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FISCHER-CLIFFORD THEORY APPLIED TO A NON-SPLIT EXTENSION GROUP $2^{5\cdot}GL_4(2)$

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Abstract. The non-split extension $\overline{G} = 2^{5\cdot}GL_4(2)$ is a subgroup of the Dempwolff group $2^{5\cdot}GL_5(2)$ of index 496 and has order 645120. In this paper, the author computes the character table of $\overline{G}$ using the technique of Fischer-Clifford matrices. Very interesting results are obtained on the projective character tables of the inertia factor groups of $\overline{G}$.

1 Introduction

The Dempwolff group $D = 2^{5\cdot}GL_5(2)$ [10] is the second largest maximal subgroup of the sporadic simple Thompson group Th as listed in the ATLAS [9]. In the ATLAS we found the specification $N_{Th}((2A)^5) \cong 2^{5\cdot}GL_5(2)$, where the generators of the elementary abelian 2-group 2^5 are 5 commuting involutions which are found in the class $2A$ of involutions of the Thompson group Th. Note that there are two nonisomorphic maximal subgroups $\overline{G_1}$ and $\overline{G_2}$ of the type $2^{5\cdot}(2^4{:}GL_4(2))$ in D. The group $\overline{G_1}$ is the centralizer $C_D(2A)$ of elements in the class $2A$ of involutions in D whereas the other group $\overline{G_2}$ is the stabilizer in D of a subspace of dimension four in 2^5. Also, $\overline{G_2}$ is the normalizer in Th of a radical 2-subgroup of Th (see [31]). Each of the groups $\overline{G_1}$ and $\overline{G_2}$ has a class of maximal subgroups of the type $2^{5\cdot}GL_4(2)$. Suitable representations of $\overline{G_1}$ and $\overline{G_2}$ are obtained from Wilson's online ATLAS of Group Representations [30] to verify, with the aid of GAP [28] or MAGMA [8], that the classes of maximal subgroups of the type $2^{5\cdot}GL_4(2)$ contained in $\overline{G_1}$ and $\overline{G_2}$, are isomorphic to each other. It should be mentioned that the group $2^{5\cdot}GL_4(2)$ is also a 2-fold cover of the maximal subgroup $2^{4\cdot}A_8$ of the Conway group Co_3.

In this paper, the technique of Fischer-Clifford matrices [11] is applied to compute the ordinary character table of $2^{5\cdot}GL_4(2)$. This article belongs to a series of papers (see for example [2], [3], [4], [5], [7], [12], [21], [23], [24] and [25]) on the application of the said technique to compute the character tables of extension type groups. A great deal of research work in the development of the technique of Fischer-Clifford matrices has been carried out by J. Moori and his Post-graduate students (see [1], [6], [22], [27], [29] and [32]). The paper contains original and quite interesting results on the projective character tables of the inertia factor group $2^3{:}GL_3(2)$ of $2^{5\cdot}GL_4(2)$. Also, this paper serves as a good example for the application of Fischer-Clifford theory, where more than one set of projective characters of the inertia factors groups of an appropriate non-split extension group are required. Most of our computations were carried out with the aid of the computer algebra systems MAGMA and GAP. Our notation is standard and readers may refer to the ATLAS.

2 Projective characters and the Schur Multiplier

In this section, some basic concepts in projective character theory will be defined. The concepts and ideas discussed here were taken from [1], [14], [15], [16], [17], [18] and [26], where G denotes a finite group.

Definition 2.1. A function $\alpha : G \times G \to \mathbb{C}^*$ is called a factor set of G if $\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$.

The set of all equivalence classes of factor sets of G forms a finite abelian group [16], called the Schur Multiplier, and is denoted by $M(G)$.

Definition 2.2. A projective representation of a group G of degree n over the complex numbers is a map $P : G \to GL(n, \mathbb{C})$, such that

(i) $P(1) = I_n$, and

(ii) given $x, y \in G$, there exists $\alpha(x, y) \in \mathbb{C}^*$ such that $P(x)P(y) = \alpha(x, y)P(xy)$.

The map α is called the factor set associated with P.

Let P be a projective representation of G with factor set α. Define $\kappa(g) = Trace(P(g))$ for all $g \in G$. Then κ is called a projective character of G. We say that κ is irreducible if P is, and κ has a factor set α, where α is the factor set of P.

We let $IrrProj(G, \alpha)$ denote the set of irreducible projective characters of G associated with the factor set α. An element $x \in G$ is said to be α-regular if $\alpha(x, g) = \alpha(g, x)$ for all $g \in C_G(x)$. It is well known that $g \in G$ is α-regular if and only if $\kappa(g) \neq 0$ for some $\kappa \in IrrProj(G, \alpha)$ or equivalently that g is not α-regular if and only if $\kappa(g) = 0$ for all $\kappa \in IrrProj(G, \alpha)$. The number of irreducible projective characters with factor set α equals the number of α-regular classes of a group G (see [18](Theorem 3.6.7)). Projective characters also satisfy the usual orthogonality relations and have analogues to ordinary characters (see [14] and [17]).

Definition 2.3. A group R is a representation group for G if there exists a homomorphism π from R onto G such that (i) $A = ker(\pi) \cong M(G)$, and (ii) $A \leq Z(R) \cap R^{'}$.

A covering group C for G will normally be a quotient of R by a subgroup B of A. If A/B has order n we sometimes refer to the covering group as a n-fold cover of G. Projective representations of G are found in the representation group R for all the equivalence classes of factors sets in $M(G)$. However, in a n-fold cover C of G only the n equivalence classes which C covers will be represented.

3 Theory of Fischer-Clifford Matrices

Since the character table of $2^{5\cdot}GL_4(2)$ will be constructed by the technique of Fischer-Clifford matrices, we will give a brief theoretical background of this technique.

