Palestine Journal of Mathematics

Vol. 7(1)(2018) , 73–79 © Palestine Polytechnic University-PPU 2018

General Gamma type operators in L^p spaces

Alok Kumar

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 41A25, 26A15, Secondary 41A36, 40A35.

Keywords and phrases: Gamma type operators, Approximation order, Modulus of smoothness, Fubini theorem.

The author is grateful to the referee for a careful reading of the manuscript, suggestions and comments which improved the presentation of the paper.

Abstract In the present paper, we investigate the convergence and the approximation order of general Gamma type operators in $L^p(1 \leq p \leq \infty)$ spaces. The results are given in terms of some Ditzian-Totik modulus of smoothness.

1 Introduction

In [6], İzgi and B$\ddot{u}$y$\ddot{u}$kyazici introduced the following Gamma type linear and positive operators

$$\begin{aligned} L_n(f;x) &= \int_0^\infty \int_0^\infty g_{n+2}(x,u)g_n(u,t)f(t)dudt \\ &= \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t)dt, \ x > 0. \end{aligned}$$

Approximation properties of L_n were examined by several researchers (see [4], [5], [11], [12], [13], [14], [16], [17]).
In the year 2007, Mao [21] defined the following generalized Gamma type operators

$$\begin{aligned} M_{n,k}(f;x) &= \int_0^\infty \int_0^\infty g_n(x,u)g_{n-k}(u,t)f(t)dudt \\ &= \frac{(2n-k+1)!}{n!(n-k)!} x^{n+1} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t)dt, \ x > 0, \end{aligned} \tag{1.1}$$

for any f for which the above integral is convergent.
The rate of convergence of these operators for functions with derivatives of bounded variation were studied in [15]. Some approximation results for these operators based on $q-$integers were obtained in [18]. Global approximation theorems for these operators were obtained in [8].
Recently, Alok Kumar [7] obtained the following result.

Lemma 1.1. *[7] If r^{th} derivative $f^{(r)}(r = 0, 1, 2...)$ exists continuously, then we get*

$$M^{(r)}_{n,k}(f;x) = \beta_n x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} f^{(r)}(t)dt, \ x \in (0,\infty),$$

where

$$\beta_n = \frac{(2n-k+1)!}{n!(n-k)!}.$$

The Voronovskaja type theorem and local rate of convergence for the operators $M^{(r)}_{n,k}$ was given in [7]. In [10], some approximation properties of $M^{(r)}_{n,k}$ in polynomial weighted spaces

were studied.
In this paper we shall study a global rate of convergence of $M_{n,k}^{(r)}$ in L^p spaces in terms of the modulus of smoothness. We prove a direct approximation theorem for the operator $M_{n,k}^{(r)}$ using an equivalence between the Peetre $\mathcal{K}$-functional and the modulus of smoothness.
Let L^p denote the set of the Lebesgue measurable functions functions f defined on $(0,\infty)$ such that

$$\int_0^\infty |f(t)|^p dt < \infty, \quad 1 \le p < \infty,$$

and f is bounded almost everywhere on $(0,\infty)$ if $p=\infty$.
The weighted modulus of smoothness for $f \in L^p$ is defined as

$$\omega_{2,\varphi}\left(f;\sqrt{\delta}\right)_p = \sup_{0<|h|\le\sqrt{\delta}} \| \triangle_{h\varphi}^2 (f;.)\|_{L^p}, \quad \delta > 0,$$

where $\varphi(x) = x$ and

$$\triangle_{h\varphi}^2(f;x) = f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)), \; x,h \in (0,\infty).$$

For AC_{loc} (set of all locally absolutely continuous functions on $(0,\infty)$), we consider the following Peetre $\mathcal{K}$-functional:

$$\mathcal{K}_{2,\varphi}(f;\delta)_p = \inf_{g\in W_{2,\varphi}^p} \left\{\| f-g \|_{L^p} + \delta \| \varphi^2 g'' \|_{L^p}\right\},$$

where $\delta > 0$ and $W_{2,\varphi}^p = \left\{g \in L^p : g' \in AC_{loc}, \varphi^2 g'' \in L^p\right\}$. By Theorem 3.1.2, p. 24, [2], it follows that

$$\mathcal{C}^{-1}\omega_{2,\varphi}\left(f;\sqrt{\delta}\right)_p \le \mathcal{K}_{2,\varphi}(f;\delta)_p \le \mathcal{C}\omega_{2,\varphi}\left(f;\sqrt{\delta}\right)_p, \tag{1.2}$$

for some constant $\mathcal{C} > 0$.

