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Modular Relations for J-invariant and Explicit evaluations

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Abstract. On pages $392-393$ of his second note book Ramanujan defined J-invariant and recorded 14 values of J_n leaving "J_{99} as $J_{99} = \cdots$". All these values have been proved by using known values of Ramanujan-Weber class invariant G_n. Motivated by this, in this paper we establish modular relations connecting J-invariant J_n with five other J-invariants J_{r^2n} for $r = 3, 5, 7, 11, 13$. As an application, we find several explicit evaluations of J_n for different values of n. At last we give some relations connecting L_n and R_n for different values of n, where L_n and R_n represent Eisenstein series.

1 Introduction

On pages $392-393$ of his second note book Ramanujan defined J-invariant and recorded 14 values of J_n, $n \equiv 3 \pmod 4$, leaving J_{99} as "$J_{99} = \cdots$". He also records factors of certain polynomials in J_n.
The invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathbb{H} := \{\tau : Im\,\tau > 0\}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \text{ and } j(\tau) = 1728J(\tau), \tag{1.1}$$

where

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau), \tag{1.2}$$

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau+n)^{-4} \tag{1.3}$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau+n)^{-6}. \tag{1.4}$$

It can be shown that Δ is a modular form of weight twelve, and g_2 one of weight four, so that its third power is also of weight twelve. Thus their quotient, and therefore J, is a modular function of weight zero, in particular a meromorphic function $\mathbf{H} \to \mathbf{C}$(where $\mathbf{H}$ is the Upper half-plane) invariant under the action of $SL(2, \mathbf{Z})$.
Moreover, the function $\gamma(\tau)$ is defined by[4, p. 249]

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)}, \tag{1.5}$$

At the top of page 392 in his second notebook [10], which is inexplicably is printed upside down, Ramanujan defines $J := J_n$ and $u := u_n$ by

$$J_n = \frac{1-16\alpha_n(1-\alpha_n)}{8(4\alpha_n(1-\alpha_n))^{1/3}} \text{ and } J_n = \frac{\sqrt[3]{4u_n}}{2^3}, \tag{1.6}$$

where n is a natural number. In the theory of elliptic functions, $k, 0 < k < 1$, denotes the modulus, then $\sqrt{\alpha_n} := k_n$ is the singular modulus. Recall that Ramanujan's definition of the class invariant G_n,

$$G_n := 2^{-1/4} q_n^{-1/24} \prod_{j=1}^{\infty} \left(1 + q_n^{2j-1}\right), \tag{1.7}$$

where n is a positive rational number and $q_n = \exp\left(-\pi\sqrt{n}\right)$. Moreover,

$$G_n = \left(4\alpha_n \left(1 - \alpha_n\right)\right)^{-1/24}, \tag{1.8}$$

Now, from (1.6) and (1.8), we find that

$$J_n = \frac{1}{8} G_n^8 \left(1 - 4G_n^{-24}\right). \tag{1.9}$$

We now define J_n with γ_2. For $q = \exp\left(2\pi i\tau\right)$,

$$\gamma_2\left(\tau\right) = 2^8 \frac{q^{2/3} f^{16}\left(-q^2\right)}{f^{16}\left(-q\right)} + \frac{f^8\left(-q\right)}{q^{1/3} f^8\left(-q^2\right)}. \tag{1.10}$$

Setting $\tau = \left(3 + \sqrt{-n}\right)/2$, from (1.7) and (1.10) we deduce that

$$\gamma_2\left(\frac{3 + \sqrt{-n}}{2}\right) = \frac{2^8 - 2^6 G_n^{24}}{2^4 G_n^{16}} = -4G_n^8 \left(1 - 4G_n^{-24}\right). \tag{1.11}$$

Hence, from (1.5), (1.9) and (1.11),

$$J_n = -\frac{1}{32}\gamma_2\left(\frac{3 + \sqrt{-n}}{2}\right) = -\frac{1}{32}\sqrt[3]{j\left(\frac{3 + \sqrt{-n}}{2}\right)}. \tag{1.12}$$

In section 2, we collect results which are very useful in proving our main identities. In section 3, we derive modular equations connecting J_n and $J_{r^2 n}$ for different values of r and n. Using these modular equations, in section 5 we evaluate several values of J_n.

Lemma 1.1. *[3, Ch. 34, p.319] If*

$$h\left(q\right) = \frac{f^{12}\left(-q^3\right)}{q f^6\left(-q\right) f^6\left(-q^9\right)} \quad \textit{and} \quad s = \frac{f^3\left(-q\right)}{q f^3\left(-q^9\right)}, \tag{1.13}$$

where

$$f\left(-q\right) := \prod_{n=1}^{\infty} \left(1 - q^n\right) = q^{-1/24}\eta\left(\tau\right), \tag{1.14}$$

$q = \exp\left(2\pi i\tau\right), \tau\epsilon\mathbb{H}$, *and* $\eta\left(\tau\right)$ *denotes the Dedekind eta-function. Then*

$$\gamma_2\left(3\tau\right) = h\left(q\right) - 6 - 27h^{-1}\left(q\right) \tag{1.15}$$

and

$$h\left(q\right) = s + 9 + \frac{27}{s}. \tag{1.16}$$

The Eisenstein series in terms of $L(q)$ and $R(q)$ are defined by

$$L(q) := 1 + 240\sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \tag{1.17}$$

and

$$R(q) := 1 - 504\sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}, \tag{1.18}$$

where $|q| < 1$.