Let $\overline{G} = N{\cdot}G$ be an extension of N by G, where N is normal in $\overline{G}$ and $\overline{G}/N \cong G$. Denote the set of all irreducible characters of a finite group G_1 by $Irr(G_1)$. Also, define $\overline{H} = \{x \in \overline{G} | \theta^x = \theta\} = I_{\overline{G}}(\theta)$ as the inertia group of $\theta \in Irr(N)$ in $\overline{G}$ then N is normal in $\overline{H}$. Let $\overline{g} \in \overline{G}$ be a lifting of $g \in G$ under the natural homomorphism $\overline{G} \longrightarrow G$ and $[g]$ be a conjugacy class of elements with representative g. Let $X(g) = \{x_1, x_2, \cdots, x_{c(g)}\}$ be a set of representatives of the conjugacy classes of $\overline{G}$ from the coset $N\overline{g}$ whose images under the natural homomorphism $\overline{G} \longrightarrow G$ are in $[g]$ and we take $x_1 = \bar{g}$. Now let $\theta_1 = 1_N, \theta_2, \cdots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$ such that for $1 \leq i \leq t$, we have $\overline{H_i}$ with corresponding inertia factors H_i. By Gallagher [17] we obtain

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\psi_i \overline{\beta})^{\overline{G}} | \beta \in IrrProj(H_i), \text{ with factor set } \alpha_i^{-1}\},$$

where ψ_i is a projective character of $\overline{H}_i$ with factor set $\overline{\alpha}_i$ such that $\psi_i \downarrow_N = \theta_i$. Observe that α_i and $\overline{\beta}$ are obtained from $\overline{\alpha_i}$ and β, respectively. We have that $\overline{H_1} = \overline{G}$ and $H_1 = G$. Choose $y_1, y_2, .., y_r$ to be representatives of the α_i^{-1}-conjugacy classes of elements of H_i that fuse to $[g]$ in G. We define

$$R(g) = \{(i, y_k) \,|\, 1 \leq i \leq t, H_i \cap [g] \neq \emptyset, 1 \leq k \leq r\}$$

and we note that y_k runs over representatives of the α_i^{-1}-conjugacy classes of elements of H_i which fuse into $[g]$ in G. We define $y_{l_k} \in \overline{H_i}$ such that y_{l_k} ranges over all representatives of the conjugacy classes of elements of $\overline{H}_i$ which map to y_k under the homomorphism $\overline{H}_i \longrightarrow H_i$ whose kernel is N. Then we define the Fischer-Clifford matrix by $M(g) = (a^j_{(i,y_k)})$, where

$$a^j_{(i,y_k)} = \sum_l{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k}),$$

with columns indexed by $X(g)$ and rows indexed by $R(g)$ and where $\sum_l'$ is the summation over all l for which $y_{l_k} \sim x_j$ in $\overline{G}$. We also write the Fischer-Clifford matrix for the class $[g]$ as

$$M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix}$$

where, if $H_i \cap [g] = \emptyset$, then the submatrix $M_i(g)$ (corresponding to the inertia group $\overline{H_i}$ and its inertia factor H_i) is not defined and is omitted from $M(g)$. $M(g)$ is a $l \times c(g)$ matrix, where l is the number of α_i^{-1}- regular conjugacy classes of the inertia factors H_i's, $1 \leq i \leq t$, which fuse into $[g]$ in G and $c(g)$ is the number of conjugacy classes of $\overline{G}$ which correspond to the coset $N\overline{g}$. Then the partial character table of $\overline{G}$ on the classes $\{x_1, x_2, \cdots, x_{c(g)}\}$ is given by

$$\begin{bmatrix} C_1(g)\, M_1(g) \\ C_2(g)\, M_2(g) \\ \vdots \\ C_t(g)\, M_t(g) \end{bmatrix}$$

where the Fischer-Clifford matrix $M(g)$ is divided into blocks $M_i(g)$ with each block corresponding to an inertia group $\overline{H}_i$ and $C_i(g)$ is the partial character table of H_i with factor set α_i^{-1} consisting of the columns corresponding to the α_i^{-1}-regular classes that fuse into $[g]$ in G. We obtain the characters of $\overline{G}$ by multiplying the relevant columns of the projective characters of H_i with factor set α_i^{-1} by the rows of $M(g)$. Hence the full character table of $\overline{G}$ will be

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_t \end{bmatrix},$$

where $\Delta_i = [C_i(1)M_i(1)|C_i(g_2)M_i(g_2)|...|C_i(g_k)M_i(g_k)]$ with $\{1, g_1, g_2, ..., g_k\}$ the set representatives of conjugacy classes of G. We can also observe that the number of irreducible characters of $\overline{G}$ is the sum of the number of projective characters of the inertia factors H_i's with factor set α_i^{-1}, for all $i \in \{1, 2, .., t\}$. The reader is referred to Ali and Moori [3] for results on split and non-split cosets and further properties of the Fischer-Clifford matrices, which are helpful and fundamental in deducing the entries of these matrices.

4 The group $\overline{G} = 2^{5\cdot}GL_4(2)$

The Dempwolff group $D = 2^{5\cdot}GL_5(2)$ is represented as permutations on 7440 points in MAGMA, by making use of Wilson's online ATLAS of Group Representations [30]. Next, we construct the group $\overline{G_2} = 2^{5\cdot}(2^4{:}GL_4(2))$ as the normalizer $N_D(2^4)$ in D of a subspace of dimension four in 2^5. The group of interest $2^{5\cdot}GL_4(2)$ is computed as the centralizer $C_{\overline{G_2}}(2B)$, where $2B$ is a class of involutions of $\overline{G_2}$. The MAGMA command "IsMaximal($\overline{G_2}, C_{\overline{G_2}}(2B)$)" confirms that $2^{5\cdot}GL_4(2)$ is a maximal subgroup of $\overline{G_2}$. Then the normal subgroup $N = 2^5$ of

$2^{5\cdot}GL_4(2)$ is represented as a permutation group on 7440 points. The MAGMA command "Complements($C_{\overline{G_2}}(2B), N$)" computes the complements of N in $2^{5\cdot}GL_4(2)$, where an empty set "[]" is returned confirming that the extension is non-split. Having obtained a permutation representation of $2^{5\cdot}GL_4(2)$, the conjugacy classes of $2^{5\cdot}GL_4(2)$ are also computed in MAGMA and it is found that the group has exactly 39 classes.