2 Auxiliary results

In this section we give some preliminary results which will be used in the main part of this paper.
In [7], the author defined the sequence of linear and positive operators $\{M_{n,k,r}^*\}$ as

$$M_{n,k,r}^*(g;x) = \frac{\beta_n}{b(n,k,r)} x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} g(t)dt, \tag{2.1}$$

where

$$b(n,k,r) = \beta_n x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} dt = \frac{(n-r)!(n-k+r)!}{n!(n-k)!}.$$

Let us consider

$$e_m(t) = t^m, \;\; \phi_{x,m}(t) = (t-x)^m, \;\; m \in N_0, \; x,t \in (0,\infty).$$

Lemma 2.1. *[7] For any $m \in N_0$, $m+r \le n$ and $r \le n$ we have*

$$M_{n,k,r}^*(e_m;x) = \frac{(n-r-m)!(n-k+r+m)!}{(n-r)!(n-k+r)!} x^m,$$

and

$$M_{n,k,r}^*(\phi_{x,m};x) = \left(\sum_{j=0}^m (-1)^j \binom{m}{j} \frac{(n-r-m+j)!(n-k+r+m-j)!}{(n-r)!(n-k+r)!}\right) x^m,$$

for each $x \in (0,\infty)$.

Lemma 2.2. *[7] For $m = 0, 1, 2, 3, 4$, one has*

(i) $M^*_{n,k,r}(\phi_{x,0}; x) = 1$,

(ii) $M^*_{n,k,r}(\phi_{x,1}; x) = \dfrac{2r - k + 1}{n - r} x$,

(iii) $M^*_{n,k,r}(\phi_{x,2}; x) = \dfrac{4r^2 + 4r(2 - k) + 2n + k^2 - 5k + 4}{(n - r)(n - r - 1)} x^2$,

(iv) $M^*_{n,k,r}(\phi_{x,3}; x) = \dfrac{c_{n,k,r}}{(n - r)(n - r - 1)(n - r - 2)} x^3$,

(v) $M^*_{n,k,r}(\phi_{x,4}; x) = \dfrac{d_{n,k,r}}{(n - r)(n - r - 1)(n - r - 2)(n - r - 3)} x^4$,

where $c_{n,k,r} = 8r^3 + r^2(36 - 2k) + r(51 + 14n - 42k + 6k^2) - k^3 + 12k^2 - 34k - n^2 + n(17 - 6k - 6k^2 + 2kr) + 21$ *and*
$d_{n,k,r} = 16r^4 + r^3(128 - 32k) + r^2(348 + 48n - 216k + 24k^2) + r(366 + 177n + k(6n^2 - 54n - 440) + 120k^2 - 8k^3) + k^4 + k^3(4n - 22) + 139k^2 - k(245 + 116n) + 24n^2 + 131n + 100.$

3 Main Results

In this section we give a theorem on the degree of approximation of the function $f \in L^p$, $1 \leq p \leq \infty$ by the operators $M^*_{n,k,r}$.

