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A RESULT ON FUNCTIONAL EQUATIONS RELATED TO DERIVATION ON UNITAL SEMIPRIME RINGS

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Abstract The purpose of this paper is to prove the following result. Let R be $n!$-torsion free semiprime unital ring and let $D, G : R \to R$ be additive mappings satisfying the relations $D(x^n) = D(x^{n-1})x + x^{n-1}G(x)$ and $G(x^n) = G(x^{n-1})x + x^{n-1}D(x)$ for all $x \in R$ and some integer $n > 1$. In this case D and G are derivations and $D = G$.

1 Introduction

Throughout, R will represent an associative ring with center $Z(R)$. As usual we write $[x, y]$ for $xy-yx$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime, if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by Q_s the symmetric Martindale ring of quotients. For explanation of Q_s, we refer the reader to [1]. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists such $a \in R$ that $D(x) = [x, a]$ holds for all $x \in R$. Every derivation is a Jordan derivation, but the converse is in general not true. A classical result of Herstein [6] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [2]. Cusack [5] generalized Herstein's result to 2-torsion free semiprime rings (see also [3] for an alternative proof). An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation on an arbitrary 2-torsion free ring R is a Jordan triple derivation (see for example [2])

Brešar [4] has proved the following result.

Theorem A. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be a Jordan triple derivation. In this case D is a derivation.*

Since as we mentioned above, any Jordan derivation D on an arbitrary 2-torsion free ring is a Jordan triple derivation. One can conclude that Theorem A generalizes Cusack's generalization of Herstein's theorem.

Motivated by Theorem A, Vukman, Kosi-Ulbl and Eremita [9] have proved the following result.

Theorem B. *Let R be a 2-torsion free semiprime ring. Suppose there exists an additive mapping $T : R \to R$ such that*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x)$$

for all $x, y \in R$. Then there exists $q \in Q_s$ such that $2T(x) = qx + xq$, for all $x \in R$.

In the same article Vukman, Kosi-Ulbl and Eremita proved the result given below, which is an immediate consequence of Theorem A and Theorem B.

Theorem C. *Let R be a 2-torsion free semiprime ring. If $D, G : R \to R$ are additive mappings such that*

$$D(xyx) = D(x)yx - xG(y)x + xyD(x)$$
$$G(xyx) = G(x)yx - xD(y)x + xyG(x)$$

for all $x, y \in R$. Then there exists a derivation $S : R \to R$ and $q \in Q_s$ such that

$$4D(x) = qx + xq + S(x)$$

$$4G(x) = qx + xq - S(x)$$

for all $x \in R$.

Motivated by Theorem A, Vukman [10] recently proved the following result.

Theorem D. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping. Suppose that either of the relations*

$$D(xyx) = D(xy)x + xyD(x), \tag{1.1}$$
$$D(xyx) = D(x)yx + xD(yx)$$

holds for all pairs $x, y \in R$. In both cases D is a derivation.

The substitution $y = x^{n-2}$ in relations (1.1), gives

$$D(x^n) = D(x^{n-1})x + x^{n-1}D(x), \tag{1.2}$$
$$D(x^n) = D(x)x^{n-1} + xD(x^{n-1}).$$

It is our aim in this paper to prove the following result which is related to the functional equation (1.2) and give an affirmative answer to our conjecture in [7] for the case semiprime unital ring.

Theorem 1.1. *Let R be $n!$-torsion free semiprime unital ring and let $D, G : R \to R$ be additive mappings satisfying either the relations*

$$D(x^n) = D(x^{n-1})x + x^{n-1}G(x),$$
$$G(x^n) = G(x^{n-1})x + x^{n-1}D(x)$$

or the relations

$$D(x^n) = D(x)x^{n-1} + xG(x^{n-1}),$$
$$G(x^n) = G(x)x^{n-1} + xD(x^{n-1})$$

for all $x \in R$ and some integer $n > 1$. In both cases D and G are derivations and $D = G$.