Since $2^{5\cdot}GL_4(2)$ is represented as a permutation group, the MAGMA commands "M:= GModule($C_{\overline{G_2}}(2B)$,2^5)" and "M:Maximal" are used to represent $GL_4(2) \cong A_8$ as a matrix group of dimension 5 over the Galois field $GF(2)$. The generators g_1 and g_2 of $GL_4(2)$, with respective orders of 2 and 7, are as follows:

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

5 The inertia factors of $2^{5\cdot}GL_4(2)$

Throughout the remainder of this paper, let $\overline{G} = 2^{5\cdot}GL_4(2)$ be a non-split extension of $N = 2^5$ by $G = GL_4(2)$, where N is the vector space of dimension 5 over $GF(2)$ on which the linear group $G = < g_1, g_2 >$ acts. When G acts on the conjugacy classes of elements of 2^5, we obtain four orbits of lengths 1, 1, 15 and 15 with respective point stabilizers of the types $GL_4(2)$, $GL_4(2)$, $2^3{:}GL_3(2)$ and $2^3{:}GL_3(2)$. The structures of the stabilizers are easily determined by checking the indices of the maximal subgroups of G in the ATLAS. Also, with the aid of MAGMA it is determined that the stabilizers of type $2^3{:}GL_3(2)$ are contained in the same class of maximal subgroups of G.

Let $\chi(G|N)$ be the permutation character of G on N. Then, from the ATLAS it is obtained that $\chi(G|2^3{:}GL_3(2)) = 1a + 14a$ is the permutation character of G on the classes of $2^3{:}GL_3(2)$. Hence $\chi(G|N) = \sum_{i=1}^{4} I_{P_i}^{G} = \sum_{i=1}^{4} \chi(G|P_i) = 4 \times 1a + 2 \times 14a$, where $I_{P_i}^{G}$ are the identity characters of the point stabilizers P_i induced to G. Therefore, $\chi(G|2^5)(g)$ will give the number k of points of N fixed by each $g \in G$ such that $k = 2^n$, where $n \in \{0, 1, 2, 3, 4, 5\}$. These values of k are listed in Table 5.

Since G has four orbits on N, then by Brauer's Theorem [13] the action of G on $Irr(N)$ will also has four orbits. The lengths of these four orbits will be 1, r, s and t, where $r+s+t = 31$, with respective inertia factor groups H_1, H_2, H_3 and H_4 as subgroups of G such that $[G : H_1] = 1$, $[G : H_2] = r$, $[G : H_3] = s$ and $[G : H_4] = t$. After checking the indices of the maximal subgroups of G in the ATLAS, it is deduced that $r = 1$ and $s = t = 15$. Hence, there are four inertia groups $\overline{H}_i = 2^{5\cdot}H_i$ in $2^{5\cdot}GL_4(2)$, $i \in \{1, 2, 3, 4\}$, with corresponding inertia factor groups $H_1 = H_2 = G$ and $H_3 = H_4 = 2^3{:}GL_3(2)$. The group $2^3{:}GL_3(2)$ is constructed from elements within G and the generators are as follows:

- $2^3{:}GL_3(2) = \langle \alpha_1, \alpha_2 \rangle$, $\alpha_1 \in 7A$, $\alpha_2 \in 7B$ where

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \quad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We obtain the fusion of the inertia factor $2^3{:}GL_3(2)$ into G by using the permutation character of $2^3{:}GL_3(2)$ in G of degree 15 and if necessary direct computation in MAGMA. The fusion map of $2^3{:}GL_3(2)$ into G is shown in Table 1.

6 The projective character tables of the inertia factor groups

In Section 4 it was found that $\overline{G}$ has exactly 39 classes. Hence $|Irr(\overline{G})| = |Irr(G)| + |IrrProj\ (H_2, \alpha_2^{-1})| + |IrrProj(H_3, \alpha_3^{-1})| + |IrrProj(H_4, \alpha_4^{-1})| = 39$. Since $|Irr(\overline{G})| = 14$, it follows that $|IrrProj(H_2, \alpha_2^{-1})| + |IrrProj(H_3, \alpha_3^{-1})| + |IrrProj(H_4, \alpha_4^{-1})| = 25$. Therefore it is necessary to determine all projective character tables of G and $2^3{:}GL_3(2)$ for us to find the appropriate sets of characters, $IrrProj(H_i, \alpha_i^{-1})$, which are needed in the construction of the character table of $\overline{G}$.

The first step to find all the projective character tables of G and $2^3{:}GL_3(2)$, with corresponding factor sets, is to compute their Schur multipliers $M(G)$ and $M(2^3{:}GL_3(2))$. From the ATLAS we obtained that $M(G) \cong 2$, the cyclic group C_2 of order 2, hence G has two set of character tables. The 14 irreducible ordinary characters of G is one set and the other set consists of the irreducible projective characters with factor set α^{-1} such that $\alpha^2 \sim 1$. The set $IrrProj(G, \alpha^{-1})$, consisting of 9 projective characters, is obtained from the ordinary character table of the 2-fold cover $2 \cdot G \cong 2{\cdot}A_8$ of $A_8 \cong GL_4(2)$ found in [9] and is listed in Table 2.

The group $2^3{:}GL_3(2)$ is represented as permutations on 8 points in MAGMA . The sequence of MAGMA commands found in [7](page 52) is used to compute the Schur multiplier $M(2^3{:}GL_3(2)) \cong C_2 \times C_2 \cong 2^2$ of $2^3{:}GL_3(2)$ and also the ordinary character table of the full representation group $R = 2^2{\cdot}(2^3{:}GL_3(2))$ of $2^3{:}GL_3(2)$. Since $M(2^3{:}GL_3(2)) \cong 2^2$, we found that there are 3 sets of projective characters of $2^3{:}GL_3(2)$ with non-trivial factor sets β_i^{-1}, $i = 1, 2, 3$, such that $\beta_i^2 \sim 1$. We obtained that $|Irr(R)| = 29$, where 11 of these are the ordinary characters of $2^3{:}GL_3(2)$ and so we deduce that $\sum_{i=1}^{3} |IrrProj(2^3{:}GL_3(2), \beta_i^{-1})| = 18$.

