_{n-1}^{(n)}(x;\lambda) = \frac{1}{(2z)^{(\alpha-1)n}}\sum_{l=0}^{(\alpha-1)n}\binom{(\alpha-1)n}{l}\lambda^l\mathcal{G}_{n-1}^{(\alpha n)}(x+l;\lambda). \tag{2.10}$$

Further, let us consider the following associated sequences:

$$\frac{x}{(2^\mu\ z^\nu)^n}\mathcal{F}_{n-1}^{(n)}(x;\lambda;\mu,\nu) \sim (1, z(\lambda e^z+1)),\ \ p_n(x) \sim \Big(1, \Big(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\Big)^\alpha z\Big),\quad (\alpha \neq 0). \tag{2.11}$$

For $x_n \sim (1, z)$, by (1.7) and (2.11), we get

$$p_n(x) = x\Bigg(\frac{z}{z\Big(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\Big)^\alpha}\Bigg)^n\ x^{-1}x\ =\ x\Big(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\Big)^{\alpha n}\ x^{n-1}$$

$$= \frac{x}{(2^\mu\ z^\nu)^{\alpha n}}\sum_{l=0}^{\alpha n}\binom{\alpha n}{l}\lambda^l(x+l)^{n-1}. \tag{2.12}$$

For $n \geq 1$, by (1.7) and (2.11), we get

$$p_n(x) = x\left(\frac{z(\lambda e^z+1)}{z\left(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\right)^\alpha}\right)^n\ x^{-1}\frac{x}{(2^\mu\ z^\nu)^n}\mathcal{F}_{n-1}^{(n)}(x;\lambda;\mu,\nu)$$

$$= x\frac{1}{(2^\mu\ z^\nu)^{(\alpha+1)n}}(\lambda e^z+1)^{(\alpha+1)n}\mathcal{F}_{n-1}^{(n)}(x;\lambda;\mu,\nu). \tag{2.13}$$

By (2.12) and (2.13), we get

$$\sum_{l=0}^{\alpha n}\binom{\alpha n}{l}\lambda^l(x+l)^{n-1} = \frac{1}{(2^\mu\ z^\nu)^n}(\lambda e^z+1)^{(\alpha+1)n}\mathcal{F}_{n-1}^{(n)}(x;\lambda;\mu,\nu). \tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.5. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$, we have :

$$\sum_{l=0}^{\alpha n}\binom{\alpha n}{l}(x+l)^{n-1} = \frac{1}{(2^\mu\ z^\nu)^n}\sum_{l=0}^{(\alpha+1)n}\binom{(\alpha+1)n}{l}\mathcal{F}_{n-1}^{(n)}(x+l;\lambda;\mu,\nu). \tag{2.15}$$

Further in view of relation (1.23), we deduce the following result from (2.15)

Corollary 2.6. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$, we have :

$$\sum_{l=0}^{\alpha n}\binom{\alpha n}{l}(x+l)^{n-1} = \frac{1}{(z)^n}\sum_{l=0}^{(\alpha+1)n}\binom{(\alpha+1)n}{l}\mathcal{B}_{n-1}^{(n)}(x+l;\lambda). \tag{2.16}$$

Also in view of relation (1.24), we deduce the following result from (2.15)

Corollary 2.7. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$, we have :

$$\sum_{l=0}^{\alpha n}\binom{\alpha n}{l}(x+l)^{n-1} = \frac{1}{(2)^n}\sum_{l=0}^{(\alpha+1)n}\binom{(\alpha+1)n}{l}\mathcal{E}_{n-1}^{(n)}(x+l;\lambda). \tag{2.17}$$

Furthermore, in view of relation (1.25), we deduce the following result from (2.15)

Corollary 2.8. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$, we have :

$$\sum_{l=0}^{\alpha n}\binom{\alpha n}{l}(x+l)^{n-1} = \frac{1}{(2\ z)^n}\sum_{l=0}^{(\alpha+1)n}\binom{(\alpha+1)n}{l}\mathcal{G}_{n-1}^{(n)}(x+l;\lambda). \tag{2.18}$$

Let us consider the following associated sequences:

$$(x)_n \sim (1, e^z-1),\quad x\mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda) \sim \left(1, z\left(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\right)^\alpha\right),\quad (\alpha \neq 0), \tag{2.19}$$

By (1.7) and (2.19), we get

$$x\mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda) = x\left(\frac{e^z-1}{z\left(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\right)^\alpha}\right)^n x^{-1}(x)_n, \tag{2.20}$$

$$= x\left(\frac{e^z-1}{z}\right)^n\left(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\right)^{\alpha n}(x-1)_{n-1}.$$