Theorem 3.1. *Let $f \in L^p$, $1 \leq p \leq \infty$. Then, there exists a positive constant $\mathcal{C}$ such that*

$$\|M^*_{n,k,r}(f)\|_{L^p} \leq \mathcal{C}\|f\|_{L^p}.$$

Proof. Let $f \in L^p$, $1 \leq p \leq \infty$. For $p = 1$, we have

$$\begin{aligned} |M^*_{n,k,r}(f; x)| &= \left| \frac{\beta_n}{b(n,k,r)} x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} f(t)dt \right| \\ &\leq \frac{\beta_n}{b(n,k,r)} \int_0^\infty \frac{x^{n+1-r} t^{n-k+r}}{(x+t)^{2n-k+2}} |f(t)|dt. \end{aligned}$$

Then

$$\begin{aligned} \|M^*_{n,k,r}(f)\|_{L^1} &= \int_0^\infty |M^*_{n,k,r}(f; x)|dx \\ &\leq \frac{\beta_n}{b(n,k,r)} \int_0^\infty \left(\int_0^\infty \frac{x^{n+1-r} t^{n-k+r}}{(x+t)^{2n-k+2}} |f(t)|dt \right) dx. \end{aligned}$$

Using

$$\int_0^\infty \frac{x^{b-1}dx}{(1+ax)^{b+c}} = \frac{(b-1)!(c-1)!}{a^b(b+c-1)!}, \quad a, b, c \in (0, \infty), \tag{3.1}$$

we obtain

$$\int_0^\infty \frac{x^{n+1-r} t^{n-k+r}}{(x+t)^{2n-k+2}} dx = \frac{(n+1-r)!(n-k+r-1)!}{(2n-k+1)!}.$$

Then, we get

$$\begin{aligned} \|M^*_{n,k,r}(f)\|_{L^1} &\leq \frac{\beta_n}{b(n,k,r)} \times \frac{(n+1-r)!(n-k+r-1)!}{(2n-k+1)!} \|f\|_{L^1} \\ &\leq \frac{n+1-r}{n-k+r} \|f\|_{L^1} \leq \mathcal{C}\|f\|_{L^1} \end{aligned}$$

where $\mathcal{C}$ is a positive constant.
Let $p=\infty$. Observe that

$$\begin{aligned}|M^*_{n,k,r}(f;x)| &\leq \frac{\beta_n}{b(n,k,r)}\int_0^\infty \frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}|f(t)|dt\\ &\leq \|f\|_{L^\infty}\frac{\beta_n}{b(n,k,r)}\int_0^\infty \frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}dt=\|f\|_{L^\infty}.\end{aligned}$$

Then, we have

$$\|M^*_{n,k,r}(f)\|_{L^\infty}=\sup_{x\in(0,\infty)} ess|M^*_{n,k,r}(f;x)|\leq \|f\|_{L^\infty}.$$

Thus, by applying the Riesz-Thorin theorem (see [1]), we get the required result. □

Lemma 3.2. *[2] Let $g\in W^p_{2,\varphi}, 1\leq p\leq\infty$. Then, there exists a positive constant $\mathcal{C}$ such that*

$$\|\varphi g'\|_{L^p}\leq \mathcal{C}\left(\|g\|_{L^p}+\|\varphi^2 g''\|_{L^p}\right).$$

Lemma 3.3. *Let $\xi(f;x)(t)=\int_x^t(t-v)f(v)dv$. If $f\in L^p, 1\leq p\leq\infty$, then*

$$\|M^*_{n,k,r}(\xi(f;.);.)\|_{L^p}\leq \frac{\mathcal{C}}{n}\|\varphi^2 f\|_{L^p},$$

where $\mathcal{C}$ is a positive constant independent of f.

Proof. For $1<p<\infty$, proof is follows by the Riesz-Thorin theorem.
Let $p=1$. Using the Fubini theorem, we obtain

$$\begin{aligned}\left\|M^*_{n,k,r}(\xi(f;.))\right\|_{L^1} &\leq \int_0^\infty M^*_{n,k,r}(|\xi(f;x)|;x)dx\\ &\leq \int_0^\infty\left(\int_0^x \frac{\beta_n}{b(n,k,r)}\frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}\left(\int_t^x(v-t)|f(v)|dv\right)dt\right)dx\\ &- \int_0^\infty\left(\int_x^\infty \frac{\beta_n}{b(n,k,r)}\frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}\left(\int_x^t(v-t)|f(v)|dv\right)dt\right)dx\\ &= \int_0^\infty |f(v)|W(v)dv,\end{aligned}$$

where

$$W(v)=\left(\int_v^\infty\int_0^v-\int_0^v\int_v^\infty\right)\frac{\beta_n}{b(n,k,r)}\frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}(v-t)dtdx.$$