Lemma 1.2. *[1, Ch.15 p.367] For each positive rational n,*

$$\left(\left(\frac{8}{3}J_n\right)^3+1\right)L_n^3-\left(\frac{8}{3}J_n\right)^3R_n^2=0, \tag{1.19}$$

where J_n, $L_n := L(-e^{-\pi\sqrt{-n}})$ and $R_n := R(-e^{-\pi\sqrt{-n}})$ are defined as in (1.12), (1.17), and (1.18) respectively.

Using the Lemma 1.2, at the end of section 4 we derive nine relations connecting L_n and R_n for different values of n.
Finally, we end this section by defining modular equation. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_nx^n}{(c)_nn!},$$

where $(a)_0=1,\ (a)_n=a(a+1)(a+2)...(a+n-1),$ for $n\geq 1,\ \mid x\mid<1.$
Let $z(r):=z(r;x):={}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;x\right)$ and

$$q_r:=q_r(x):=exp\left(-\pi csc\left(\frac{\pi}{r}\right)\frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-x)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;x)}\right),$$

where $r=2,3,4,6$ and $0<x<1$.
Let n denote a fixed rational number, and assume that

$$n\frac{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;\beta\right)}, \tag{1.20}$$

where $r=2,3,4$ or 6. Then a modular equation of degree n in the theory of elliptic functions of signature r is a relation between α and β induced by (1.20). For more details on modular equations one can see [5], [6], [7], [8], and [9].

2 Preliminary Results

In this section, we lists some identities which are useful in proving our main results.

Lemma 2.1. *If $P=\dfrac{f(-q)}{q^{1/3}f(-q^9)}$ and $Q=\dfrac{f(-q^3)}{qf(-q^{27})}$, then*

$$\begin{aligned}&(PQ)^3+\left(\frac{9}{PQ}\right)^3+27\left[\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3\right]+243\left(\frac{1}{P^3}+\frac{1}{Q^3}\right)\\&+9\left(P^3+Q^3\right)+81=\left(\frac{Q}{P}\right)^6.\end{aligned} \tag{2.1}$$

Lemma 2.2. *[7] If $P=\dfrac{f(-q)f(-q^5)}{q^2f(-q^9)f(-q^{45})}$ and $Q=\dfrac{f(-q)f(-q^{45})}{q^{-4/3}f(-q^9)f(-q^5)}$, then*

$$\begin{aligned}&Q^3+\frac{1}{Q^3}=\left(P^2+\frac{9^2}{P^2}\right)+5\left(\sqrt{P}+\frac{3}{\sqrt{P}}\right)\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)\\&+5\left(P+\frac{9}{P}\right)+10.\end{aligned} \tag{2.2}$$

Lemma 2.3. *[7] If $P=\dfrac{f(-q)f(-q^7)}{q^{8/3}f(-q^9)f(-q^{63})}$ and $Q=\dfrac{f(-q)f(-q^{63})}{q^{-2}f(-q^9)f(-q^7)}$, then*

$$\begin{aligned}&Q^4+\frac{1}{Q^4}-14\left(Q^3+\frac{1}{Q^3}\right)+28\left(Q^2+\frac{1}{Q^2}\right)+7\left(Q+\frac{1}{Q}\right)=P^3+\frac{9^3}{P^3}\\&+7\left(\sqrt{P^3}+\frac{27}{\sqrt{P^3}}\right)\left[\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)-\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)\right]+98.\end{aligned} \tag{2.3}$$

Lemma 2.4. *[7] If* $P = \dfrac{f(-q)f(-q^{11})}{q^4 f(-q^9)f(-q^{99})}$ *and* $Q = \dfrac{f(-q)f(-q^{99})}{q^{-10/3} f(-q^9)f(-q^{11})}$, *then*

$$\begin{aligned} Q^6 + \frac{1}{Q^6} &= 165\left(P + \frac{9}{P}\right) + 66\left(P^2 + \frac{9^2}{P^2}\right) + 11\left(P^3 + \frac{9^3}{P^3}\right) + 1848 \\ &+ \left(P^5 + \frac{9^5}{P^5}\right) + 22\left(Q^3 + \frac{1}{Q^3}\right)\left[2\left(P^2 + \frac{9^2}{P^2}\right) + 3\left(P + \frac{9}{P}\right) + 26\right] \\ &+ 55\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\left(\sqrt{Q^9} + \frac{1}{\sqrt{Q^9}}\right) + 11\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\left[\left(\sqrt{P^7} + \frac{3^7}{\sqrt{P^7}}\right)\right. \\ &\left. + \left(\sqrt{P^5} + \frac{3^5}{\sqrt{P^5}}\right) + 14\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right) + 15\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\right]. \end{aligned} \tag{2.4}$$