Replacing x by $x+1$, we have

$$(x+1)\mathcal{F}_{n-1}^{(\alpha n)}(x+1;\lambda)=(x+1)\left(\frac{e^z-1}{z}\right)^n\left(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\right)^{\alpha n}(x)_{n-1},$$

or

$$\mathcal{F}_{n-1}^{(\alpha n)}(x+1;\lambda)=\left(\frac{e^z-1}{z}\right)^n\left(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\right)^{\alpha n}\sum_{l=0}^{n-1}S_1(n-1,l)x^l,$$

$$=\sum_{l=0}^{n-1}\sum_{k=0}^{l}S_1(n-1,l)S_2(k+n,n)\frac{n!}{(k+n)!}(l)_k\mathcal{F}_{l-k}^{(\alpha n)}(x,\lambda;\mu;\nu)z^k. \tag{2.21}$$

Therefore by (2.21), we obtain the following theorem

Theorem 2.9. For $n\geq 1$ and $\alpha\in\mathbb{Z}_+$, we have :

$$\mathcal{F}_{n-1}^{(\alpha n)}(x+1;\lambda)=\sum_{l=0}^{n-1}\sum_{k=0}^{l}S_1(n-1,l)S_2(k+n,n)\frac{\binom{l}{k}}{\binom{(k+n)}{n}}\mathcal{F}_{l-k}^{(\alpha n)}(x,\lambda;\mu;\nu). \tag{2.22}$$

Further in view of relation (1.23), we deduce the following result from (2.22)

Corollary 2.10. For $n\geq 1$ and $\alpha\in\mathbb{Z}_+$, we have :

$$\mathcal{B}_{n-1}^{(\alpha n)}(x+1;\lambda)=\sum_{l=0}^{n-1}\sum_{k=0}^{l}S_1(n-1,l)S_2(k+n,n)\frac{\binom{l}{k}}{\binom{(k+n)}{n}}\mathcal{B}_{l-k}^{(\alpha n)}(x,\lambda). \tag{2.23}$$

Also in view of relation (1.24), we deduce the following result from (2.22)

Corollary 2.11. For $n\geq 1$ and $\alpha\in\mathbb{Z}_+$, we have :

$$\mathcal{E}_{n-1}^{(\alpha n)}(x+1;\lambda)=\sum_{l=0}^{n-1}\sum_{k=0}^{l}S_1(n-1,l)S_2(k+n,n)\frac{\binom{l}{k}}{\binom{(k+n)}{n}}\mathcal{E}_{l-k}^{(\alpha n)}(x,\lambda). \tag{2.24}$$

Furthermore, in view of relation (1.25), we deduce the following result from (2.22)

Corollary 2.12. For $n\geq 1$ and $\alpha\in\mathbb{Z}_+$, we have :

$$\mathcal{G}_{n-1}^{(\alpha n)}(x+1;\lambda)=\sum_{l=0}^{n-1}\sum_{k=0}^{l}S_1(n-1,l)S_2(k+n,n)\frac{\binom{l}{k}}{\binom{(k+n)}{n}}\mathcal{G}_{l-k}^{(\alpha n)}(x,\lambda). \tag{2.25}$$

Let

$$x\mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda;\mu;\nu)\sim\left(1,z\left(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\right)^\alpha\right),(\alpha\neq 0)$$
$$(x)_n\sim(1,e^z-1), \tag{2.26}$$

Then we have

$$(x)_n=x\left(\frac{z\left(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\right)^\alpha}{e^z-1}\right)^n x^{-1}\ x\ \mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda;\mu;\nu)$$
$$=x\left(\frac{z}{e^z-1}\right)^n\left(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\right)^{\alpha n}\mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda;\mu;\nu), \tag{2.27}$$
$$=xB_{n-1}^{(n)}(x).$$

and

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l\ =\ x\sum_{l=0}^{n-1}S_1(n,l+1)x^l,\qquad(n\geq 1). \tag{2.28}$$

Therefore by (2.27) and (2.28), we get the following theorem

Theorem 2.13. For $n \geq 1$ and $0 \leq l \leq n-1$, we have :

$$S_1(n, l+1) = \binom{n-1}{l} \mathcal{B}_{n-l-1}^{(n)}. \tag{2.29}$$

Also from (2.27), we note that

$$\left(\frac{e^z-1}{z}\right)^n (x-1)_{n-1} = (\lambda e^z+1)^{\alpha n}(2^\mu\ z^\nu)^{-\alpha n}\mathcal{F}_{n-1}^{(\alpha n)}(x;\lambda;\mu;\nu), (n \geq 1). \tag{2.30}$$

L.H.S. of (2.30)

$$= \left(\frac{e^z-1}{z}\right)^n \sum_{l=0}^{n-1} S_1(n-1,l)(x-1)^l$$

$$= \sum_{l=0}^{n-1} S_1(n-1,l) \sum_{k=0}^{l} \frac{\binom{l}{k}}{\binom{k+n}{n}} S_2(k+n,n)(x-1)^{l-k} \qquad . \tag{2.31}$$

and R.H..S of (2.30).

$$= (2^\mu\ z^\nu)^{-\alpha n} \sum_{l=0}^{\alpha n} \binom{\alpha n}{l} \lambda^l \mathcal{F}_{n-1}^{(\alpha n)}(x+l;\lambda;\mu;\nu) \qquad . \tag{2.32}$$

Therefore, by (2.30), (2.31) and (2.32), we obtain the following theorem.