Observe that
$W(v)$

$$\begin{aligned}&= \left(\int_0^\infty\int_0^v-\int_0^v\int_0^v-\int_0^v\int_0^\infty+\int_0^v\int_0^v\right)\frac{\beta_n}{b(n,k,r)}.\frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}(v-t)dtdx\\ &= \int_0^v(v-t)\left(\int_0^\infty\frac{\beta_n}{b(n,k,r)}\frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}dx\right)dt-\int_0^v\int_0^\infty\frac{(v-t)\beta_n}{b(n,k,r)}\frac{x^{n+1-r}t^{n-k+r}}{(x+t)^{2n-k+2}}dtdx.\end{aligned}$$

Therefore, using (3.1) we get

$$W(v)=v^2\left(\frac{2n+4r^2-kr-39k+21}{2(n-r)(n-k+r)}\right)\leq\mathcal{C}\frac{v^2}{n},$$

where $\mathcal{C}$ is a positive constant.
Consequently

$$\|M^*_{n,k,r}(\xi(f;.))\|_{L^1} \leq \frac{\mathcal{C}}{n}\int_0^\infty v^2|f(v)|dv = \frac{\mathcal{C}}{n}\|\varphi^2 f\|_{L^1}.$$

If $p=\infty$, then using Lemma 9.6.1, [2] we can write

$$\begin{aligned}|\xi(f;x)(t)| &= \left|\int_x^t (t-v)f(v)dv\right| \\ &\leq \frac{|t-x|}{x}\left(\frac{1}{x}+\frac{1}{t}\right)\left|\int_x^t \varphi^2(v)f(v)dv\right| \\ &\leq \|\varphi^2 f\|_{L^\infty}\frac{(t-x)^2}{x}\left(\frac{1}{x}+\frac{1}{t}\right).\end{aligned}$$

Then, we have

$$\begin{aligned}\left|M^*_{n,k,r}(\xi(f;x);x)\right| &\leq M^*_{n,k,r}(|\xi(f;x)|;x) \\ &\leq \|\varphi^2 f\|_{L^\infty}\left(\frac{1}{x^2}M^*_{n,k,r}(\phi_{x,2};x)+\frac{1}{x}M^*_{n,k,r}\left(\frac{\phi_{x,2}}{e_1};x\right)\right)\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\left|M^*_{n,k,r}(\xi(f;x);x)\right| \leq \|\varphi^2 f\|_{L^\infty}\left(\frac{1}{x^2}M^*_{n,k,r}(\phi_{x,2};x)+\frac{1}{x}\sqrt{M^*_{n,k,r}(\phi_{x,4};x)}\times\sqrt{M^*_{n,k,r}\left(\frac{1}{e_2};x\right)}\right).$$

From Lemma 2.2, we get

$$M^*_{n,k,r}(\phi_{x,2};x) \leq \mathcal{C}_1\frac{x^2}{n},\ \ \mathcal{C}_1>0,$$

$$M^*_{n,k,r}(\phi_{x,4};x) \leq \mathcal{C}_2\frac{x^4}{n^2},\ \ \mathcal{C}_2>0.$$

By elementary calculation we obtain

$$M^*_{n,k,r}\left(\frac{1}{e_2};x\right) = \frac{(n+1-r)(n+2-r)}{(n-k+r)(n-k+r-1)}x^{-2} \leq \frac{\mathcal{C}_3}{x^2},$$

where $\mathcal{C}_3>0$. From the above, we have

$$\begin{aligned}\left|M^*_{n,k,r}(\xi(f;x);x)\right| &\leq \|\varphi^2 f\|_{L^\infty}\left(\frac{\mathcal{C}_1}{n}+\frac{1}{x}\sqrt{\mathcal{C}_2\frac{x^4}{n^2}}\times\sqrt{\frac{\mathcal{C}_3}{x^2}}\right) \\ &\leq \frac{\mathcal{C}_4}{n}\|\varphi^2 f\|_{L^\infty},\end{aligned}$$

where $\mathcal{C}_4>0$, which gives the result for $p=\infty$. □

Now, we can formulate the following approximation theorem.