Lemma 2.5. *[7] If* $P = \dfrac{f(-q)f(-q^{13})}{q^{14/3} f(-q^9)f(-q^{117})}$ *and* $Q = \dfrac{f(-q)f(-q^{117})}{q^{-4} f(-q^9)f(-q^{13})}$, *then*

$$\begin{aligned} &Q^7 + \frac{1}{Q^7} - 65\left(Q^6 + \frac{1}{Q^6}\right) + 910\left(Q^5 + \frac{1}{Q^5}\right) - 1417\left(Q^4 + \frac{1}{Q^4}\right) \\ &- 6994\left(Q^3 + \frac{1}{Q^3}\right) + 10049\left(Q^2 + \frac{1}{Q^2}\right) + 6981\left(Q + \frac{1}{Q}\right) - 17472 \\ &+ P^6 + \frac{9^6}{P^6} - 13\left(\sqrt{P^9} + \frac{3^9}{\sqrt{P^9}}\right)\left[\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - \left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\right] \\ &- 13\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left[139\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - 179\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\right. \\ &\left. -2\left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}}\right) + 52\left(\sqrt{Q^7} + \frac{1}{\sqrt{Q^7}}\right) + 10\left(\sqrt{Q^9} + \frac{1}{\sqrt{Q^9}}\right)\right] \\ &+ 13\left(P^3 + \frac{9^3}{P^3}\right)\left[5\left(Q^3 + \frac{1}{Q^3}\right) - 12\left(Q^2 + \frac{1}{Q^2}\right) - 6\left(Q + \frac{1}{Q}\right) + 26\right]. \end{aligned} \tag{2.5}$$

3 Modular relations for $J-$invariant

In this section, we establish some new modular relations connecting J_n and $J_{k^2 n}$ for $k = 3, 5, 7, 11$, and 13.

Lemma 3.1. *If* J_n *is defined as in* (1.9), *then*

$$J_n = J_{1/n}. \tag{3.1}$$

Proof. Using the fact that $G_n = G_{1/n}$ and the equation (1.9), we arrive at (3.1). □

Theorem 3.2. *If* $M = \dfrac{J_n}{J_{3^2 n}}$ *and* $N = J_n J_{3^2 n}$, *then*

$$\begin{aligned} &M^6 + \frac{1}{M^6} - \left(M^3 + \frac{1}{M^3}\right)\left[1069956 - \frac{421875}{N^3}\right] - \left(\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right) \\ &\left[18\left(4063232\sqrt{N^3} + \frac{15089625}{\sqrt{N^3}}\right) + \frac{3^3.5^9}{\sqrt{N^9}}\right] - \frac{3^2.5^3}{\sqrt{N^3}}\left(\sqrt{M^9} + \frac{1}{\sqrt{M^9}}\right) \\ &- 1073741824N^3 - \frac{717906250}{N^3} + 2587918086 = 0. \end{aligned} \tag{3.2}$$

Proof. The equation (1.16) can be written as

$$s = \frac{1}{2}(h - 9 - a). \tag{3.3}$$

If $P = \dfrac{f(-q)}{q^{1/3} f(-q^9)}$, the equation (3.3) becomes

$$P^3 = \frac{1}{2}(h - 9 - a), \tag{3.4}$$

similarly, we have

$$Q^3 = \frac{1}{2}(g - 9 - b), \tag{3.5}$$

where

$$a^2 := (h-9)^2 - 108 \ \text{ and } \ b^2 := (g-9)^2 - 108. \tag{3.6}$$

Using the equations (3.4) and (3.5) in (2.1), we find that

$$\begin{aligned} &8abgh^2 + 8ag^2h^2 + 8bgh^3 + 8g^2h^3 - 72abgh - 72ag^2h - 72agh^2 - 144bgh^2 \\ &- 144g^2h^2 - 72gh^3 + 648agh - 16bg^2 + 216bgh - 16g^3 + 216g^2h + 1296gh^2 \\ &+ 288bg + 432g^2 - 1944gh - 864b - 2592g = 0. \end{aligned} \tag{3.7}$$

Using the equation (3.6) in (3.7), after collecting the terms containing a on one side and then squaring both sides and similarly for b also, we obtain

$$\begin{aligned} &8503056g^5h^5 - 153055008\left(g^5h^4 + g^4h^5\right) - 8503056\left(g^6h^2 + g^2h^6\right) \\ &+ 229582512\left(g^5h^3 + g^3h^5\right) - 3443737680g^4h^4 + 306110016\left(g^5h^2 + g^2h^5\right) \\ &- 4132485216\left(g^4h^3 + g^3h^4\right) - 3214155168\left(g^4h^2 + g^2h^4\right) - 6198727824 \\ &+ 6207230880g^3h^3 + 8111915424\left(g^3h^2 + g^2h^3\right) + 229582512\left(g^3h + gh^3\right) \\ &+ 2754990144g^2h^2 - 4132485216\left(g^2h + gh^2\right) + 6198727824gh = 0. \end{aligned} \tag{3.8}$$

Using the equation (1.5) in (1.15), we find that

$$h = -16J_n + 3 + r, \tag{3.9}$$

similarly, we have

$$g = -16J_{9n} + 3 + k, \tag{3.10}$$

where

$$r^2 = (-16J_n + 3)^2 + 27 \ \text{ and } \ k^2 = (-16J_{9n} + 3)^2 + 27. \tag{3.11}$$

Using the equations (3.9) and (3.10) in (3.8) and eliminating r and k using (3.11) and then setting $M = \dfrac{J_n}{J_{9n}}$ and $N = J_n J_{9n}$, we arrive at (3.2). □