Haggarty and Humphreys [14] show that is possible to determine the projective characters of $2^3{:}GL_3(2)$ with a given factor set $\beta_i^{-1}, i = 1, 2, 3$, without the full representation group R of $2^3{:}GL_3(2)$. We proceed computationally in MAGMA by first computing the center $Z(R)$ of R. We obtained that $Z(R) \cong M(2^3{:}GL_3(2)) \cong 2^2$. Next, we compute the three non-conjugate normal subgroups N_i of 2^2 of order two. Then the three factor groups $R_i \cong R/N_i$ are the 2-fold covers of $2^3{:}GL_3(2)$. Thus the three sets of projective characters of $2^3{:}GL_3(2)$ with factor sets β_i^{-1} can be determined from the ordinary character tables of R_i. We compute the character tables of the groups R_i and found that $|Irr(R_1)| = 19$, $|Irr(R_2)| = |Irr(R_3)| = 16$, where 11 of these in each group R_i are the ordinary irreducible characters of $2^3{:}GL_3(2)$. Thus the number of projective characters of $2^3{:}GL_3(2)$ associated with each non- trivial factor set β_1^{-1}, β_2^{-1} and β_3^{-1} is 8, 5 and 5, respectively. Since $R_2 \cong R_3$ it follows that each of the factor sets β_2 and β_3 gives rise to the same projective character table for $2^3{:}GL_3(2)$.

We deduce that $|IrrProj(H_2, \alpha_2^{-1})| = |IrrProj(G, \alpha^{-1})| = 9$, $|IrrProj(H_3, \alpha_3^{-1})| = |Irr(2^3{:}GL_3(2)| = 11$, $|IrrProj(H_4, \alpha_4^{-1})| = |IrrProj(2^3{:}GL_3(2), \beta_2^{-1})| = 5$, after we have considered the number of ordinary characters and the number of the projective characters of G and $2^3{:}GL_3(2)$. Hence we will use the sets $Irr(G)$, $IrrProj(G, \alpha^{-1})$, $Irr(2^3{:}GL_3(2))$ and $IrrProj(2^3{:}GL_3(2), \beta_2^{-1})$ to construct the ordinary character table of $\overline{G}$. The set $IrrProj(2^3{:}GL_3(2), \beta_2^{-1})$ is given in Table 3.

Note that the respective irreducible projective characters of the groups G and $2^3{:}GL_3(2)$ take zero values on the so-called α_i^{-1}- irregular classes. Also, note in Tables 2 and 3 that the number of α_i^{-1}- regular classes is equal to the number of irreducible projective characters .

Table 1. The fusion of $2^3{:}GL_3(2)$ into G

$[h]_{2^3:GL_3(2)}$	$\longrightarrow$	$[g]_{GL_4(2)}$	$[h]_{2^3:GL_3(2)}$	$\longrightarrow$	$[g]_{GL_4(2)}$
$1A$		$1A$	$4B$		$4A$
$2A$		$2A$	$4C$		$4B$
$2B$		$2A$	$6A$		$6B$
$2C$		$2B$	$7A$		$7B$
$3A$		$3B$	$7B$		$7A$
$4A$		$4A$			

Table 2. Projective character table of $GL_4(2)$ with factor set α^{-1}

$[h]_{GL_4(2)}$	$1A$	$2A$	2B	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$	$7A$	$7B$	$15A$	$15B$
$\lvert C_{GL_4(2)}(h)\rvert$	20160	192	96	180	18	16	8	15	12	6	7	7	15	15
ϕ_1	8	0	0	-4	2	0	0	-2	0	0	1	1	1	1
ϕ_2	24	0	0	-6	0	0	0	-1	0	0	a	$\bar{a}$	-1	-1
ϕ_3	24	0	0	-6	0	0	0	-1	0	0	$\bar{a}$	a	-1	-1
ϕ_4	48	0	0	6	0	0	0	-2	0	0	-1	-1	1	1
ϕ_5	56	0	0	-4	-1	0	0	1	0	$\sqrt{3}i$	0	0	1	1
ϕ_6	56	0	0	-4	-1	0	0	1	0	-$\sqrt{3}i$	0	0	1	1
ϕ_7	56	0	0	2	2	0	0	1	0	0	0	0	b	$\bar{b}$
ϕ_8	56	0	0	2	2	0	0	1	0	0	0	0	$\bar{b}$	b
ϕ_9	64	0	0	4	-2	0	0	-1	0	0	1	1	-1	-1

where a = $-\frac{1}{2}(1 - i\sqrt{7})$ and b = $-\frac{1}{2}(1 - i\sqrt{15})$

Table 3. Projective character table of $2^3{:}GL_3(2)$ with factor set β_2^{-1}

$[h]_{2^3:GL_3(2)}$	$1A$	$2A$	2B	$2C$	$3A$	$4A$	$4B$	$4C$	$6A$	$7A$	$7B$
$\lvert C_{2^3:GL_3(2)}(h)\rvert$	1344	192	32	32	6	16	8	8	6	7	7
ψ_1	8	0	0	0	2	0	0	0	0	1	1
ψ_2	8	0	0	0	-1	0	0	0	$\sqrt{3}i$	1	1
ψ_3	8	0	0	0	-1	0	0	0	-$\sqrt{3}i$	1	1
ψ_4	24	0	0	0	0	0	0	0	0	$\bar{c}$	c
ψ_5	24	0	0	0	0	0	0	0	0	c	$\bar{c}$

where c = $-\frac{1}{2}(1 + i\sqrt{7})$

7 The Fischer-Clifford matrices of $2^{5\cdot}(2^4{:}GL_4(2))$

The sizes of the Fischer-Clifford matrices $M(g)$, ranging from 1 to 3, for each conjugacy class representative g of G are determined by the class fusions of the inertia factors H_i into G. Note for H_2 and H_4 we are only using the fusion of the α_i^{-1}- regular classes into the classes of G. Hence, we have the number of conjugacy classes of $\overline{G}$ lying above a class $[g]$ of G and the centralizer orders of these classes is given by the equation $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f_j}$ which is obtained from the method of coset analysis (see[19], [20] and [22]). We are writing k for the number of orbits being formed when $N = 2^5$ is acting by conjugation on the coset $N\overline{g}$ and where f_j of these orbits fused under the action of $\{\overline{h}{:}h \in C_G(g)\}$ to give a class of $\overline{G}$ with representative x. Here $\overline{g}$ denotes a lifting of g in $\overline{G}$ under the natural homomorphism $\lambda{:}\overline{G} \to G$. Also, note that k is obtained from the value of the permutation character $\chi(G|2^5)$ on the class $[g]$ of G.