Theorem 3.4. *Let $f\in L^p$, $1\leq p\leq\infty$. Then, there exists a positive constant $\mathcal{C}$ such that*

$$\|M^*_{n,k,r}(f)-f\|_{L^p} \leq \mathcal{C}\left(\omega_{2,\varphi}\left(f;n^{-1/2}\right)_p+\frac{1}{n}\|f\|_{L^p}\right).$$

Proof. Let $g \in W^p_{2,\varphi}$. For every $x \in (0,\infty)$, we have

$$|M^*_{n,k,r}(f;x) - f(x)| \leq |M^*_{n,k,r}(f-g;x)| + |M^*_{n,k,r}(g;x) - M^*_{n,k,r}(g(x);x)| + |g(x) - f(x)|.$$

From this and by Theorem 3.1, we have

$$\|M^*_{n,k,r}(f) - f\|_{L^p} \leq \|M^*_{n,k,r}(g - g(.);.)\|_{L^p} + \mathcal{C}_1\|g - f\|_{L^p}, \tag{3.2}$$

where $\mathcal{C}_1$ is some positive constant.
Using Taylor's theorem we get

$$g(t) - g(x) = (t-x)g'(x) + \xi(g'';x)(t),$$

where $\xi(g'';x)(t) = \int_x^t (t-v)g''(v)dv$ is integral remainder.
Then, we get

$$M^*_{n,k,r}(g - g(x);x) = g'(x)M^*_{n,k,r}(\phi_{x,1};x) + M^*_{n,k,r}(\xi(g'';x);x). \tag{3.3}$$

By Lemma 2.2, we have

$$M^*_{n,k,r}(\phi_{x,1};x) = \frac{2r-k+1}{n-r}x \leq \mathcal{C}_2\frac{x}{n},$$

where $\mathcal{C}_2 > 0$.
Using (3.3), Lemma 3.2 and Lemma 3.3, we obtain

$$\begin{aligned}\|M^*_{n,k,r}(g) - g\|_{L^p} &\leq \frac{\mathcal{C}_2}{n}\|\varphi g'\|_{L^p} + \|M^*_{n,k,r}(\xi(g'';.))\|_{L^p} \\ &\leq \frac{\mathcal{C}_3}{n}\left(\|g\|_{L^p} + \|\varphi^2 g''\|_{L^p}\right),\end{aligned}$$

for some $\mathcal{C}_3 > 0$. Together with (3.2) this leads to

$$\|M^*_{n,k,r}(f) - f\|_{L^p} \leq \mathcal{C}_4\|g - f\|_{L^p} + \frac{\mathcal{C}_3}{n}\left(\|g - f\|_{L^p} + \|f\|_{L^p} + \|\varphi^2 g''\|_{L^p}\right)$$

where $\mathcal{C}_4 > 0$. Taking the infimum over all $g \in W^p_{2,\varphi}$ we get

$$\|M^*_{n,k,r}(f) - f\|_{L^p} \leq \mathcal{C}_5\left(\mathcal{K}_{2,\varphi}\left(f;\frac{1}{n}\right)_p + \frac{1}{n}\|f\|_{L^p}\right)$$

for some $\mathcal{C}_5 > 0$. Using (1.2), we get

$$\|M^*_{n,k,r}(f) - f\|_{L^p} \leq \mathcal{C}\left(\omega_{2,\varphi}\left(f;n^{-1/2}\right)_p + \frac{1}{n}\|f\|_{L^p}\right).$$

Hence, the proof is completed. □

Lemma 3.5. *If r^{th} derivative $f^{(r)}(r = 0,1,2...)$ exists continuously and $f^{(r)} \in L^p, 1 \leq p \leq \infty$, there exists a positive constant $\mathcal{C}$ such that*

$$\left\|\frac{1}{b(n,k,r)}M^{(r)}_{n,k}(f) - f^{(r)}\right\|_{L^p} \leq \mathcal{C}\left(\omega_{2,\varphi}\left(f^{(r)};n^{-1/2}\right)_p + \frac{1}{n}\|f^{(r)}\|_{L^p}\right).$$

References

[1] J. Bergh and J. Löfström, Interpolation Spaces, An Introduction, *Springer-Verlag, Berlin, Heidelberg, New York* (1976).

