Coefficient estimates for a subclass of analytic bi-univalent functions by means of Faber polynomial expansions

Serap Bulut

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Abstract In this work, considering a subclass of analytic bi-univalent functions, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in this class. For this purpose, we use the Faber polynomial expansions. In certain cases, our estimates improve some of those existing coefficient bounds.

1 Introduction

Let \( \mathcal{A} \) denote the class of all functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). We also denote by \( \mathcal{S} \) the class of all functions in the normalized analytic function class \( \mathcal{A} \) which are univalent in \( \mathbb{U} \).

It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), which is defined by

\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{U})
\]

and

\[
f(f^{-1}(w)) = w \quad (|w| < r_0(f) : r_0(f) \geq \frac{1}{4}).
\]

In fact, the inverse function \( g = f^{-1} \) is given by

\[
g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_3 - a_3) w^3 - (5a_4 - 5a_2 a_3 + a_4) w^4 + \cdots
\]

\[
= w + \sum_{n=2}^{\infty} A_n w^n.
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{U} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1.1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [25], where it was proved that \( |a_2| < 1.51 \). Brannan and Clunie [5] improved Lewin's result to \( |a_2| \leq \sqrt{2} \) and later Netanyahu [27] proved that \( |a_2| \leq 4/3 \). Brannan and Taha [6] and Taha [33] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \). For a brief history and interesting examples of functions in the class \( \Sigma \), see [31] (see also [6]). In fact, the aforementioned work of Srivastava et al. [31] essentially revived the investigation of various subclasses of the bi-univalent function class \( \Sigma \) in recent years; it was followed by such works as those by Frasin and Aouf [17], Xu et al. [35, 36], Hayami and Owa [22], and others (see, for example, [3, 7, 8, 9, 10, 14, 18, 26, 28, 29, 30]).

Not much is known about the bounds on the general coefficient \( |a_n| \) for \( n > 3 \). This is because the bi-univalency requirement makes the behavior of the coefficients of the function \( f \) and \( f^{-1} \) unpredictable. Here, in this paper, we use the Faber polynomial expansions for a general subclass of analytic bi-univalent functions to determine estimates for the general coefficient bounds \( |a_n| \).
The Faber polynomials introduced by Faber [16] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [19] and [21] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions.

In the literature, there is only a few works determined the general coefficient bounds \(|a_n|\) for the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions, [2, 11, 12, 13, 20, 23, 24, 32].

Now, we consider a subclass of analytic bi-univalent functions defined by Murugusundaramoorthy et al. [26].

**Definition 1.1.** (See [26]) A function \(f \in \Sigma\) given by (1.1) is said to be in the class \(\mathcal{M}_\Sigma (\alpha, \lambda)\) if the following conditions are satisfied:

\[
\Re \left( \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \right) > \alpha \tag{1.3}
\]

and

\[
\Re \left( \frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) > \alpha \tag{1.4}
\]

where \(0 \leq \alpha < 1; 0 \leq \lambda < 1; z, w \in \mathbb{U}\) and \(g = f^{-1}\) is defined by (1.2).

Note that, for \(\lambda = 0\), the class \(\mathcal{M}_\Sigma (\alpha, \lambda)\) reduces to \(S_\Sigma^* (\alpha)\) bi-starlike functions of order \(\alpha (0 \leq \alpha < 1)\).

Murugusundaramoorthy et al. [26] obtained the following coefficient estimates for the functions belonging the class \(\mathcal{M}_\Sigma (\alpha, \lambda)\).

**Theorem 1.2.** [26] Let \(f (z)\) given by (1.1) be in the class \(\mathcal{M}_\Sigma (\alpha, \lambda), 0 \leq \alpha < 1\) and \(0 \leq \lambda < 1\). Then

\[
|a_2| \leq \frac{\sqrt{2} (1-\alpha)}{1-\lambda}, \tag{1.5}
\]

\[
|a_3| \leq \frac{4 (1-\alpha)^2}{(1-\lambda)^2} \left[ 1 - \frac{1-\alpha}{1-\lambda} \right]. \tag{1.6}
\]

Later Bulut [9] improved these results as follows:

**Theorem 1.3.** [9] Let the function \(f (z)\) given by the Taylor-Maclaurin series expansion (1.1) be in the function class \(\mathcal{M}_\Sigma (\alpha, \lambda)\). Then

\[
|a_2| \leq \begin{cases} 
\frac{\sqrt{2(1-\alpha)}}{1-\lambda}, & 0 \leq \alpha \leq \frac{1}{2} \\
\frac{2(1-\alpha)}{1-\lambda}, & \frac{1}{2} \leq \alpha < 1
\end{cases}
\]

and

\[
|a_3| \leq \begin{cases} 
\frac{2(1-\alpha)}{(1-\lambda)^2}, & 0 \leq \alpha \leq \frac{3-\lambda}{4} \\
\frac{4(1-\alpha)^2}{(1-\lambda)^2} + \frac{1-\alpha}{1-\lambda}, & \frac{3-\lambda}{4} \leq \alpha < 1
\end{cases}
\]

The object of the present paper is to give an upper bound for the coefficients \(|a_n|\) of analytic bi-univalent functions in the class \(\mathcal{M}_\Sigma (\alpha, \lambda)\) by using Faber polynomials.

## 2 Coefficient estimates

Using the Faber polynomial expansion of functions \(f \in \mathcal{A}\) of the form (1.1), the coefficients of its inverse map \(g = f^{-1}\) may be expressed as, [1]:

\[
g (w) = f^{-1} (w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n} (a_2, a_3, \ldots, a_n) w^n, \tag{2.1}
\]
where
\[
K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)! (n-1)!} a_2^{n-1} + \frac{(-n)!}{(2 (-n+1))! (n-3)!} a_2^{n-3} a_3
\]
\[
+ \frac{(-n)!}{(-2n+3)! (n-4)!} a_2^{n-4} a_4
\]
\[
+ \frac{(-n)!}{(2 (-n+2))! (n-5)!} a_2^{n-5} [a_5 + (-n+2) a_3^2]
\]
\[
+ \frac{(-n)!}{(-2n+5)! (n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4]
\]
\[
+ \sum_{j \geq 7} a_2^{n-j} V_j,
\]
\[(2.2)\]
such that \(V_j\) \((7 \leq j \leq n)\) is a homogeneous polynomial in the variables \(a_2, a_3, \ldots, a_n\), [4]. In particular, the first three terms of \(K_{n-1}^{-n}\) are
\[
K_1^{-2} = -2a_2,
\]
\[
K_2^{-3} = 3 (2a_2^2 - a_3),
\]
\[
K_3^{-4} = -4 (5a_2^3 - 5a_2 a_3 + a_4).
\]
In general, for any \(p \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}\), an expansion of \(K^p\) is as, [1],
\[
K_{n-1}^p = p a_n + \frac{p(p-1)}{2} D_{n-1}^2 + \frac{p!}{(p-3)! 3!} D_{n-1}^3 + \cdots + \frac{p!}{(p-n+1)! (n-1)!} D_{n-1}^{n-1}, \quad (2.3)
\]
where \(D_{n-1}^p = D_{n-1}^p (a_2, a_3, \ldots, a_n)\), and by [34],
\[
D_{n-1}^m (a_2, \ldots, a_n) = \sum \frac{m!}{i_1! \cdots i_{n-1}!} a_2^{i_1} \cdots a_n^{i_{n-1}}
\]
and the sum is taken over all non-negative integers \(i_1, \ldots, i_{n-1}\) satisfying
\[
i_1 + i_2 + \cdots + i_{n-1} = m
\]
\[
i_1 + 2i_2 + \cdots + (n-1) i_{n-1} = n-1.
\]
It is clear that \(D_{n-1}^{n-1} (a_2, \ldots, a_n) = a_2^{n-1}\).

Consequently, for functions \(f \in \mathcal{M}_E (\alpha, \lambda)\) of the form (1.1), we can write:
\[
\frac{zf' (z)}{F_\lambda (z)} = 1 + \sum_{n=2}^{\infty} F_{n-1} (b_2, b_3, \ldots, b_n) z^{n-1}, \quad (2.4)
\]
where
\[
F_\lambda (z) = (1 - \lambda) f (z) + \lambda z f' (z) = z + \sum_{n=2}^{\infty} b_n z^n
\]
with
\[
b_n = [1 + (n-1) \lambda] a_n.
\]
So we get
\[
F_{n-1} (b_2, b_3, \ldots, b_n) = (n a_n - b_n) + \sum_{j=1}^{n-2} K_j^{-1} (b_2, b_3, \ldots, b_{j+1}) [(n-j) a_{n-j} - b_{n-j}].
\]