Theorem 2.14. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+$, we have :

$$(2^\mu\ z^\nu)^{-\alpha n} \sum_{l=0}^{\alpha n} \binom{\alpha n}{l} \lambda^l \mathcal{F}_{n-1}^{(\alpha n)}(x+l;\lambda;\mu;\nu) = \sum_{l=0}^{n-1}\sum_{k=0}^{l} \frac{\binom{l}{k}}{\binom{k+n}{n}} S_1(n-1,l)S_2(k+n,n)(x-1)^{l-k} \tag{2.33}$$

Further in view of relation (1.23), we deduce the following result from (2.33)

Corollary 2.15. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+$, we have :

$$z^{-\alpha n} \sum_{l=0}^{\alpha n} \binom{\alpha n}{l} (-\lambda)^l \mathcal{B}_{n-1}^{(\alpha n)}(x+l;\lambda) = \sum_{l=0}^{n-1}\sum_{k=0}^{l} \frac{\binom{l}{k}}{\binom{k+n}{n}} S_1(n-1,l)S_2(k+n,n)(x-1)^{l-k} \tag{2.34}$$

Also in view of relation (1.24), we deduce the following result from (2.33)

Corollary 2.16. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+$, we have :

$$2^{-\alpha n} \sum_{l=0}^{\alpha n} \binom{\alpha n}{l} \lambda^l \mathcal{E}_{n-1}^{(\alpha n)}(x+l;\lambda) = \sum_{l=0}^{n-1}\sum_{k=0}^{l} \frac{\binom{l}{k}}{\binom{k+n}{n}} S_1(n-1,l)S_2(k+n,n)(x-1)^{l-k} \tag{2.35}$$

Furthermore, in view of relation (1.25), we deduce the following result from (2.33)

Corollary 2.17. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+$, we have :

$$(2z)^{-\alpha n} \sum_{l=0}^{\alpha n} \binom{\alpha n}{l} (\lambda)^l \mathcal{G}_{n-1}^{(\alpha n)}(x+l;\lambda) = \sum_{l=0}^{n-1}\sum_{k=0}^{l} \frac{\binom{l}{k}}{\binom{k+n}{n}} S_1(n-1,l)S_2(k+n,n)(x-1)^{l-k} \tag{2.36}$$

Let

$$p_n(x) \sim \Big(1, \Big(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\Big)^\alpha z\Big), (x)_n \sim (1, e^z-1), (\alpha \neq 0). \tag{2.37}$$

By (2.12), we have

$$\begin{aligned} p_n(x) &= \frac{x}{(2^\mu\ z^\nu)^{\alpha n}} \sum_{l=0}^{\alpha n} \binom{\alpha n}{l} (x+l)^{n-1} \\ &= \frac{1}{(2^\mu\ z^\nu)^{\alpha n}}\ x \sum_{k=0}^{\alpha n} \binom{\alpha n}{k} \sum_{l=0}^{n-1} \binom{n-1}{l} k^{n-1-l} (x)^1. \end{aligned} \tag{2.38}$$

From (1.7) and (2.37), we have

$$\begin{aligned} (x)_n &= x\Bigg(\frac{z\Big(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\Big)^\alpha}{e^z-1}\Bigg)^n x^{-1} p_n(x) \\ &= x\Big(\frac{z}{\lambda e^z+1}\Big)^n \Big(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\Big)^{\alpha n} \Big(\frac{1}{2^\mu\ z^\nu}\Big)^{\alpha n} \sum_{k=0}^{\alpha n}\sum_{l=0}^{n-1} \binom{\alpha n}{k}\binom{n-1}{l} k^{n-1-l} (x)^1 \end{aligned}$$

$$= \Big(\frac{1}{2^\mu\ z^\nu}\Big)^{\alpha n} x \sum_{p=0}^{n-1} \Big(\sum_{k=0}^{\alpha n}\sum_{l=p}^{n-1}\sum_{m=p}^{l} \binom{\alpha n}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p} k^{n-1-l} \mathcal{F}_{l-m}^{(\alpha n)}(x;\lambda;\mu;\nu)\mathcal{B}_{m-p}^{(n)}\Big) x^p. \tag{2.39}$$

Therefore, by (2.28) and (2.39), we obtain the following theorem.