Proof. We will restrict our attention on the first system of relations. The proof in case the second system of relations holds is similar and will therefore be omitted. We have

$$D(x^n) = D(x^{n-1})x + x^{n-1}G(x), \tag{1.3}$$
$$G(x^n) = G(x^{n-1})x + x^{n-1}D(x).$$

Subtracting the two relations of equation (1.3), we obtain

$$T(x^n) = T(x^{n-1})x - x^{n-1}T(x), \tag{1.4}$$

where $T = D - G$. We will denote the identity element of the ring R by e. Putting e for x in the above relation gives

$$T(e) = 0. \tag{1.5}$$

Let y be any element of the center $Z(R)$. Putting $x+y$ for x in the relation (1.4), we get

$$\sum_{i=0}^{n}\binom{n}{i}T(x^{n-i}y^{i}) = \Big(\sum_{i=0}^{n-1}\binom{n-1}{i}T(x^{n-1-i}y^{i})\Big)(x+y) - \Big(\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^{i}\Big)T(x+y).$$

Using (1.4) and rearranging the above relation in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y, that is

$$\begin{aligned} f_i(x,y) &= \binom{n}{i}T(x^{n-i}y^{i}) \\ &\quad - \binom{n-1}{i}\Big(T(x^{n-1-i}y^{i})x - (x^{n-1-i}y^{i})T(x)\Big) \\ &\quad - \binom{n-1}{i-1}\Big(T(x^{n-i}y^{i})y - (x^{n-i}y^{i})T(y)\Big). \end{aligned}$$

Replacing x by $x+2y$, $x+3y$,..., $x+(n-1)y$ in turn in the relation (1.4) and expressing the resulting system of $(n-1)$ homogeneous equations of variables $f_i(x,y)$, $i=1,2,\ldots,n-1$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \ldots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of the above matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular putting the identity element e for y, we obtain

$$\begin{aligned} f_{n-1}(x,e) &= \binom{n}{n-1}T(x) - \binom{n-1}{n-1}\Big(T(e)x - eT(x)\Big) \\ &\quad - \binom{n-1}{n-2}\Big(T(x)e - xT(e)\Big) = 0. \end{aligned}$$

Using relation (1.5) in above relation we get $nT(x) = (n-2)T(x)$, which implies using torsion restriction $T(x) = 0$ and gives $D = G$. This ascertainment enables us to combine the given two relations into only one relation

$$D(x^n) = D(x^{n-1})x + x^{n-1}D(x). \tag{1.6}$$

Putting e for x in the above relation we obtain

$$D(e) = 0. \tag{1.7}$$

Let y be any element of center $Z(R)$. Putting $x+y$ for x in the relation (1.6), we get

$$\sum_{i=0}^{n}\binom{n}{i}D(x^{n-i}y^{i}) = \Big(\sum_{i=0}^{n-1}\binom{n-1}{i}D(x^{n-1-i}y^{i})\Big)(x+y) + \Big(\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^{i}\Big)D(x+y).$$

Using (1.6) and rearranging the above relation in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y, that is

$$\begin{aligned} f_i(x,y) &= \tbinom{n}{i} D(x^{n-i}y^i) \\ &\quad - \tbinom{n-1}{i}\Big(D(x^{n-1-i}y^i)x + (x^{n-1-i}y^i)D(x)\Big) \\ &\quad - \tbinom{n-1}{i-1}\Big(D(x^{n-i}y^i)y + (x^{n-i}y^i)D(y)\Big). \end{aligned}$$

Replacing x by $x+2y$, $x+3y$,..., $x+(n-1)y$ in turn in the relation (1.6) and expressing the resulting system of $(n-1)$ homogeneous equations of the variables $f_i(x,y)$, $i = 1, 2, \ldots, n-1$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \ldots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular putting the identity element e for y, we obtain

$$\begin{aligned} f_{n-2}(x,e) &= \tbinom{n}{n-2} D(x^2) - \tbinom{n-1}{n-2}\Big(D(x)x + xD(x)\Big) \\ &\quad - \tbinom{n-1}{n-3}\Big(D(x^2)e + x^2D(e)\Big) = 0. \end{aligned}$$

Using (1.7) in the above relation it reduces to

$$\frac{n(n-1)}{2} D(x^2) = (n-1)(D(x)x + xD(x)) + \frac{(n-1)(n-2)}{2} D(x^2),$$

which implies

$$(n-1)(D(x^2) - D(x)x - xD(x)) = 0.$$

Since R is $n!$-torsion free, the above relation reduces to

$$D(x^2) = D(x)x + xD(x)$$

for all $x \in R$. In other words, D is a Jordan derivation on R. According to Cusack's generalization of Herstein theorem, one can conclude that D is a derivation, which completes the proof of the theorem.

Remark 1.2. In view of the above theorem it is an obvious question, whether the result can be proved without identity element. Unfortunately, we were unable to prove the above theorem in general. We leave an open question whether or not the above theorem can be proved without assuming that a ring possesses the identity element.

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