[2] Z. Ditzian and V. Totik, Moduli of Smoothness, *Springer-Verlag, New York* (1987).

[3] R. A. DeVore and G. G. Lorentz, Constructive Approximation, *Springer, Berlin* (1993).

[4] A. İzgi, Voronovskaya type asymptotic approximation by modified gamma operators, *Appl. Math. Comput.* **217**, 8061–8067 (2011).

[5] A. İzgi, Approximation of L^p-integrable functions by Gamma type operators, *Journal of Analysis and Computation*, **7(2)**, 113–124 (2011).

[6] A. İzgi and I. Büyükyazici, Approximation and rate of approximation on unbounded intervals, *Kastamonu Edu. J. Okt.*, **11**, 451–460 (2003)(in Turkish).

[7] Alok Kumar, Voronovskaja type asymptotic approximation by general Gamma type operators, *Int. J. of Mathematics and its Applications*, **3(4-B)**, 71–78 (2015).

[8] Alok Kumar and D. K. Vishwakarma, Global approximation theorems for general Gamma type operators, *Int. J. of Adv. in Appl. Math. and Mech.* **3(2)**, 77–83 (2015).

[9] Alok Kumar, Artee, D. K. Vishwakarma and Rajat Kaushik, On general Gamma-Taylor operators on weighted spaces, *Int. J. Adv. Appl. Math. and Mech.*, **3(4)**, 9–15 (2016).

[10] Alok Kumar, Artee and D. K. Vishwakarma, Approximation properties of general gamma type operators in polynomial weighted space, *Int. J. Adv. Appl. Math. and Mech.*, **4(3)**, 7–13 (2017).

[11] G. Krech, A note on the paper "Voronovskaja type asymptotic approximation by modified gamma operators", *Appl. Math. Comput.*, **219**, 5787–5791 (2013).

[12] G. Krech, Modified Gamma operators in L^p spaces, *Lith. Math. J.*, **54(4)**, 454–462 (2014).

[13] G. Krech, On the rate of convergence for modified Gamma operators, *Rev. Un. Mat. Argentina*, **55(2)**, 123–131 (2014).

[14] H. Karsli, Rate of convergence of a new Gamma type operators for the functions with derivatives of bounded variation, *Math. Comput. Modell.*, **45(5-6)**, 617–624 (2007).

[15] H. Karsli, On convergence of general Gamma type operators, *Anal. Theory Appl.*, **27(3)**, 288–300 (2011).

[16] H. Karsli and M. A. Özarslan, Direct local and global approximation results for operators of gamma type, *Hacet. J. Math. Stat.*, **39**, 241–253 (2010).

[17] H. Karsli, V. Gupta and A. Izgi, Rate of pointwise convergence of a new kind of gamma operators for functions of bounded variation, *Appl. Math. Letters*, **22**, 505–510 (2009).

[18] H. Karsli, P. N. Agrawal and M. Goyal, General Gamma type operators based on q-integers, *Appl. Math. Comput.*, **251**, 564–575 (2015).

[19] A. Lupas and M. Müller, Approximationseigenschaften der Gammaoperatören, *Mathematische Zeitschrift*, **98**, 208–226 (1967).

[20] S. M. Mazhar, Approximation by positive operators on infinite intervals, *Math. Balkanica*, **5(2)**, 99–104 (1991).

[21] L. C. Mao, Rate of convergence of Gamma type operator, *J. Shangqiu Teachers Coll.*, **12**, 49–52 (2007).

Author information

Alok Kumar, Department of Computer Science, Dev Sanskriti Vishwavidyalaya, Haridwar-249411, Uttarakhand, India.
E-mail: alokkpma@gmail.com

Received: April 16, 2016.

Accepted: February 2, 2017.