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Uniqueness in von Neumann Regular Unital Rings

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Abstract. Forty years ago, Hartwig and Luh proved in (Pac. J. Math., 1977) that a unit-regular ring having unique inner inverse unit for each its element is either a boolean ring or a division ring, and vice versa. Extending their result, we define and explore certain other variations of uniquely elements in von Neumann regular rings.

1 Introduction and Background

Everywhere in the text of the present paper all our rings are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are mainly in agreement with those from [3] and [10]. For instance, the following three classical concepts are well-known and stated in [2] (all of them being equivalent in the commutative case).

We will hereafter use for simpleness the term "regular" instead of "von Neumann regular".

Definition 1.1. A ring R is called *regular* if, for any $r \in R$, there is an element $a \in R$ such that $r = rar$.

More restrictive properties are due to the following.

Definition 1.2. A ring R is called *unit-regular* if, for any $r \in R$, there is a unit $u \in R$ such that $r = rur$.

It is a long time known that unit-regular rings are always regular, but this cannot be reversed in the non-commutative case, however. Likewise, a ring R is unit-regular if, and only if, every its element r can be written as $r = ve$ for some unit v and idempotent e. Indeed, it is elementarily to verify that the element ur is an idempotent, say e, and consequently $r = u^{-1}e$ setting $u^{-1} = v$.

Little more restrictive properties can be viewed by the following.

Definition 1.3. A ring R is called *strongly regular* if, for any $r \in R$, there is an element $a \in R$ such that $r = r^2a$.

As it is known, a can be chosen to be a unit with $ar = ra$, so that $r = ar^2$ (that is, the initial equality $r = r^2a$ is left-right symmetric) and thus strongly regular rings are unit-regular, but the converse fails in general. Actually, strongly regular rings are precisely the reduced regular rings, i.e., exactly the abelian regular rings. Also, in a more concrete flavor, every strongly regular ring is a subdirect product (sum) of division rings, and visa versa (cf. [3], [6] and [7]).

The leitmotif of this article is to examine in different aspects the existence of unique elements in the defined above three classes of regular rings.

2 Uniqueness in Regular Rings

The next notion, which corresponds to Definition 1.1, is pivotal for our further work.

Definition 2.1. A ring R is said to be *uniquely regular* if, for any non-zero $r \in R$, there is a unique element $a \in R$ such that $r = rar$.

We are now ready to proceed by proving with the following assertion.

Theorem 2.2. *A ring R is uniquely regular if, and only if, R is a division ring.*

Proof. Let $r \in R$ be an arbitrary element and let there exist a unique element $a \in R$ with $r = rar$. Since it is readily checked that $r^2ar = r^2 = rar^2$, it follows that we can write

$$r(a(1 - r(1 - ra)))r = rar = r((1 - (1 - ar)r)a)r.$$

Utilizing the uniqueness, we therefore deduce that

$$a - ar(1 - ra) = a = a - (1 - ar)ra.$$

This allows us to derive that $ar = ar^2a = ra$. Hence it follows at once that $r = rar = r^2a$ and thus R is strongly regular. This means that any element r in R can be written as $r = ue$ for some unit u and idempotent e (with $ue = eu$). But R does not possess non-trivial idempotents as well. In fact, if e is a non-zero idempotent, then $e = e.1.e = e.e.e$ which forces that $e = 1$. So, this arbitrary element r of R is either 0 or u, which substantiates our claim after all.

Reciprocally, suppose now that R is a division ring. Given $0 \neq r \in R$, it follows that r inverts in R and thus $r = rr^{-1}r$ seeing that r^{-1} is unique for r, as needed. □

Remark 2.3. Another argument to show that R is a division ring could be like this: For all $0 \neq r \in R$, the equality $r = rar$ is tantamount to $r = r(a+(1-ar))r = r(a+(1-ra))r = rar$. Hence the uniqueness gives that $a + 1 - ar = a + 1 - ra = a$, that is, $ar = 1 = ra$, i.e., all r are invertible elements, as required.

3 Uniqueness in Unit-Regular Rings

The next notion, which corresponds to Definition 1.2, is key for our further work.

Definition 3.1. A ring R is said to be *uniquely unit-regular* if, for any non-zero $r \in R$, there is a unique unit $u \in R$ such that $r = rur$.

This is obviously tantamount to the following: A ring R is uniquely unit-regular if, for any non-zero $r \in R$, there is a unique unit $v \in R$ with $r = ve$ for some idempotent $e \in R$. In fact, $v = u^{-1}$ and $e = ur$.

Nevertheless, the situation is rather complicated when we ask for the uniqueness of the existing idempotent. Specifically, one can state the following:

Definition 3.2. A ring R is said to be *pseudo uniquely unit-regular* if, for each non-zero $r \in R$, there is a unique idempotent e such that $r = ue$ for some unit $u \in R$.

We now come to the following principal known statement ([5, Theorem 4]).

Theorem 3.3. *A ring R is uniquely unit-regular if, and only if, R is either a boolean ring or a division ring.*

We may sharp this to the following.

Theorem 3.4. *A ring R is pseudo uniquely unit-regular if, and only if, R is a subdirect product of division rings.*

Proof. Suppose first that R is pseudo uniquely unit-regular. As above, for an arbitrary $r \in R$, we write

$$r(u(1 - r(1 - ru)))r = rur = r((1 - (1 - ur)r)u)r,$$

where u is a unit. It is not too hard to be checked that $u(1-r(1-ru))$ is a unit with the inverse $(1+r(1-ru))u^{-1}$ and $(1-(1-ur)r)u$ is a unit with the inverse $u^{-1}(1+(1-ur)r)$.