Since $\overline{G}$ has index 496 in $2^{5\cdot}GL_5(2)$, the action of $2^{5\cdot}GL(5,2)$ on the cosets of $\overline{G}$ gives rise to a permutation character $\chi(2^{5\cdot}GL_5(2)|\overline{G})$ of degree 496. We deduce from the character table of $2^{5\cdot}GL_5(2)$ in [7] (also available in the GAP library) that $\chi(2^{5\cdot}GL_5(2)|\overline{G}) = 1a + 2 \times 30a + 155a + 280a$. Using the relevant properties of Fischer-Clifford matrices found in [3], the equation $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f_j}$, the values of $\chi(2^{5\cdot}GL_5(2)|\overline{G})$ on the classes of $\overline{G}$, the fusion map of G into $GL_5(2)$ (see Table 4), and Proposition 7.5.1 in [22] we are able to determine fully the centralizers orders of the conjugacy classes $[x]_{\overline{G}}$ of $\overline{G}$ coming from a coset $N\overline{g}$.

Table 4. The fusion of $GL_4(2)$ into $GL_5(2)$

$[g]_{GL_4(2)}$ $\longrightarrow$	$[z]_{GL_5(2))}$	$[g]_{GL_4(2)}$ $\longrightarrow$	$[z]_{GL_5(2))}$
$1A$	$1A$	$5A$	$5A$
$2A$	$2A$	$6A$	$6B$
$2B$	$2B$	$6B$	$6A$
$3A$	$3B$	$7A$	$7A$
$3B$	$3A$	$7B$	$7B$
$4A$	$4A$	$15A$	$15A$
$4B$	$4C$	$15B$	$15B$

Having obtained the centralizer orders of classes of $\overline{G}$ associated with a coset $N\overline{g}$ together with the row and column orthogonality relations of Fischer-Clifford matrices in [3], the entries of the Fischer-Clifford matrix $M(g)$ of $\overline{G}$ are completed. The fusion of $\overline{G}$ into $2^{5\cdot}GL_5(2)$ and the restriction of some characters of $2^{5\cdot}GL_5(2)$ to $\overline{G}$ enable us to determine the orders of the elements of $\overline{G}$ coming from a coset $N\overline{g}$. The computations involved in obtaining the desired Fischer-Clifford matrix and classes of $\overline{G}$ corresponding to a coset $N\overline{g}$ were made easy, due to the relatively small sizes of the Fischer-Clifford matrices.

For example, consider the conjugacy class $2A$ of G. Observe that the only class fusions into $2A$ are from the two classes of involutions $2A$ and $2B$ of H_3. Hence the Fischer-Clifford matrix $M(2A)$ will be of size 3. Therefore the coset $N\overline{g}$, for a class representative g in $2A$, is splitting into 3 classes $[x_1]_{\overline{G}}$, $[x_2]_{\overline{G}}$ and $[x_3]_{\overline{G}}$ of $\overline{G}$. Then we obtain that $M(2A)$ has the following form with corresponding weights attached to the rows and columns:

$$M(2A) = \begin{array}{c} \\ |C_{H_1}(2A)| \\ |C_{H_3}(2A)| \\ |C_{H_3}(2B)| \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & |C_{\overline{G}}(x_3)| \end{array} \\ \left(\begin{array}{ccc} a & d & g \\ b & e & h \\ c & f & i \end{array} \right) \\ \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \end{array}.$$

By Theorem 1.3, property (c) on page 304 and property (i) of Lemma 2.8, all found in [3], we have the following form of $M(2A)$:

$$M(2A) = \begin{array}{c} \\ 192 \\ 192 \\ 32 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} 1536 & |C_{\overline{G}}(x_2)| & |C_{\overline{G}}(x_3)| \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & e & h \\ 6 & f & i \end{array} \right) \\ \begin{array}{ccc} 4 & 12 & 16 \end{array} \end{array}.$$

Consider the equation $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f_j} = \frac{16\times 192}{f_j} = \frac{3072}{f_j}$, where $k = \chi(G|2^5)(2A) = (4\times 1a + 2\times 14a)(2A) = 16$ is obtained from the value of the permutation character $\chi(G|2^5)$ on the class $2A$ of G. Since $|C_{\overline{G}}(x_1)| = \frac{3072}{f_j} = 1536$, it follows that $f_1 = 2$. Also $\sum f_j = k = 16$, then we must have that $f_2 + f_3 = 14$. We observe from Table 4 that the class $2A$ is the only class of G that fuse into the class $2A$ of $GL_5(2)$. Also, we obtained from [7] that the class $2A$ of $GL_5(2)$ splits into two classes, $[2B]_D$ and $[4A]_D$, of the Dempwolff group $D = 2^{5\cdot}GL_5(2)$ when the technique of coset analysis is applied. Hence $[x_1]_{\overline{G}}$, $[x_2]_{\overline{G}}$ and $[x_3]_{\overline{G}}$ are the only classes of $\overline{G}$ that will fused into $[2B]_D$ and $[4A]_D$. Using the values $\chi(D|\overline{G})([2B]_D) = \chi(D|\overline{G})([4A]_D) = 112$ of the permutation character $\chi(D|\overline{G})$ on the classes $[x_1]_{\overline{G}}$, $[x_2]_{\overline{G}}$ and $[x_3]_{\overline{G}}$ we deduce that $f_2 = 6$ and $f_3 = 8$ and hence the corresponding centralizer orders of $[x_2]_{\overline{G}}$ and $[x_3]_{\overline{G}}$ can only be 512 and 384. Thus we obtained that

$$M(2A) = \begin{array}{c} \\ 192 \\ 192 \\ 32 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} 1536 & 512 & 384 \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & e & h \\ 6 & f & i \end{array} \right) \\ \begin{array}{ccc} 4 & 12 & 16 \end{array} \end{array}.$$

By the orthogonality relations for columns and rows in [3](properties (b) and (c) on page 304) and the remaining properties given in Lemma 2.8 of [3] we obtained that

$$M(2A) = \begin{array}{c} \\ 192 \\ 192 \\ 32 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} 1536 & 512 & 384 \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{array} \right) \\ \begin{array}{ccc} 4 & 12 & 16 \end{array} \end{array}.$$

Let $248a$ be the irreducible character of $2^{5\cdot}GL_5(2)$ of degree 248 which we restrict to $\overline{G}$ ($(248a)_{\overline{G}} = \chi_{15} + \chi_{24} + \chi_{29} + \chi_{35}$) by the method of set intersection (see [19] and [21]). Then the shape of $M(2A)$ is forced (see Table 6) after considering the restriction of $248a$ to the partial character table of $\overline{G}$ associated with the coset $N\overline{g}$ of the class $2A$ of G. Hence we have fusion of the classes $[x_1]_{\overline{G}}$ and $[x_2]_{\overline{G}}$ into $[4A]_D$ and $[x_3]_{\overline{G}}$ into $[2B]_D$ (see Table 5).