Our first theorem introduces an upper bound for the coefficients \(|a_n|\) of analytic bi-univalent functions in the class \(\mathcal{M}_E (\alpha, \lambda)\).
Theorem 2.1. For $0 \leq \lambda < 1$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{M}_\Sigma (\alpha, \lambda)$ be given by (1.1). If $a_k = 0$ ($2 \leq k \leq n - 1$), then

$$|a_n| \leq \frac{2(1 - \alpha)}{(n - 1)(1 - \lambda)} \quad (n \geq 4).$$

Proof. For the function $f \in \mathcal{M}_\Sigma (\alpha, \lambda)$ of the form (1.1), we have the expansion (2.4) and for the inverse map $g = f^{-1}$, considering (1.2), we obtain

$$\frac{wg'(w)}{G_\lambda (w)} = 1 + \sum_{n=2}^{\infty} F_{n-1} (B_2, B_3, \ldots, B_n) w^{n-1}$$

where

$$G_\lambda (w) = (1 - \lambda) g(w) + \lambda wg'(w) = z + \sum_{n=2}^{\infty} B_n w^n$$

with

$$B_n = [1 + (n - 1) \lambda] A_n.$$

On the other hand, since $f \in \mathcal{M}_\Sigma (\alpha, \lambda)$ and $g = f^{-1} \in \mathcal{M}_\Sigma (\alpha, \lambda)$, by definition, there exist two positive real part functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}$$

and

$$q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{A},$$

where

$$\Re (p(z)) > 0 \quad \text{and} \quad \Re (q(w)) > 0$$

in $U$ so that

$$\frac{zf'(z)}{F_\lambda (z)} = \alpha + (1 - \alpha) p(z) \quad (2.6)$$

and

$$\frac{wg'(w)}{G_\lambda (w)} = \alpha + (1 - \alpha) q(w). \quad (2.7)$$

Note that, by the Carathéodory lemma (e.g., [15]), $|c_n| \leq 2$ and $|d_n| \leq 2$ ($n \in \mathbb{N} := \{1, 2, \ldots\}$). Comparing the corresponding coefficients of (2.4) and (2.6), for any $n \geq 2$, yields

$$(n a_n - b_n) + \sum_{j=1}^{n-2} K_j^{-1} (b_2, b_3, \ldots, b_{j+1}) [(n - j) a_{n-j} - b_{n-j}] = (1 - \alpha) c_{n-1} \quad (2.8)$$

and similarly, from (2.5) and (2.7) we find

$$(n A_n - B_n) + \sum_{j=1}^{n-2} K_j^{-1} (B_2, B_3, \ldots, B_{j+1}) [(n - j) A_{n-j} - B_{n-j}] = (1 - \alpha) d_{n-1}. \quad (2.9)$$

Note that for $a_k = 0$ ($2 \leq k \leq n - 1$), we have $A_n = -a_n$ and so

$$n a_n - b_n = (n - 1) (1 - \lambda) a_n = (1 - \alpha) c_{n-1},$$

$$n A_n - B_n = - (n - 1) (1 - \lambda) a_n = (1 - \alpha) d_{n-1}.$$

Taking the absolute values of the above equalities, we obtain

$$|a_n| = \frac{(1 - \alpha)|c_{n-1}|}{(n - 1)(1 - \lambda)} = \frac{(1 - \alpha)|d_{n-1}|}{(n - 1)(1 - \lambda)} \leq \frac{2(1 - \alpha)}{(n - 1)(1 - \lambda)},$$

which completes the proof of the Theorem 2.1. \qed
The following corollary is immediate consequence of the above theorem.

**Corollary 2.2.** Let the function $f \in S_2^\alpha$ ($0 \leq \alpha < 1$) be given by (1.1). If $a_k = 0$ ($2 \leq k \leq n - 1$), then

$$|a_\alpha| \leq \frac{2(1 - \alpha)}{n - 1} \quad (n \geq 4).$$

**Theorem 2.3.** For $0 \leq \lambda < 1$ and $0 \leq \alpha < 1$, let the function $f \in M_\lambda^\alpha$ ($\alpha$) be given by (1.1). Then one has the following

$$|a_2| \leq \begin{cases} \frac{\sqrt{2(1-\alpha)}}{1-\lambda}, & 0 \leq \alpha < \frac{1}{2}, \\ \frac{2(1-\alpha)}{1-\lambda}, & \frac{1}{2} \leq \alpha < 1 \end{cases},$$