Since ur, $u(1-r(1-ru))r$ and $(1-(1-ur)r)ur$ are idempotents, say e, f and g, respectively, it follows directly that $r = u^{-1}e = (1+r(1-ru))u^{-1}f = u^{-1}(1+(1-ur)r)g$. Thus the idempotent's uniqueness implies that $e = f = g$, i.e.,

$$u(1-r(1-ru))r = ur = (1-(1-ur)r)ur.$$

Canceling by u the first equality, we infer that $(1-r(1-ru))r = r$ which amounts to $r(1-ru)r = 0$, that is, $r^2 - rrur = r^2 - r^2 = 0$ which is always fulfilled and so there is nothing new.

As treating the second equality, it leads to $(1-ur)rur = (1-ur)r = 0$, i.e., $r = ur^2$. This shows that R has to be a strongly regular ring. We furthermore apply [3] or [10] to get that R is a subdirect product of division rings, indeed.

Suppose conversely that R is a subdirect product of division rings. This again in view of [3] or [10] means that R is a strongly regular ring, that is, every element $r \in R$ can be written as $r = ue = eu$, where u is a unit and e is an idempotent. We shall show now that e is unique. To that goal, writing $r = ue = vf$ for some unit v and idempotent f with $vf = fv$, it follows that $e = f$. In fact, denoting $v^{-1}u = w$, we have that $we = f$. Thus $wewe = we$, i.e., $ewe = e$ which enables us that $ev^{-1}ue = e$, that is, $ev^{-1}vf = e$ and $ef = e$. A similar trick with $e = w^{-1}f$ insures that $w^{-1}fw^{-1}f = w^{-1}f$ is amounting to $fw^{-1}f = f$, i.e., $fu^{-1}vf = f$ which assures that $fu^{-1}ue = f$ and so $fe = f$. But all idempotents in strongly regular rings are central, so that $ef = fe$ which finally ensures that $e = f$, as wanted. □

Actually, the last affirmation states that a ring is pseudo uniquely unit-regular if, and only if, it is strongly regular.

4 Uniqueness in Strongly Regular Rings

The next notion, which corresponds to Definition 1.3, is basic for our further work.

Definition 4.1. A ring R is said to be *uniquely strongly regular* if, for any non-zero $r \in R$, there is a unique unit $u \in R$ such that $r = r^2u$.

Certainly, since such a ring is strongly regular, it must be that $ru = ur$ and so this ring is definitely uniquely unit-regular. It is also elementarily seen that boolean rings and division rings are themselves uniquely strongly regular.

Definition 4.2. A ring R is said to be *pseudo uniquely strongly regular* if, for any non-zero $r \in R$, there is a unique idempotent e such that $r = ue$ for some unit $u \in R$ with $ue = eu$.

This makes such a ring pseudo uniquely unit-regular. It is also routinely observed that the direct product of division rings is pseudo uniquely strongly regular.

We have now all the ingredients necessary to establish the following.

Theorem 4.3. *A ring R is uniquely strongly regular if, and only if, R is either a boolean ring or a division ring.*

Proof. It follows immediately from the proof of the aforementioned [5, Lemma 4 and Theorem 4]. □

We may now sharpen this to the following.

Theorem 4.4. *A ring R is pseudo uniquely strongly regular if, and only if, R is a subdirect product of division rings.*

Proof. It follows in the same manner as that of Theorem 3.4 alluded to above. □

5 Concluding Discussion

It is worthwhile noticing that our results quoted above could be generalized in the class of π-regular rings, but we leave that for the interested reader.

We also state here some more fundamentals: A ring R is called *clean* if each its element is the sum of a unit and an idempotent; if these two elements do commute, the clean ring is said to be *strongly clean*. It was proved in [1] that unit-regular rings are clean. Very optimistically, it was asked in [8] whether or not unit-regular rings are even strongly clean. However, recently was constructed a counter-example in [9] that there exists a special unit-regular ring which is not strongly clean. On the other vein, it was shown in [4] that there is a regular ring which need not be clean.

In conclusion, all of the considered above (pseudo) uniquely (unit-) regular rings are necessarily strongly regular, that is, a subdirect product of division rings, and hence they are strongly clean. This adequately leads us to the following:

Conjecture. *If R is a unit-regular ring with a finite number of unit inner inverses for each its element, then R is strongly clean. In particular, unit-regular rings with a finite number of units are strongly clean.*

References

[1] V.P. Camillo and D. Khurana, *A characterization of unit regular rings*, Commun. Algebra **29** (2001), 2293–2295.

[2] G. Ehrlich, *Unit-regular rings*, Portugal. Math. **27** (1968), 209–212.

[3] K.R. Goodearl, Von Neumann Regular Rings, second edition, Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991.

[4] D. Handelman, *Perspectivity and cancellation in regular rings*, J. Algebra **48** (1977), 1–16.

[5] R.E. Hartwig and J. Luh, *A note on the group structure of unit regular ring elements*, Pac. J. Math. **71** (1977), 449–461.

[6] S. Lajos and F. Szász, *Characterizations of strongly regular rings*, Proc. Japan Acad. (Math.) **46** (1970), 38–40.

[7] J. Luh, *A note on strongly regular rings*, Proc. Japan Acad. (Math.) **40** (1964), 74–75.

[8] W.K. Nicholson, *Strongly clean rings and Fitting's lemma*, Commun. Algebra **27** (1999), 3583–3592.

[9] P.P. Nielsen and J. Šter, *Connections between unit-regularity, regularity, cleanness and strong cleannessof elements and rings*, Trans. Amer. Math. Soc. (2018), in press.

[10] A.A. Tuganbaev, Rings Close to Regular, Mathematics and its Applications **545**, Kluwer Academic Publishers, Dordrecht, 2002.

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