Theorem 2.18. For $n \geq 1$, $\alpha \in \mathbb{Z}_+$ and $0 \leq l \leq n-1$ we have :

$$S_1(n, p+1) = \Big(\frac{1}{2^\mu\ z^\nu}\Big)^{\alpha n} \sum_{k=0}^{\alpha n}\sum_{l=p}^{n-1}\sum_{m=p}^{l} \binom{\alpha n}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p} k^{n-1-l} \mathcal{F}_{l-m}^{(\alpha n)}(x;\lambda;\mu;\nu)\mathcal{B}_{m-p}^{(n)}. \tag{2.40}$$

Further in view of relation (1.23), we deduce the following result from (2.40)

Corollary 2.19. For $n \geq 1$, $\alpha \in \mathbb{Z}_+$ and $0 \leq l \leq n-1$ we have :

$$S_1(n, p+1) = \Big(\frac{1}{z}\Big)^{\alpha n} \sum_{k=0}^{\alpha n}\sum_{l=p}^{n-1}\sum_{m=p}^{l} \binom{\alpha n}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p} k^{n-1-l} \mathcal{B}_{l-m}^{(\alpha n)}(x;\lambda)\mathcal{B}_{m-p}^{(n)}. \tag{2.41}$$

Also in view of relation (1.24), we deduce the following result from (2.40)

Corollary 2.20. For $n \geq 1$, $\alpha \in \mathbb{Z}_+$ and $0 \leq l \leq n-1$ we have :

$$S_1(n, p+1) = \Big(\frac{1}{2}\Big)^{\alpha n} \sum_{k=0}^{\alpha n}\sum_{l=p}^{n-1}\sum_{m=p}^{l} \binom{\alpha n}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p} k^{n-1-l} \mathcal{E}_{l-m}^{(\alpha n)}(x;\lambda)\mathcal{B}_{m-p}^{(n)}. \tag{2.42}$$

Furthermore, in view of relation (1.25), we deduce the following result from (2.40)

Corollary 2.21. For $n \geq 1$, $\alpha \in \mathbb{Z}_+$ and $0 \leq l \leq n-1$ we have :

$$S_1(n, p+1) = \Big(\frac{1}{2\ z}\Big)^{\alpha n} \sum_{k=0}^{\alpha n}\sum_{l=p}^{n-1}\sum_{m=p}^{l} \binom{\alpha n}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p} k^{n-1-l} \mathcal{G}_{l-m}^{(\alpha n)}(x;\lambda)\mathcal{B}_{m-p}^{(n)}. \tag{2.43}$$

In the next section, we use alternating ways to get new identities for the Apostol type polynomials.

3 Connections with families of polynomials

In this section, we present several identities involving the Apostol type polynomials and some other families of polynomials. In order to established these identities we use umbral calculus to have alternative ways. For instance, we obtain the connection of Apostol type polynomials with Hermite and Touchard polynomials.

3.1 Connections to the Hermite polynomials

Let us consider the following Sheffer sequences:

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \sim \left(\left(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\right)^\alpha,\ z\right), \qquad H_n(x) \sim (e^{\frac{z^2}{4}},\ \frac{z}{2}). \tag{3.1.1}$$

Then, by (1.8), we assume that

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \sum_{k=0}^{n} C_{n,k} H_k(x). \tag{3.1.2}$$

From (1.9) and (3.1.2), we have

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\langle \frac{e^{\frac{z^2}{4}}}{\left(\frac{\lambda e^z+1}{2^\mu\ z^\nu}\right)^\alpha}\left(\frac{z}{2}\right)^k | x^n\rangle \\ &= \frac{1}{k!2^k}\langle e^{\frac{z^2}{4}}\left(\frac{2^\mu\ z^\nu}{\lambda e^z+1}\right)^\alpha |\ z^k x^n\rangle \\ &= 2^{-k}\binom{n}{k}\sum_{l=0}^{\frac{n-k}{2}}\frac{1}{2^{2l}l!}(n-k)_{2l}\mathcal{F}_{n-k-2l}^{(\alpha)}(0;\lambda;\mu;\nu). \end{aligned} \tag{3.1.3}$$

Therefore, by (3.1.2) and (3.1.3), we obtain the following theorem

Theorem 3.1. For $n \geq 0$, we have :

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = n!\sum_{k=0}^{n}\sum_{0\leq l\leq n-k, l:even}\frac{\mathcal{F}_{n-k-l}^{(\alpha)}(\lambda;\mu;\nu)}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!}H_k(x). \tag{3.1.4}$$

Further in view of relation (1.23), we deduce the following result from (3.1.4)

Corollary 3.2. For $n \geq 0$, we have :

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = n!\sum_{k=0}^{n}\sum_{0\leq l\leq n-k, l:even}\frac{\mathcal{B}_{n-k-l}^{(\alpha)}(\lambda)}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!}H_k(x). \tag{3.1.5}$$

which for $\lambda = 1$ gives the result of [16; Theo. 2.2].