We use similar types of arguments as in the case of $M(2A)$ to find the classes of $\overline{G}$ and the Fischer-Clifford matrix $M(g)$ associated with each coset $N\overline{g}$, for a class representative $g \in G$. These classes and Fischer-Clifford matrices of $\overline{G}$ are listed in Tables 5 and 6, respectively. Note that the last column in Table 5 represents the fusion of classes of $\overline{G}$ into $2^{5\cdot}GL_5(2)$.

Table 5. The conjugacy classes of elements of $\overline{G}$

$[g]_G$	k	f_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	$\to [y]_{2^{5\cdot}GL_5(2)}$
$1A$	32	$f_1=1$	$1A$	645120	$1A$
		$f_2=1$	$2A$	645120	$2A$
		$f_3=15$	$2B$	43008	$2A$
		$f_4=15$	$2C$	43008	$2A$
$2A$	16	$f_1=2$	$4A$	1536	$4A$
		$f_2=6$	$4B$	512	$4A$
		$f_2=8$	$2D$	384	$2B$
$2B$	8	$f_1=2$	$4C$	384	$4B$
		$f_2=6$	$4D$	128	$4B$
$3A$	2	$f_1=1$	$3A$	360	$3B$
		$f_2=1$	$6A$	360	$6B$
$3B$	8	$f_1=1$	$3B$	144	$3A$
		$f_2=1$	$6B$	144	$6A$
		$f_3=3$	$6C$	48	$6A$
		$f_4=3$	$6D$	48	$6A$
$4A$	8	$f_1=2$	$8A$	64	$8A$
		$f_2=2$	$8B$	64	$8A$
		$f_3=4$	$8C$	32	$8A$
$4B$	4	$f_1=2$	$8D$	16	$8B$
		$f_2=2$	$8E$	16	$8B$
$5A$	2	$f_1=1$	$5A$	30	$5A$
		$f_2=1$	$10A$	30	$10A$
$6A$	2	$f_1=2$	$12A$	12	$12C$
$6B$	4	$f_1=1$	$6E$	24	$6C$
		$f_2=1$	$6F$	24	$6C$
		$f_3=1$	$12B$	24	$12A$
		$f_4=1$	$12C$	24	$12B$
$7A$	4	$f_1=1$	$7A$	28	$7A$
		$f_2=1$	$14A$	28	$14A$
		$f_3=1$	$14B$	28	$14A$
		$f_4=1$	$14C$	28	$14A$
$7B$	4	$f_1=1$	$7B$	28	$7B$
		$f_2=1$	$14D$	28	$14B$
		$f_3=1$	$14E$	28	$14B$
		$f_4=1$	$14F$	28	$14B$
$15A$	2	$f_1=1$	$15A$	30	$15A$
		$f_2=1$	$30A$	30	$30A$
$15B$	2	$f_1=1$	$15B$	30	$15B$
		$f_2=1$	$30B$	30	$30B$

Table 6. The Fischer-Clifford Matrices of $2^{5\cdot}GL_4(2)$

$M(g)$	$M(g)$
$M(1A)=\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\15&15&-1&-1\\15&-15&1&-1\end{pmatrix}$	$M(2A)=\begin{pmatrix}1&1&1\\1&1&-1\\-6&2&0\end{pmatrix}$
$M(2B)=\begin{pmatrix}1&1\\3&-1\end{pmatrix}$	$M(3A)=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$
$M(3B)=\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\3&3&-1&-1\\3&-3&1&-1\end{pmatrix}$	$M(4A)=\begin{pmatrix}1&1&1\\1&1&-1\\2&-2&0\end{pmatrix}$
$M(4B)=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$	$M(5A)=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$
$M(6A)=\begin{pmatrix}1\end{pmatrix}$	$M(6B)=\begin{pmatrix}1&1&1&1\\-1&1&-1&1\\-1&-1&1&1\\1&-1&-1&1\end{pmatrix}$
$M(7A)=\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}$	$M(7B)=\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&1&-1&-1\\1&-1&1&-1\end{pmatrix}$
$M(15A)=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$	$M(15B)=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$

8 The character table of $2^{5\cdot}GL_4(2)$

Having obtained the Fischer-Clifford matrices of $\overline{G}$, the projective characters of the inertia factors H_2 and H_4, the ordinary characters of H_1 and H_3, the fusion of the α_i^{-1}- regular classes of H_2 and H_4 into G, and the fusion of the classes of H_1 and H_3 into G, we are able to construct the ordinary character table of $2^{5\cdot}GL_4(2)$ as outlined in Section 3. The set of irreducible characters of $\overline{G}$ will be partitioned into four blocks $\triangle_1$, $\triangle_2$, $\triangle_3$ and $\triangle_4$ corresponding to the inertia factor groups H_1, H_2, H_3 and H_4, respectively. In fact, $\triangle_1 = \{\chi_j | 1 \leq j \leq 14\}$, $\triangle_2 = \{\chi_j | 15 \leq j \leq 23\}$, $\triangle_3 = \{\chi_j | 24 \leq j \leq 34\}$ and $\triangle_4 = \{\chi_j | 35 \leq j \leq 39\}$ where $\chi_i \in Irr(\overline{G})$ such that $Irr(\overline{G}) = \triangle_1 \cup \triangle_2 \cup \triangle_3 \cup \triangle_4$. The consistency and accuracy of the character table of $2^{5\cdot}GL_4(2)$ (see Table 7) have been tested by using the GAP commands labelled as Programme E in [27].