Theorem 3.3. *If* $M = \dfrac{J_n}{J_{5^2 n}}$ *and* $N = J_n J_{5^2 n}$, *then*

$$\begin{aligned} &M^3 + \frac{1}{M^3} + 21114880N + \frac{223908975}{N} - 2\left(2^{19}N^2 + \frac{39375963}{N^2}\right) + \frac{4851495}{N^3} \\ &- 20\left(\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right)\left[1984\sqrt{N} + \frac{103455}{\sqrt{N}} + \frac{998541}{\sqrt{N^3}}\right] - 125915650 = 0. \end{aligned} \tag{3.12}$$

Proof. Proof of the equation (3.12) is similar to the proof of the equation (3.2); except that in place of the equation (2.1), we use the equation (2.2). □

Theorem 3.4. *If* $M = \dfrac{J_n}{J_{7^2 n}}$ *and* $N = J_n J_{7^2 n}$*, then*

$$\begin{aligned}
&M^4 + \frac{1}{M^4} - 401660\left(M^3 + \frac{1}{M^3}\right) + 24762303370\left(M^2 + \frac{1}{M^2}\right)\\
&-\left(M + \frac{1}{M}\right)\left[10422120833264 - \frac{10576844578125}{N^3}\right] - 98\left(\sqrt{M} + \frac{1}{\sqrt{M}}\right)\\
&\left[2^{19}\cdot 4721\sqrt{N^3} + \frac{1698059062125}{\sqrt{N^3}}\right] - 1073741824N^3 + \frac{19642711359375}{N^3}\\
&-7\left(\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right)\left[2^{18}\cdot 31\sqrt{N^3} + \frac{4577752675125}{\sqrt{N^3}}\right]\\
&-\frac{1063491990750}{\sqrt{N^3}}\left(\sqrt{M^5} + \frac{1}{M^5}\right) + 49402229030035 = 0.
\end{aligned} \tag{3.13}$$

Proof. Proof of the equation (3.13) is similar to the proof of the equation (3.2); except that in place of the equation (2.1), we use the equation (2.3). □

Theorem 3.5. *If* $M = \dfrac{J_n}{J_{11^2 n}}$ *and* $N = J_n J_{11^2 n}$*, then*

$$\begin{aligned}
&M^6 + \frac{1}{M^6} + \frac{2^3\cdot 3^{12}\cdot 5^{12}\cdot 11\cdot 17^3\cdot 29^3}{N^6} - 2^{50}N^5 - \frac{2\cdot 3^{11}\cdot 5^{11}\cdot 17\cdot 29\cdot 3761410391}{N^5}\\
&-1210\left(2^{29}\cdot 3042163477N^3 + \frac{1729103636909295291515625}{N^3}\right)\\
&+5082\left(1147088135325598351 36N^2 + \frac{99762718750783765525 6875}{N^2}\right)\\
&-14520\left(2^{12}\cdot 969058196940645767N + \frac{315672079330410336180675}{N}\right)\\
&+264\left(2^{38}\cdot 35521N^4 + \frac{3^8\cdot 5^8\cdot 538753758144107}{N^4}\right) - \left(\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right)\\
&\times\left[29040\left(2^6\cdot 111064952346422216 09\sqrt{N} + \frac{152597697388165606 16745}{\sqrt{N}}\right)\right.\\
&+6050\left(140805407764474757 12\sqrt{N^3} - \frac{577408934247280653633 75}{\sqrt{N^3}}\right)\\
&+4840\left(2^{27}\cdot 46577929\sqrt{N^5} - \frac{454987854519923096156 25}{\sqrt{N^5}}\right)\\
&\left.+22\left(2^{37}\cdot 31\sqrt{N^7} + \frac{3^9\cdot 5^9\cdot 11\cdot 13602087532679}{\sqrt{N^7}}\right) - \frac{3^9\cdot 5^{10}\cdot 73467730691443}{\sqrt{N^9}}\right]\\
&-\left(\sqrt{M^9} + \frac{1}{\sqrt{M^9}}\right)\left[880\left(2^6\cdot 7\cdot 19\cdot 1291\sqrt{N} + \frac{14668948511415}{\sqrt{N}}\right)\right.\\
&\left.+\frac{301585797480510025 3125}{\sqrt{N^3}}\right] + 1318298884208744202274348806\\
&+\left(M^3 + \frac{1}{M^3}\right)\left[7260\left(2^{12}\cdot 108130107511N + \frac{81657125024102611 6275}{N}\right)\right.\\
&-\left(208065580 2368N^2 + \frac{3629115261076367882110 5000}{N^2}\right)\\
&\left.-4974463399905062614 7204 + \frac{5^8\cdot 3^7\cdot 49807509355935617}{N^3}\right] = 0.
\end{aligned} \tag{3.14}$$

Proof. Proof of the equation (3.14) is similar to the proof of the equation (3.2); except that in place of the equation (2.1), we use the equation (2.4). □