Also in view of relation (1.24), we deduce the following result from (3.1.4)

Corollary 3.3. For $n \geq 0$, we have :

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = n!\sum_{k=0}^{n}\sum_{0\leq l\leq n-k, l:even}\frac{\mathcal{E}_{n-k-l}^{(\alpha)}(\lambda)}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!}H_k(x). \tag{3.1.6}$$

which for $\lambda = 1$ gives the result of [16; Theo. 2.1].

Furthermore, in view of relation (1.25), we deduce the following result from (3.1.4)

Corollary 3.4. For $n \geq 0$, we have :

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = n! \sum_{k=0}^{n} \sum_{0\leq l\leq n-k, l:even} \frac{\mathcal{F}_{n-k-l}^{(\alpha)}(\lambda)}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!} H_k(x). \tag{3.1.7}$$

Further, note that

$$H_n(x) \sim (e^{\frac{z^2}{4}}, \frac{z}{2}).$$

Thus we have

$$e^{\frac{z^2}{4}} H_n(x) \sim (1, \frac{z}{2}) \quad and \quad (2x)^n \sim (1, \frac{z}{2}). \tag{3.1.8}$$

From (3.1.8), we have

$$e^{\frac{z^2}{4}} H_n(x) = (2x)^n \quad \Longleftrightarrow \quad H_n(x) = e^{-\frac{z^2}{4}} (2x)^n. \tag{3.1.9}$$

Also let us assume that

$$H_n(x) = \sum_{k=0}^{n} C_{n,k} \mathcal{F}_k^{(\alpha)}(x;\lambda;\mu;\nu). \tag{3.1.10}$$

From (1.9), (3.1.9) and (3.1.10), we get

$$C_{n,k} = \frac{1}{k!} \langle \frac{\left(\frac{\lambda e^{2z}+1}{2^{\mu+\nu}\ z^{\nu}}\right)^{\alpha}}{e^{\frac{(2z)^2}{4}}} (2z)^k | x^n \rangle$$

$$= \frac{1}{k!} \langle \left(\frac{\lambda e^{z}+1}{2^{\mu}\ z^{\nu}}\right)^{\alpha} (z)^k | e^{-\frac{(z)^2}{4}} (2x)^n \rangle$$

$$C_{n,k} = \frac{1}{2^{\mu r} k!} \ \langle (\lambda e^z + 1)^{\alpha} (z)^{k-\nu r} | H_n(x) \rangle$$

$$= \frac{1}{2^{\mu r}} \binom{n}{k} 2^{k-\nu r} \sum_{j=0}^{r} \binom{r}{j} \lambda^j \langle e^{jz} | H_{n-k+\nu r}(x) \rangle. \tag{3.1.11}$$

Therefore, by (3.1.10) and (3.1.11), we obtain the following theorem

Theorem 3.5. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^{(\mu+\nu)r}} \sum_{k=0}^{n} \binom{n}{k} 2^k \sum_{j=0}^{r} \binom{r}{j} \lambda^j H_{n-k+\nu r}(j) \mathcal{F}_k^{(\alpha)}(x;\lambda;\mu;\nu). \tag{3.1.12}$$

Further in view of relation (1.23), we deduce the following result from (3.1.12)

Corollary 3.6. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^r} \sum_{k=0}^{n} \binom{n}{k} 2^k \sum_{j=0}^{r} \binom{r}{j} \lambda^j \ H_{n-k+r}(j) \mathcal{B}_k^{(\alpha)}(x;\lambda). \tag{3.1.13}$$

Also in view of relation (1.24), we deduce the following result from (3.1.12)

Corollary 3.7. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^r} \sum_{k=0}^{n} \binom{n}{k} 2^k \sum_{j=0}^{r} \binom{r}{j} \lambda^j \, H_{n-k}(j)\mathcal{E}_k^{(\alpha)}(x;\lambda). \tag{3.1.14}$$

which for $\lambda = 1$ gives the result of [16; Theo. 2.5].

Furthermore, in view of relation (1.25), we deduce the following result from (3.1.12)

Corollary 3.8. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^{2r}} \sum_{k=0}^{n} \binom{n}{k} 2^k \sum_{j=0}^{r} \binom{r}{j} \lambda^j \, H_{n-k+r}(j)\mathcal{G}_k^{(\alpha)}(x;\lambda). \tag{3.1.15}$$

Again let us assume that

$$H_n(x) = \sum_{k=0}^{n} C_{n,k}\mathcal{F}_k^{(\alpha)}(x;\lambda;\mu;\nu). \tag{3.1.16}$$