Table 7. The Character table of $2^{5\cdot}GL_4(2)$

	1A				2A			2B		3A		3B			
	$1A$	$2A$	$2B$	$2C$	$4A$	$4B$	$2D$	$4C$	$4D$	$3A$	$6A$	$3B$	$6B$	$6C$	$6D$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	7	7	7	-1	-1	-1	3	3	4	4	1	1	1	1
χ_3	14	14	14	14	6	6	6	2	2	-1	-1	2	2	2	2
χ_4	20	20	20	20	4	4	4	4	4	5	5	-1	-1	-1	-1
χ_5	21	21	21	21	-3	-3	-3	1	1	6	6	0	0	0	0
χ_6	21	21	21	21	-3	-3	-3	1	1	-3	-3	0	0	0	0
χ_7	21	21	21	21	-3	-3	-3	1	1	-3	-3	0	0	0	0
χ_8	28	28	28	28	-4	-4	-4	4	4	1	1	1	1	1	1
χ_9	35	35	35	35	3	3	3	-5	-5	5	5	2	2	2	2
χ_{10}	45	45	45	45	-3	-3	-3	-3	-3	0	0	0	0	0	0
χ_{11}	45	45	45	45	-3	-3	-3	-3	-3	0	0	0	0	0	0
χ_{12}	56	56	56	56	8	8	8	0	0	-4	-4	-1	-1	-1	-1
χ_{13}	64	64	64	64	0	0	0	0	0	4	4	-2	-2	-2	-2
χ_{14}	70	70	70	70	-2	-2	-2	2	2	-5	-5	1	1	1	1
χ_{15}	8	-8	-8	8	0	0	0	0	0	-4	4	2	-2	-2	2
χ_{16}	24	-24	-24	24	0	0	0	0	0	-6	6	0	0	0	0
χ_{17}	24	-24	-24	24	0	0	0	0	0	-6	6	0	0	0	0
χ_{18}	48	-48	-48	48	0	0	0	0	0	6	-6	0	0	0	0
χ_{19}	56	-56	-56	56	0	0	0	0	0	-4	4	-1	1	1	-1
χ_{20}	56	-56	-56	56	0	0	0	0	0	-4	4	-1	1	1	-1
χ_{21}	56	-56	-56	56	0	0	0	0	0	2	-2	2	-2	-2	2
χ_{22}	56	-56	-56	56	0	0	0	0	0	2	-2	2	-2	-2	2
χ_{23}	64	-64	-64	64	0	0	0	0	0	4	-4	-2	2	2	-2
χ_{24}	15	15	-1	-1	-5	3	-1	3	-1	0	0	3	3	-1	-1
χ_{25}	45	45	-3	-3	9	1	-3	-3	1	0	0	0	0	0	0
χ_{26}	45	45	-3	-3	9	1	-3	-3	1	0	0	0	0	0	0
χ_{27}	90	90	-6	-6	-6	10	-6	6	-2	0	0	0	0	0	0
χ_{28}	105	105	-7	-7	-19	5	1	-3	1	0	0	3	3	-1	-1
χ_{29}	105	105	-7	-7	13	5	-7	-3	1	0	0	3	3	-1	-1
χ_{30}	105	105	-7	-7	5	-3	1	9	-3	0	0	3	3	-1	-1
χ_{31}	120	120	-8	-8	8	8	-8	0	0	0	0	-3	-3	1	1
χ_{32}	210	210	-14	-14	-14	2	2	6	-2	0	0	-3	-3	1	1
χ_{33}	315	315	-21	-21	-9	-1	3	-9	3	0	0	0	0	0	0
χ_{34}	315	315	-21	-21	15	-9	3	3	-1	0	0	0	0	0	0
χ_{35}	120	-120	8	-8	0	0	0	0	0	0	0	6	-6	2	-2
χ_{36}	120	-120	8	-8	0	0	0	0	0	0	0	-3	3	-1	1
χ_{37}	120	-120	8	-8	0	0	0	0	0	0	0	-3	3	-1	1
χ_{38}	360	-360	24	-24	0	0	0	0	0	0	0	0	0	0	0
χ_{39}	360	-360	24	-24	0	0	0	0	0	0	0	0	0	0	0

Table 7 (continued)

	$4A$			$4B$		$5A$		$6A$	6B			
	$8A$	$8B$	$8C$	$8D$	$8E$	$5A$	$10A$	$12A$	$6E$	$6F$	$12B$	$12C$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	1	1	2	2	0	-1	-1	-1	-1
χ_3	2	2	2	0	0	-1	-1	-1	0	0	0	0
χ_4	0	0	0	0	0	0	0	1	1	1	1	1
χ_5	1	1	1	-1	-1	1	1	-2	0	0	0	0
χ_6	1	1	1	-1	-1	1	1	1	0	0	0	0
χ_7	1	1	1	-1	-1	1	1	1	0	0	0	0
χ_8	0	0	0	0	0	-2	-2	1	-1	-1	-1	-1
χ_9	-1	-1	-1	-1	-1	0	0	1	0	0	0	0
χ_{10}	1	1	1	1	1	0	0	0	0	0	0	0
χ_{11}	1	1	1	1	1	0	0	0	0	0	0	0
χ_{12}	0	0	0	0	0	1	1	0	-1	-1	-1	-1
χ_{13}	0	0	0	0	0	-1	-1	0	0	0	0	0
χ_{14}	-2	-2	-2	0	0	0	0	-1	1	1	1	1
χ_{15}	0	0	0	0	0	-2	2	0	0	0	0	0
χ_{16}	0	0	0	0	0	-1	1	0	0	0	0	0
χ_{17}	0	0	0	0	0	-1	1	0	0	0	0	0
χ_{18}	0	0	0	0	0	-2	2	0	0	0	0	0
χ_{19}	0	0	0	0	0	1	-1	0	A	$-A$	A	$-A$
χ_{20}	0	0	0	0	0	1	-1	0	$-A$	A	$-A$	A
χ_{21}	0	0	0	0	0	1	-1	0	0	0	0	0
χ_{22}	0	0	0	0	0	1	-1	0	0	0	0	0
χ_{23}	0	0	0	0	0	-1	1	0	0	0	0	0
χ_{24}	-3	1	1	1	-1	0	0	0	-1	-1	1	1
χ_{25}	-1	3	-1	1	-1	0	0	0	0	0	0	0
χ_{26}	-1	3	-1	1	-1	0	0	0	0	0	0	0
χ_{27}	-2	-2	2	0	0	0	0	0	0	0	0	0
χ_{28}	-1	3	-1	-1	1	0	0	0	1	1	-1	-1
χ_{29}	3	-1	-1	-1	1	0	0	0	-1	-1	1	1
χ_{30}	3	-1	-1	1	-1	0	0	0	1	1	-1	-1
χ_{31}	0	0	0	0	0	0	0	0	1	1	-1	-1
χ_{32}	2	2	-2	0	0	0	0	0	-1	-1	1	1
χ_{33}	1	-3	1	1	-1	0	0	0	0	0	0	0
χ_{34}	-3	1	1	-1	1	0	0	0	0	0	0	0
χ_{35}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{36}	0	0	0	0	0	0	0	0	$-A$	A	A	$-A$
χ_{37}	0	0	0	0	0	0	0	0	A	$-A$	$-A$	A
χ_{38}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{39}	0	0	0	0	0	0	0	0	0	0	0	0