Theorem 3.6. *If* $M = \dfrac{J_n}{J_{13^2 n}}$ *and* $N = J_n J_{13^2 n}$, *then*

$$
\begin{aligned}
&M^7 + \frac{1}{M^7} - 39725995010\left(M^6 + \frac{1}{M^6}\right) + 25187082552210651494 5\left(M^5 + \frac{1}{M^5}\right)\\
&- 313175466368248037844755405 2\left(M^4 + \frac{1}{M^4}\right) - 2^6\left(2^{54}N^6 - \frac{3^{21}\cdot 5^{12}\cdot 11^6\cdot 23^3}{N^6}\right)\\
&- 849076929439631208783010060485124\left(M^2 + \frac{1}{M^2}\right)\\
&+ 13\left(2^{33}\cdot 139788452862322661N^3 + \frac{3^9\cdot 5^6\cdot 5363832419034840795436019}{N^3}\right)\\
&+ 78\left(M + \frac{1}{M}\right)\left[\frac{3^{20}\cdot 5^{12}\cdot 689746645984}{N^6} + 12573541945430201834892911006970 1\right.\\
&\left.-8\left(268430598176165634192179 2N^3 + \frac{4790328047278248382915085156 25}{N^3}\right)\right]\\
&+ 390\left(M^2 + \frac{1}{M^2}\right)\left[2^{32}\cdot 200108322229N^3 + \frac{3^8\cdot 5^6\cdot 1512470935209442551026 09}{N^3}\right]\\
&+ 52\left(M^3 + \frac{1}{M^3}\right)\left[1555837466169461689791351289 70\right.\\
&\left.-64932389322752N^3 + \frac{452866355325576839840450442187 5}{N^3}\right]\\
&+ \frac{852510438372404284022557416093 75}{N^3}\left(M^4 + \frac{1}{M^4}\right) - \left(\sqrt{M} + \frac{1}{\sqrt{M}}\right)\\
&\times\left[15730000\left(5735052650479616\sqrt{N^9} - \frac{656658633991780358553656 25}{\sqrt{N^9}}\right)\right.\\
&+52\left(2^{19}\cdot 3047863736984892693435613\sqrt{N^3}\right.\\
&\left.\left.-\frac{3^7\cdot 5^4\cdot 14575095389698164645889486 1}{\sqrt{N^3}}\right)\right] - 52\left(\sqrt{M^3} + \frac{1}{\sqrt{M^3}}\right)\\
&\left[101059611570062061575230717952\sqrt{N^3} + \frac{3^7\cdot 5^3\cdot 995738047857537295565191489}{\sqrt{N^3}}\right.\\
&\left.+2\left(2^{45}\cdot 31\sqrt{N^9} - \frac{3^{18}\cdot 5^9\cdot 10652401537069457}{\sqrt{N^9}}\right)\right] - 52\left(\sqrt{M^5} + \frac{1}{\sqrt{M^5}}\right)\\
&\times\left[187003535742050739364560896\sqrt{N^3} - \frac{1606680436848000517404685429537 5}{\sqrt{N^3}}\right.\\
&\left.+\frac{145308782324308504540907812500 0}{\sqrt{N^9}}\right] - 52\left(\sqrt{M^7} + \frac{1}{\sqrt{M^7}}\right)\\
&\times\left[69249205830242218803 2\sqrt{N^3} + \frac{1765003190847531638397313361625}{\sqrt{N^3}}\right]\\
&- 52\left(\sqrt{M^9} + \frac{1}{\sqrt{M^9}}\right)\left[583155318784\sqrt{N^3} + \frac{6100611010501994620512827625}{\sqrt{N^3}}\right]\\
&- \frac{16060135922226053531879850 0}{\sqrt{N^3}}\left(\sqrt{M^{11}} + \frac{1}{\sqrt{M^{11}}}\right)\\
&- 2061081880149706509438288237881550 0 = 0.
\end{aligned}
\tag{3.15}
$$

Proof. Proof of the equation (3.15) is similar to the proof of the equation (3.2); except that in place of the equation (2.1), we use the equation (2.5). □

4 Explicit evaluation for J-invariant

Theorem 4.1. *We have*

$$\text{(i)}\ \ J_3 = 0, \tag{4.1}$$

$$\text{(ii)}\ \ J_5 = \frac{13\sqrt{5}}{16} - \frac{25}{16}, \tag{4.2}$$

$$\text{(iii)}\ \ J_7 = \frac{15}{32}, \tag{4.3}$$

$$\textit{(iv)}\ \ J_{11} = 1, \tag{4.4}$$

$$\textit{(v)}\ \ J_{13} = -\frac{465}{16} + \frac{135\sqrt{13}}{16}, \tag{4.5}$$

$$\textit{(vi)}\ \ J_{27} = 5(3^{\frac{1}{3}}). \tag{4.6}$$

Proof of (i). Setting $n = 1/3$ and using the fact that $J_n = J_{1/n}$ in (3.2), we obtain the required result. □

Proof of (ii). Putting $n = 1/5$ and using the fact that $J_n = J_{1/n}$ in (3.12), we find that

$$\left(64v^2 + 200v - 55\right)(v-1)^2 (16v+33)^2 (v-3)^2 (8v+3)^2 = 0, \tag{4.7}$$

where $v := J_5$. We observe that the first factor of (4.7) vanishes for specific value of q, whereas the other factor does not vanish. Hence, we deduce that

$$64v^2 + 200v - 55 = 0. \tag{4.8}$$

Solving the equation (4.8) for v, we arrive at (4.2). □

Proof of (iii). Setting $n = 1/7$ and using the fact that $J_n = J_{1/n}$ in (3.13), we arrive at $J_7 = \frac{15}{32}$. □

Proof of (iv). Fixing $n = 1/11$ and using the fact that $J_n = J_{1/n}$ in (3.14), we obtain that $J_{11} = 1$. □