From (1.9), (3.1.9) and (3.1.16), we get

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\langle \frac{\left(\frac{\lambda e^{2z}+1}{2^{\mu+\nu}\ z^{\nu}}\right)^{\alpha}}{e^{\frac{(2z)^2}{4}}}(2z)^k | x^n\rangle \\
&= \frac{1}{k!}2^n\langle\left(\frac{\lambda e^z+1}{2^{\mu}\ z^{\nu}}\right)^{\alpha} e^{-}\frac{(z)^2}{|}4\ (z)^k x^n\rangle \\
C_{n,k} &= 2^{n-r}\binom{n}{k-r}\ \langle(\lambda e^z+1)^{\alpha}|\sum_{l=0}^{\infty}\frac{(-1)^l\ z^{2l}}{l!2^{2l}}x^{n-k-r}\rangle \\
&= \frac{1}{2^r}\sum_{j=0}^{r}\sum_{l=0}^{\frac{n-k-r}{2}} 2^{k-r}\binom{n}{k-r}\binom{r}{j}\frac{(-1)^l(n-k+r)!}{l!(n-k+r-2l)!}\lambda^j|(2j)^{n-k+r-2l}.
\end{aligned} \tag{3.1.17}$$

Therefore, by (3.1.16) and (3.1.17), we obtain the following theorem

Theorem 3.9. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\sum_{l=0}^{\frac{n-k-r}{2}} 2^{k-r}\binom{n}{k-r}\binom{r}{j}\frac{(-1)^l(n-k+r)!}{l!(n-k+r-2l)!}\lambda^j(2j)^{n-k+r-2l}\mathcal{F}_k^{(\alpha)}(x;\lambda;\mu;\nu). \tag{3.1.18}$$

Further in view of relation (1.23), we deduce the following result from (3.1.18)

Corollary 3.10. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\sum_{l=0}^{\frac{n-k-r}{2}} 2^{k-r}\binom{n}{k-r}\binom{r}{j}\frac{(-1)^l(n-k+r)!}{l!(n-k+r-2l)!}(-\lambda)^j(2j)^{n-k+r-2l}\mathcal{B}_k^{(\alpha)}(x;\lambda). \tag{3.1.19}$$

Also in view of relation (1.24), we deduce the following result from (3.1.18)

Corollary 3.11. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^r} \sum_{j=0}^{r} \sum_{l=0}^{\frac{n-k-r}{2}} 2^{k-r} \binom{n}{k-r} \binom{r}{j} \frac{(-1)^l (n-k+r)!}{l!(n-k+r-2l)!} \lambda^j (2j)^{n-k+r-2l} \mathcal{E}_k^{(\alpha)}(x;\lambda). \tag{3.1.20}$$

Furthermore, in view of relation (1.25), we deduce the following result from (3.1.18)

Corollary 3.12. For $n \geq 0$, we have :

$$H_n(x) = \frac{1}{2^r} \sum_{j=0}^{r} \sum_{l=0}^{\frac{n-k-r}{2}} 2^{k-r} \binom{n}{k-r} \binom{r}{j} \frac{(-1)^l (n-k+r)!}{l!(n-k+r-2l)!} \lambda^j (2j)^{n-k+r-2l} \mathcal{G}_k^{(\alpha)}(x;\lambda). \tag{3.1.21}$$

3.2 Connections to the Touchard polynomials

Using Similar technique used in previous theorems, we can express our 2-variable Apostol type polynomials in terms of other families. For instance we can obtain Apostol type polynomials in terms of Touchard polynomials $T_n(x)$ Using the facts that $T_n(x) \sim (1, log(t+1))$ and

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \sum_{k=0}^{n} C_{n,k} T_k(x). \tag{3.2.1}$$

with (2.1) and (1.9) we obtain the following result

Theorem 3.13. For $n \geq 0$, we have :

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{n}{l} \mathcal{F}_{n-l}^{(\alpha)} S_1(l,k)\, T_k(x). \tag{3.2.2}$$

Further in view of relation (1.23), we deduce the following result from (3.2.2)

Corollary 3.14. For $n \geq 0$, we have :

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{n}{l} \mathcal{B}_{n-l}^{(\alpha)} S_1(l,k)\, T_k(x). \tag{3.2.3}$$

which for $\lambda = \alpha = 1$ gives the result of [12; p.42(Theo. 3.1)].

Also in view of relation (1.24), we deduce the following result from (3.2.2)

Corollary 3.15. For $n \geq 0$, we have :

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{n}{l} \mathcal{E}_{n-l}^{(\alpha)} S_1(l,k)\, T_k(x). \tag{3.2.4}$$

which for $\lambda = \alpha = 1$ gives the result of [12; p.43(Theo. 3.3)].