(2.10)

$$|a_3| \leq \begin{cases} \frac{2(1-\alpha)}{(1-\lambda)^3}, & 0 \leq \alpha \leq \frac{3-\lambda}{4}, \\ \frac{4(1-\alpha)^2}{(1-\lambda)^3} + \frac{1-\alpha}{4}, & \frac{3-\lambda}{4} \leq \alpha < 1 \end{cases}.$$\hspace{1cm} (2.11)

**Proof.** If we set $n = 2$ and $n = 3$ in (2.8) and (2.9), respectively, we get

$$(1 - \lambda) a_2 = (1 - \alpha) c_1,$$\hspace{1cm} (2.12)

$$(1 - \lambda) [2 a_3 - (1 + \lambda) a_2^2] = (1 - \alpha) c_2,$$\hspace{1cm} (2.13)

$$- (1 - \lambda) a_2 = (1 - \alpha) d_1,$$\hspace{1cm} (2.14)

$$(1 - \lambda) [(3 - \lambda) a_2^2 - 2 a_3] = (1 - \alpha) d_2.$$\hspace{1cm} (2.15)

From (2.12) and (2.14), we find (by the Carathéodory lemma)

$$|a_2| = \frac{(1 - \alpha) |c_1|}{1 - \lambda} = \frac{(1 - \alpha) |d_1|}{1 - \lambda} \leq \frac{2(1 - \alpha)}{1 - \lambda}.$$\hspace{1cm} (2.16)

Also from (2.13) and (2.15), we obtain

$$2 (1 - \lambda)^2 a_3^2 = (1 - \alpha) (c_2 + d_2).$$\hspace{1cm} (2.17)

Using the Carathéodory lemma, we get

$$|a_2| \leq \frac{\sqrt{2(1-\alpha)}}{1-\lambda},$$

and combining this with the inequality (2.16), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (2.10).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.15) from (2.13). We thus get

$$a_3 = a_3^2 + \frac{(1 - \alpha) (c_2 - d_2)}{4(1 - \lambda)}.$$\hspace{1cm} (2.18)

Upon substituting the value of $a_3^2$ from (2.12) into (2.18), it follows that

$$a_3 = \frac{(1 - \alpha)^2 c_1^2}{(1 - \lambda)^2} + \frac{(1 - \alpha) (c_2 - d_2)}{4(1 - \lambda)}.$$\hspace{1cm}

We thus find (by the Carathéodory lemma) that

$$|a_3| \leq \frac{4(1 - \alpha)^2}{(1 - \lambda)^2} + \frac{1 - \alpha}{1 - \lambda}.$$\hspace{1cm} (2.19)
On the other hand, upon substituting the value of $a_2^2$ from (2.17) into (2.18), it follows that

$$a_3 = \frac{1 - \alpha}{4(1-\lambda)^2} \left[ (3-\lambda) a_2 + (1+\lambda) d_2 \right].$$

Consequently, (by the Caratheodory lemma) we have

$$|a_3| \leq \frac{2(1-\alpha)}{(1-\lambda)^2}. \quad (2.20)$$

Combining (2.19) and (2.20), we get the desired estimate on the coefficient $|a_3|$ as asserted in (2.11). This evidently completes the proof of Theorem 2.3. $\square$

Note that, Theorem 2.3 gives another proof of Theorem 1.3. By setting $\lambda = 0$ in Theorem 2.3, we obtain the following consequence.

**Corollary 2.4.** Let the function $f \in S^*_\lambda (\alpha)$ $(0 \leq \alpha < 1)$ be given by (1.1). Then one has the following

$$|a_2| \leq \begin{cases} \sqrt{2}(1-\alpha) , \quad 0 \leq \alpha < \frac{1}{2} \\ 2(1-\alpha) , \quad \frac{1}{2} \leq \alpha < 1 \end{cases},$$

$$|a_3| \leq \begin{cases} 2(1-\alpha) , \quad 0 \leq \alpha \leq \frac{3}{4} \\ 4(1-\alpha)^2 + (1-\alpha) , \quad \frac{3}{4} \leq \alpha < 1 \end{cases}.$$ 

References


**Author information**  
Serap Bulut, Faculty of Aviation and Space Sciences, Kocaeli University, Arslanbey Campus, 41285 Kartepe, Kocaeli, TURKEY.  
E-mail: serap.bulut@kocaeli.edu.tr  

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