Proof of (v). Putting $n = 1/13$ and using the fact that $J_n = J_{1/n}$ in (3.15), we find that

$$\begin{aligned} &v\left(64v^2 + 3720v - 5175\right)(8v+3)(v-30)(16v+33)\left(64v^2 - 24v + 9\right) \\ &\left(4096v^4 - 238080v^3 + 14169600v^2 + 19251000v + 26780625\right) \\ &\left(256v^2 - 528v + 1089\right)\left(v^2 + 30v + 900\right) = 0, \end{aligned} \tag{4.9}$$

where $v := J_{13}$. We observe that the second factor of (4.9) vanishes for specific value of q, whereas the other factors does not vanish. Hence, we deduce that

$$64v^2 + 3720v - 5175 = 0. \tag{4.10}$$

Solving the above equation for v, we obtain the required result. □

Proof of (vi). Setting $n = 3$ and using the value $J_3 = 0$ in (3.2), we find that

$$v^3\left(v^3 - 375\right)^3 = 0, \tag{4.11}$$

where $v := J_{27}$, since $v \neq 0$ and solving the second factor of the above equation (4.11), we obtain (4.6). □

Theorem 4.2. *We have*

$$\text{(i)}\ \ J_{63} = \frac{5}{32}\left(\frac{3}{4}(a_1+b_1)\right)^{\frac{1}{3}}, \tag{4.12}$$

$$\text{(ii)}\ \ J_{75} = 3\left(369830 - 165393\sqrt{5}\right)^{\frac{1}{3}}, \tag{4.13}$$

$$\textit{(iii)}\ \ J_{99} = \left(\frac{1147951079}{2} + \frac{199832633\sqrt{33}}{2}\right)^{\frac{1}{3}}, \tag{4.14}$$

$$\textit{(iv)}\ \ J_{147} = \left(531745995375 + 116036489250\sqrt{21}\right)^{\frac{1}{3}}, \tag{4.15}$$

$$\textit{(v)}\ \ J_{171} = \frac{1}{4}(a_2+b_2)^{\frac{1}{3}}, \tag{4.16}$$

where

$a_1 = 180040533 + 39288067\sqrt{21}$,

$b_1 = 273\sqrt{2\left(434925969567 + 94908627499\sqrt{21}\right)}$,

$a_2 = 21187806942033 + 2806393586997\sqrt{57}$,

and

$b_2 = 2\sqrt{\frac{448923163012861107298413933}{2} + \frac{59461325524651981512667761\sqrt{57}}{2}}$.

Proof of (i). Setting $n = 7$ and using $J_7 = \frac{15}{32}$ in (3.2), we get

$$\begin{aligned} &-1152921504606846976v^{12} + 237547991405386373529600 0v^9 \\ &+ 20730609033176678400000 0v^6 + 149371330340291328000000000v^3 \\ &+ 6256903954262253662109375 = 0, \end{aligned} \tag{4.17}$$

where $v := J_{63}$. Solving the above equation for v, we obtain (4.12). □

Proof of (ii). Putting $n = 3$ and using the value $J_3 = 0$ in (3.12), we find that

$$v^6 - 19970820v^3 + 4851495 = 0,$$

where $v := J_{75}$. Solving the above equation for v, we arrive at (4.13). □

Proof of (iii). Setting $n = 11$ and using the value $J_{11} = 1$ [3] in (3.2), we obtain that

$$\left(v^6 - 1147951079v^3 - 52313624\right)(v-1)^2\left(v^2+v+1\right)^2 = 0, \tag{4.18}$$

where $v := J_{99}$. We observe that the first factor of (4.18) vanishes for specific value of q, whereas the other factor does not vanish. Hence, we deduce that

$$v^6 - 1147951079v^3 - 52313624 = 0. \tag{4.19}$$

Solving the equation (4.19) for v, we obtain (4.14). □

Proof of (iv). Setting $n = 3$ and using the value $J_3 = 0$ in (3.13), we find that

$$v^2\left(v^6 - 1063491990750v^3 + 10576844578125\right) = 0, \tag{4.20}$$

where $v := J_{147}$. We observe that the second factor of (4.20) vanishes for specific value of q, whereas the other factor does not vanish. Hence, we deduce that

$$v^6 - 1063491990750v^3 + 10576844578125 = 0. \tag{4.21}$$

Solving the above equation for v, we arrive at (4.15). □

Proof of (v). Setting $n = 19$ and using the value $J_{19} = 3$ [3, Ch.34,p.310] in (3.2), we obtain

$$v^{12} - 21187806942033v^9 + 439680430611v^6 - 238502206323v^3 - 1137893184 = 0,$$

where $v := J_{19}$. Solving the above equation for v, we arrive at (4.16) □

Theorem 4.3.

$$\textit{(i)}\ \ J_{243} = 5\left(8704c_1 + \frac{72603983653110c_1}{1447318030184058288 01} + 151022371885959\right)^{\frac{1}{3}}, \tag{4.22}$$

$$\textit{(ii)}\ \ J_{275} = \frac{544127}{2} + 121667\sqrt{5} + \frac{\sqrt{592131660065 + 264809328784\sqrt{5}}}{2}, \tag{4.23}$$

where $c_1 = 5223567527018075142287271005853^{1/3}$.