Furthermore, in view of relation (1.25), we deduce the following result from (3.2.2)

Corollary 3.16. For $n \geq 0$, we have :

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}\mathcal{G}_{n-l}^{(\alpha)}\, S_1(l,k)\, T_k(x). \tag{3.2.5}$$

Next, let us assume that

$$T_k(x) = \sum_{k=0}^{n} C_{n,k}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu). \tag{3.2.6}$$

From (1.9) and (3.2.6), we have

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\langle\Big(\frac{\lambda e^{e^t-1}+1}{2^{\mu}\,(e^t-1)^{\nu}}\Big)^{\alpha}(e^t-1)^k|x^n\rangle\\
&= \frac{1}{2^{\mu\alpha}}\langle\Big(\frac{\lambda e^{e^t-1}+1}{(e^t-1)^{\nu}}\Big)^{\alpha}\sum_{l=k}^{n}\binom{n}{l}\, S_2(l,k)|x^{n-l}\rangle\\
&= \sum_{m=0}^{n-k}\binom{n}{m}\, S_2(n-m,k)\langle\frac{(\lambda e^{e^t-1}+1)^{\alpha}}{t^{\nu\alpha}}|\sum_{j=0}^{m}B_j^{\nu\alpha}\frac{t^j}{j!}x^m\rangle\\
&= \sum_{m=0}^{n-k}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}\, S_2(n-m,k)B_{m-j}^{\nu\alpha}\langle\sum_{p=0}^{\alpha}\binom{\alpha}{p}(-\lambda)^{\alpha-p}(e^{e^t-1})^p|x^j\rangle\\
&= \sum_{m=0}^{n-k}\sum_{j=0}^{m}\sum_{p=0}^{\alpha}\binom{n}{m}\binom{m}{j}\binom{\alpha}{p}(-\lambda)^{\alpha-p}\, S_2(n-m,k)\, B_{m-j}^{\nu\alpha}\sum_{l=0}^{\infty}Bell_{j+\nu\alpha}^{(p)}\frac{1}{(j+\nu\alpha)!}.
\end{aligned} \tag{3.2.7}$$

Therefore, by (3.2.6) and (3.2.7), we obtain the following theorem

Theorem 3.17. For $n \geq 0$, we have :

$$T_k(x) = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{j=0}^{m}\sum_{p=0}^{\alpha}\binom{n}{m}\binom{m}{j}\binom{\alpha}{p}\frac{(-\lambda)^{\alpha-p}S_2(n-m,k)}{(j+\nu\alpha)!}B_{m-j}^{\nu\alpha}Bell_{j+\nu\alpha}^{(p)}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu). \tag{3.2.8}$$

Further in view of relation (1.23), we deduce the following result from (3.2.8)

Corollary 3.18. For $n \geq 0$, we have :

$$T_k(x) = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{j=0}^{m}\sum_{p=0}^{\alpha}\binom{n}{m}\binom{m}{j}\binom{\alpha}{p}\frac{(\lambda)^{\alpha-p}S_2(n-m,k)}{(j+\alpha)!}B_{m-j}^{\alpha}Bell_{j+\alpha}^{(p)}\mathcal{B}_n^{(\alpha)}(x;\lambda). \tag{3.2.9}$$

which for $\lambda = \alpha = 1$ gives the result of [12; p.43(Theo. 3.2)].

Also in view of relation (1.24), we deduce the following result from (3.2.8)

Corollary 3.19. For $n \geq 0$, we have :

$$T_k(x) = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{j=0}^{m}\sum_{p=0}^{\alpha}\binom{n}{m}\binom{m}{j}\binom{\alpha}{p}\frac{(-\lambda)^{\alpha-p}S_2(n-m,k)}{j!}Bell_j^{(p)}\mathcal{E}_n^{(\alpha)}(x;\lambda). \tag{3.2.10}$$

Furthermore, in view of relation (1.25), we deduce the following result from (3.2.8)

Corollary 3.20. For $n \geq 0$, we have :

$$T_k(x) = \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{j=0}^{m} \sum_{p=0}^{\alpha} \binom{n}{m} \binom{m}{j} \binom{\alpha}{p} \frac{(-\lambda)^{\alpha-p} S_2(n-m,k)}{(j+\alpha)!} B_{m-j}^{\alpha} Bell_{j+\alpha}^{(p)} \mathcal{G}_n^{(\alpha)}(x;\lambda). \tag{3.2.11}$$

In the next section, certain results related to some mixed form of Apostol type polynomials are explored.

4 Concluding Remarks

As a remark, in this section we consider recently introduced mixed form of Apostol type polynomials defined as Hermite Apostol type polynomials (HATP) and investigate the properties of these polynomials which are derived from umbral calculus. We can establish connection between our polynomials and several known families of polynomials. For example we explore the relation involving Hermite Apostol type polynomials with Apostol type polynomials and Bernoulli polynomials.