where $A = -i3 = -\sqrt{3}i$

Table 7 (continued)

	7A				$7B$				$15A$		15B	
	$7A$	$14A$	$14B$	$14C$	$7B$	$14D$	$14E$	$14F$	$15A$	$30A$	$15B$	$30B$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	0	0	0	0	0	0	0	0	-1	-1	-1	-1
χ_3	0	0	0	0	0	0	0	0	-1	-1	-1	-1
χ_4	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0
χ_5	0	0	0	0	0	0	0	0	1	1	1	1
χ_6	0	0	0	0	0	0	0	0	C	C	$\overline{C}$	$\overline{C}$
χ_7	0	0	0	0	0	0	0	0	$\overline{C}$	$\overline{C}$	C	C
χ_8	0	0	0	0	0	0	0	0	1	1	1	1
χ_9	0	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	B	B	B	B	$\overline{B}$	$\overline{B}$	$\overline{B}$	$\overline{B}$	0	0	0	0
χ_{11}	$\overline{B}$	$\overline{B}$	$\overline{B}$	$\overline{B}$	B	B	B	B	0	0	0	0
χ_{12}	0	0	0	0	0	0	0	0	1	1	1	1
χ_{13}	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_{14}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{15}	1	1	-1	-1	1	-1	-1	1	1	-1	1	-1
χ_{16}	B	B	$-B$	$-B$	$\overline{B}$	$-\overline{B}$	$-\overline{B}$	$\overline{B}$	-1	1	-1	1
χ_{17}	$\overline{B}$	$\overline{B}$	$-\overline{B}$	$-\overline{B}$	B	$-B$	$-B$	B	-1	1	-1	1
χ_{18}	-1	-1	1	1	-1	1	1	-1	1	-1	1	-1
χ_{19}	0	0	0	0	0	0	0	0	1	-1	1	-1
χ_{20}	0	0	0	0	0	0	0	0	1	-1	1	-1
χ_{21}	0	0	0	0	0	0	0	0	C	$-C$	$\overline{C}$	$-\overline{C}$
χ_{22}	0	0	0	0	0	0	0	0	$\overline{C}$	$-\overline{C}$	C	$-C$
χ_{23}	1	1	-1	-1	1	-1	-1	1	-1	1	-1	1
χ_{24}	1	-1	1	-1	1	1	-1	-1	0	0	0	0
χ_{25}	$\overline{B}$	$-\overline{B}$	$\overline{B}$	$-\overline{B}$	B	B	$-B$	$-B$	0	0	0	0
χ_{26}	B	$-B$	B	$-B$	$\overline{B}$	$\overline{B}$	$-\overline{B}$	$-\overline{B}$	0	0	0	0
χ_{27}	-1	1	-1	1	-1	-1	1	1	0	0	0	0
χ_{28}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{30}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	1	-1	1	-1	1	1	-1	-1	0	0	0	0
χ_{32}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	0	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	1	-1	-1	1	1	-1	1	-1	0	0	0	0
χ_{36}	1	-1	-1	1	1	-1	1	-1	0	0	0	0
χ_{37}	1	-1	-1	1	1	-1	1	-1	0	0	0	0
χ_{38}	B	$-B$	$-B$	B	$\overline{B}$	$-\overline{B}$	$\overline{B}$	$-\overline{B}$	0	0	0	0
χ_{39}	$\overline{B}$	$-\overline{B}$	$-\overline{B}$	$\overline{B}$	B	$-B$	B	$-B$	0	0	0	0

where $B = b7 = \frac{-1+\sqrt{7}i}{2}$ and $C = \text{b}15 = \frac{-1+\sqrt{15}i}{2}$

We use GAP to compute possible power maps from the character table of $\overline{G}$. The GAP commands of Programme E in [27] produces unique p-power maps (see Table 8) for our Table 7.

Table 8. The power maps of the elements of $2^{5\cdot}GL_4(2)$

$[g]_G$	$[x]_{\overline{G}}$	2	3	5	7	$[g]_G$	$[x]_{\overline{G}}$	2	3	5	7
1A	1A					2A	4A	2C			
	2A	1A					4B	2C			
	2B	1A					2D	1A			
	2C	1A									
2B	4C	2A				3A	3A		1A		
	4D	2B					6A	3A	2A		
3B	3B		1A			4A	8A	4B			
	6B	3B	2A				8B	4B			
	6C	3B	2B				8C	4B			
	6D	3B	2C								
4B	8D	4C				5A	5A			1A	
	8E	4D					10A	5A		2A	
6A	12A	6A	4C			6B	6E	3B	2D		
							6F	3B	2D		
							12B	6D	4A		
							12C	6D	4A		
7A	7A				1A	7B	7B				1A
	14A	7A			2C		14D	7B			2A
	14B	7A			2A		14E	7B			2B
	14C	7A			2B		14F	7B			2C
15A	15A		5A	3A		15B	15B		5A	3A	
	30A	15A	10A	6A			30B	15B	10A	6A	

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