Proof of (i). Putting $n = 27$ and using the equation (4.6) in (3.2), we find that

$$v(v^9 - 566333894572346 25v^6 - 3493071873703125v^3 - 94888341796875) = 0, \tag{4.24}$$

where $v := J_{243}$. since $v \neq 0$ and solving the above equation for v, we arrive at (4.22) □

Proof of (ii). Setting $n = 11$ and using the value of $J_{11} = 1$ in (3.12), we obtain that

$$(v^4 - 1088254v^3 + 16869271v^2 - 111059674v - 15119324)(v-1)^2 = 0,$$

where $v := J_{275}$. Since second factor of the above equation is not vanishes for specific value of q, therefore solving first factor for v, we arrive at (4.23). □

Theorem 4.4. *We have*

$$\text{(i)}\ \ J_{363} = 15\left(\frac{a_3}{4} + \frac{9\sqrt{2\,(b_3 + c_3)}}{4}\right)^{\frac{1}{3}}, \tag{4.25}$$

$$\textit{(ii)}\ \ J_{387} = \left(\frac{a_4}{4} + \frac{1}{2}\sqrt{\frac{b_4}{2} + \frac{c_4}{2}}\right)^{\frac{1}{3}}, \tag{4.26}$$

$$\textit{(iii)}\ \ J_{475} = \frac{127580541}{2} + 28527876\sqrt{5} + a_5, \tag{4.27}$$

$$\textit{(iv)}\ \ J_{603} = \left(\frac{a_6}{4} + \frac{1}{2}\sqrt{\frac{b_6}{2} + \frac{c_6}{2}}\right)^{\frac{1}{3}}, \tag{4.28}$$

where $a_3 = 893587548090400075 + 155553625762776261\sqrt{33}$,

$b_3 = 985800871731127224422562749215446 1$,

$c_3 = 171605904990038179720865933476463 5\sqrt{33}$,

$a_4 = 211345136395511928131 25 + 1860790168869410611875\sqrt{129}$,

$b_4 = 44666766678037540637472435599838341224120 3125$,

$c_4 = 39326895204313325954377906531132680150421875\sqrt{129}$,

$a_6 = 97331938812393474148072097625 + 686526563243390788085932537 5\sqrt{201}$,

$b_6 = 94735063129795071747526692894932777233525896625488200156 25$,

and

$c_6 = 66820961446688490903985502579209118976994974594054267187 5\sqrt{201}$.

Proof of (i). Setting $n = 3$ and using the value $J_3 = 0$ in (3.14), we find that

$$\begin{aligned} &v^{12} - 301585797480510025312 5v^9 + 425503995943090603042968 75v^6 \\ &+ 141217318671843024316406 25v^3 + 136810219174141992187 5000 = 0, \end{aligned} \tag{4.29}$$

where $v := J_{363}$. Solving the above equation for v, we obtain the value of J_{363}. □

Proof of (ii). Putting $n = 43$ and using the value $J_{43} = 30$ [3, Ch.34,p.310] in (3.2), we get that

$$\begin{aligned} &v^{12} - 2113451363955119281312 5v^9 - 1437692133256863328125v^6 \\ &- 2125796939077148437 5v^3 + 509603748046875000 = 0, \end{aligned} \tag{4.30}$$

where $v := J_{387}$. Solving the above equation for v, we arrive at required result. □

Proof of (iii). Putting $n = 19$ and using the value $J_{19} = 3$ [3, Ch.34,p.310] in (3.12), we get that

$$(v^4 - 255161082v^3 + 173131479v^2 - 84454758v - 59373324)(v-3)^2 = 0,$$

Where $v := J_{475}$. Since second factor of the above equation is not vanishes for specific value of q therefore solving first factor for v, we obtain that (4.27). □

Proof of (iv). Setting $n = 67$ and using the value $J_{67} = 30$ [3, Ch.34,p.311] in the equation (3.2), we find that

$$\begin{aligned} &v^{12} - 97331938812393474148072097625v^9 - 662973550176836375236814062 5v^6 \\ &- 969942430564957683808593 75v^3 + 407097618765144591796875000 = 0, \end{aligned} \tag{4.31}$$

where $v := J_{603}$. Solving the above equation for v, we obtain the required result. □

Theorem 4.5. *We have*

(i) $$J_{\frac{11}{9}} = -\left(\frac{11}{2}\right)^{\frac{1}{3}}\left(-104359189 + 18166603\sqrt{33}\right)^{\frac{1}{3}}, \tag{4.32}$$

(ii) $$J_{\frac{3}{25}} = 3\left(369830 - 165393\sqrt{5}\right)^{\frac{1}{3}}, \tag{4.33}$$

(iii) $$J_{\frac{11}{25}} = \frac{544127}{2} + 121667\sqrt{5} - \frac{\sqrt{11\left(53830150915 + 24073575344\sqrt{5}\right)}}{2}, \tag{4.34}$$

(iv) $$J_{\frac{3}{49}} = 15\left(157554369 - 34381182\sqrt{21}\right)^{\frac{1}{3}}. \tag{4.35}$$