The HATP ${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ are defined by the generating function [13]

$$\Big(\frac{2^{\mu}\, t^{\nu}}{\lambda e^t+1}\Big)^{\alpha} e^{2xt-t^2} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu)\frac{t^n}{n!}. \tag{4.1}$$

where ${}_H\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu) = {}_H\mathcal{F}_n^{(\alpha)}(0;\lambda;\mu,\nu)$ denotes the Hermite Apostol type numbers defined by.

$$\Big(\frac{2^{\mu}\, t^{\nu}}{\lambda e^t+1}\Big)^{\alpha} e^{-t^2} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}(\lambda;\mu,\nu)\frac{t^n}{n!}. \tag{4.2}$$

From (4.1) and (1.6), we note that

$${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) \sim \Big(e^{\frac{t^2}{4}}\Big(\frac{\lambda e^z+1}{2^{\mu}\, t^{\nu}}\Big)^{\alpha}, \frac{t}{2}\Big). \tag{4.3}$$

Also, by (1.2), (4.1)and (4.3), we have

$$\begin{aligned}
{}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) &= e^{-\frac{t^2}{4}}\Big(\frac{2^{\mu}\, t^{\nu}}{\lambda e^z+1}\Big)^{\alpha}(2x)^n \\
&= \frac{e^{-\frac{t^2}{4}}}{2^n}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) \\
&= \sum_{m=0}^{[\frac{n}{4}]} \frac{1}{m!2^n}\Big(\frac{-1}{2}\Big)^{2m}(n)_{2m}\mathcal{F}_{n-2m}^{(\alpha)}(x;\lambda;\mu,\nu)
\end{aligned}$$

$${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) = \sum_{m=0}^{[\frac{n}{4}]} \binom{n}{2m}\frac{(-1)^m\,(2m)!}{m!\,2^{n-2m}}\mathcal{F}_{n-2m}^{(\alpha)}(x;\lambda;\mu,\nu). \tag{4.4}$$

Further in view of relation (1.23), we deduce the following result from (4.4)

Corollary 4.1. For $n \geq 0$, we have :

$${}_H\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{[\frac{n}{4}]} \binom{n}{2m}\frac{(-1)^m\,(2m)!}{m!\,2^{n-2m}}\mathcal{B}_{n-2m}^{(\alpha)}(x;\lambda). \tag{4.5}$$

Also in view of relation (1.24), we deduce the following result from (4.4)

Corollary 4.2. For $n \geq 0$, we have :

$$ {}_H\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{[\frac{n}{4}]} \binom{n}{2m} \frac{(-1)^m\,(2m)!}{m!\,2^{n-2m}} \mathcal{E}_{n-2m}^{(\alpha)}(x;\lambda). \tag{4.6} $$

Furthermore, in view of relation (1.25), we deduce the following result from (4.4)

Corollary 4.3. For $n \geq 0$, we have :

$$ {}_H\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{[\frac{n}{4}]} \binom{n}{2m} \frac{(-1)^m\,(2m)!}{m!\,2^{n-2m}} \mathcal{G}_{n-2m}^{(\alpha)}(x;\lambda). \tag{4.7} $$

Let us consider the following two Sheffer sequences:

$$ \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) \sim \Big(e^{\frac{t^2}{4}}\Big(\frac{\lambda e^z+1}{2^\mu\, t^\nu}\Big)^\alpha, \frac{t}{2}\Big). $$

and

$$ \mathcal{B}_n^{(r)}(x) \sim \Big(\Big(\frac{\lambda e^z-1}{z}\Big)^r, z\Big). \tag{4.8} $$

Let us assume that

$$ {}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) = \sum_{k=0}^{n} C_{n,k}\mathcal{B}_n^{(r)}(x). \tag{4.9} $$

Then, by (1.8) and (1.9), we get

$$ \begin{aligned} C_{n,k} &= \frac{1}{m!}\langle\Big(\frac{e^z-1}{z}\Big)^r z^m | e^{-\frac{z^2}{4}}\Big(\frac{2^\mu\, z^\nu}{\lambda e^z+1}\Big)^\alpha (x)^n\rangle \\ &= \binom{n}{m}\sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+r}{r}} S_2(l+r,r)\,{}_H\mathcal{F}_{n-m-l}^{(\alpha)}. \end{aligned} \tag{4.10} $$

Therefore, by (4.9) and (4.10), we obtain the following theorem.

Theorem 4.4. For $n \geq 1$ and $\alpha \in \mathbb{Z}_+$, we have :

$$ {}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu,\nu) = \sum_{k=0}^{n}\binom{n}{m}\sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+r}{r}} S_2(l+r,r)\,{}_H\mathcal{F}_{n-m-l}^{(\alpha)}\mathcal{B}_n^{(r)}(x). \tag{4.11} $$

Further, we can derive other identities involving HATP and other families of polynomials. In our next investigation we established certain properties of other polynomials using umbral techniques.

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