Proof of (i). Setting $n = \frac{11}{9}$ and using the value $J_{11} = 1$ in (3.2), we find that

$$\left(u^6 - 1147951079u^3 - 52313624\right)(u-1)^2\left(u^2+u+1\right)^2 = 0, \tag{4.36}$$

where $u := J_{\frac{11}{9}}$. We observe that the first factor of (4.36) vanishes for specific value of q, whereas the other factor does not vanish. Hence, we deduce that

$$u^6 - 1147951079u^3 - 52313624 = 0.$$

Solving the above equation for u, we obtain (4.32). □

Proof of (ii). Putting $n = \frac{3}{25}$ and using the value of $J_3 = 0$ in (3.12), we find that

$$u^6 - 19970820u^3 + 4851495 = 0,$$

where $u := J_{\frac{3}{25}}$. Solving the above equation for u, we obtain (4.33). □

Proof of (iii). Setting $n = \frac{11}{25}$ and using the value $J_{11} = 1$ in (3.12), we find that

$$\left(u^4 - 1088254u^3 + 16869271u^2 - 111059674u - 15119324\right)(u-1)^2 = 0,$$

where $u := J_{\frac{11}{25}}$. Since second factor of the above equation doesn't vanish for specific value of q, hence solving first factor for u, we arrive at (4.34). □

Proof of (iv). Putting $n = \frac{3}{49}$ and using the value $J_3 = 0$ in (3.13), we obtain that

$$u^2\left(u^6 - 1063491990750u^3 + 10576844578125\right) = 0, \tag{4.37}$$

where $u := J_{\frac{3}{49}}$. Since $u \neq 0$ and solving second factor of the equation (4.37) for u, we arrive at (4.35). □

Theorem 4.6. *We have*

(i) $$\left[\left(13\sqrt{5}-25\right)^3 + 6^3\right] L_5^3 - \left(13\sqrt{5}-25\right)^3 R_5^2 = 0, \tag{4.38}$$

(ii) $$\left(5^3 + 4^3\right) L_7^3 - 5^3 R_7^2 = 0, \tag{4.39}$$

(iii) $$\left(m_{13}^3 + 2^3\right) L_{13}^3 - m_{13}^3 R_{13}^2 = 0, \tag{4.40}$$

where $m_{13} = -155 + 45\sqrt{13},$

(iv) $$\left(45\left(a_1+b_1\right) + 4^4\right) L_{63}^3 - 45\left(a_1+b_1\right) R_{63}^2 = 0, \tag{4.41}$$

(v) $$\left(m_{75} + 8^{-3}\right) L_{75}^3 - m_{75} R_{75}^2 = 0, \tag{4.42}$$

where $m_{75} = 369830 - 165393\sqrt{5},$

(vi) $$\left(8^3 m_{147} + 3^3\right) L_{147}^3 - 8^3 m_{147} R_{147} = 0, \tag{4.43}$$

where $m_{147} = 531745995375 + 116036489250\sqrt{21},$

(vii) $$\left[2^7\left(a_2+b_2\right) + 3^2\right] L_{171}^3 - \left[2^7\left(a_2+b_2\right)\right] R_{171}^2 = 0, \tag{4.44}$$

(viii) $$\left[40^3\left(a_3 + 9\sqrt{2\left(b_3+c_3\right)}\right) + 4\right] L_{363}^3 - 40^3\left(a_3 + 9\sqrt{2\left(b_3+c_3\right)}\right) R_{363}^2 = 0, \tag{4.45}$$

(ix) $$\left(m_{3/25} + 8^{-3}\right) L_{3/25}^3 - m_{3/25} R_{3/25}^2 = 0, \tag{4.46}$$

(x) $$\left[40^3 m_{3/49} + 1\right] L_{3/49}^3 - 40^3 m_{3/49} R_{3/49}^2. \tag{4.47}$$

where $m_{3/25} = 369830 - 165393\sqrt{5}$ *and* $m_{3/49} = \left(157554369 - 34381182\sqrt{21}\right).$

Proof. Substituting the values of J_n for $n = 3, 5, 7, 13, 63, 75, 147, 175, 363, 3/25$ and $3/49$ into the equation (1.19), we readily arrive at the equations (4.38)-(4.46). □

References

[1] George E. Andrews, B. C. Berndt, Ramanujan's Lost Notebook part II, Springer Science+Business Media, 2009.

[2] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.

[3] B. C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York, 1998.

[4] D. A. Cox, Primes of the form $x^2 + ny^2$.Wiley, New York, 1989.

[5] M. S. Mahadeva Naika, S. Chandankumar, Some new modular equations and their applications, Tamsui Oxf. J. Inf. Math. Sci., 28 (4) (2012), 437-469.

[6] M. S. Mahadeva Naika, S. Chandankumar, and B Hemanthkumar, New modular relations for cubic class invariants, Note Mat. 34(2) (2014), 75-89.

[7] M. S. Mahadeva Naika, S. Chandankumar, and M. Manjunatha, On Some new modular equations of degree 9 and their applications, Adv. Stud. Contemp. Math., 22(1) (2012), 99-115.

[8] M. S. Mahadeva Naika, S. Chandankumar, and N. P. Suman, Certain new modular identities for Ramanujan's Cubic continued fraction, New Zealand J. Math., 42 (2012), 157-175.

[9] M. S. Mahadeva Naika, D. S. Gireesh, and N. P. Suman, Some identities for Eisenstein series, (Preprint)

[10] S. Ramanujan, Notebooks. 2 vols., Tata Institute of Fundamental Research, Bombay, 1957.

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