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UNIQUENESS OF Q-SHIFT DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF ENTIRE FUNCTIONS

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Abstract In this paper, we consider the uniqueness problems of the q-shift difference-differential polynomial $[P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}]^{(k)}$, where $f(z)$ is a transcendental entire function with zero order, $P(z)$ is a nonzero polynomial of degree n, $d, s_j(j=1,\cdots,d)\in N_+$, $q_j\in\mathbb{C}\setminus\{0\}(j=1,\cdots,d)$ are constants, $c_j(c_j\neq 0, j=1,\cdots,d)$ are distinct constants. The results improve some results given by Zhang and Korhonen [14], Qi and Yang[10], Cao,Liu and Xu[3], Wang, Xu and Zhan [11].

1 Introduction

A meromorphic function $f(z)$ means meromorphic in the complex plane. If no poles occur, then $f(z)$ reduces to an entire function. We assume that the reader is familiar with the notations and the basic results of Nevanlinna theory of meromorphic functions [13]. For any nonconstant meromorphic function $f(z)$, we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite linear measure. In particular, we denote by $S_1(r,f)$ any quantity satisfying $S_1(r,f)=o(T(r,f))$ as $r\to\infty$ for all r on a set of logarithmic density 1.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and $a\in\mathbb{C}\bigcup\{\infty\}$. We define $\Theta(a,f)=1-\limsup\limits_{r\to\infty}\dfrac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}$. We say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities), provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. And if we do not consider the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities).

Definition 1.1.[8] Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\bigcup\{\infty\}$, we denote by $E_k(a,f)$ the set of all zeros of $f(z)-a$, where each zero of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a,f)=E_k(a,g)$, we say that $f(z)$ and $g(z)$ share the value a with weight k. Obviously, when $k=0$(resp.∞), $f(z)$ and $g(z)$ share the value a IM(resp.a CM).

Definition 1.2.[13] For $a\in\mathbb{C}\bigcup\{\infty\}$ and k is a positive integer or infinity. We denote by $\overline{N}_{(k}(r,\frac{1}{f-a})$ the counting function of the zeros of $f-a$ whose multiplicities are not less than k, where each zero is countied only once. Then

$$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+\cdots+\overline{N}_{(k}(r,\frac{1}{f-a}).$$

Clearly, $N_1(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})$.

Definition 1.3.[1] Suppose that f and g share 1 IM. We denote by $\overline{N}_L(r,\frac{1}{f-1})$ the reduced counting function of the zeros of $f-1$ whose multiplicities are greater than the zeros of $g-1$, where each zero is countied only once; similarly, we have $\overline{N}_L(r,\frac{1}{g-1})$. We denote by $N_{pq}(r,\frac{1}{f-1})$ the counting function of the zeros of $f-1$ and $g-1$ with multiplicity p and q respectively.

Recently, the difference variant of the Nevanlinna theory has been established independently in [2, 4, 6, 7]. With the development of difference analogue of Nevanlinna theory, many authors paid attention to the uniqueness of difference and difference operator analogs of Nevanlinna theory.

In [14], Zhang and Korhonen studied the uniqueness of q-difference polynomials of meromorphic functions and obtained the following theorems.

Theorem A.[14] Let f and g be two transcendental entire functions with zero order. Suppose that q is a nonzero constant and n is an integer satisfying $n \geq 4$. If $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share 1 CM, then $f \equiv tg$ for $t^{n+1} = 1$.

Theorem B.[[14] Let f and g be two transcendental entire functions with zero order. Suppose that q is a nonzero constant and n is an integer satisfying $n \geq 6$. If $f^n(z)(f(z)-1)f(qz)$ and $g^n(z)(g(z)-1)g(qz)$ share 1 CM, then $f \equiv g$.

In [10], Qi and Yang improved Theorem A,B and obtained the following theorems.

Theorem C.[10] Let f and g be two transcendental entire functions with zero order. Suppose that q is a nonzero constant and n is an integer satisfying $n \geq 12$. If $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share 1 IM, then $f = t_1 g$ or $fg = t_2$, for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem D.[10] Let f and g be two transcendental entire functions with zero order. Suppose that q is a nonzero constant and n is an integer satisfying $n \geq 16$. If $f^n(z)(f(z)-1)f(qz)$ and $g^n(z)(g(z)-1)g(qz)$ share 1 IM, then $f \equiv g$.

In [3], Cao et al. discussed the q-shift difference-differential polynomials and obtained the following theorems.

Theorem E.[3] Let f and g be two transcendental entire functions with zero order. Suppose that q is a nonzero constant and n is an integer satisfying $n \geq 2k+m+6$. If $[f^n(z)(f^m(z)-a)f(qz+c)]^{(k)}$ and $[g^n(z)(g^m(z)-a)g(qz+c)]^{(k)}$ share 1 CM, then $f \equiv tg$, where $t^{n+1} = t^m = 1$.

In [11], Wang et al. discussed the q-shift difference polynomials and obtained the following theorems.

Theorem F.[11] Let $f,\ g$ be two transcendental entire functions with zero order, $F(z) = P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}$ and $G(z) = P(g)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}$, where $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a nonzero polynomial of degree n, m_1 is the number of the simple zero of $P(z)$, m_2 is the number of multiple zeros of $P(z)$, $\Gamma_0 = m_1 + 2m_2$. Suppose that $n > max\{2(\Gamma_0 + 2d) - \lambda, \lambda\}$. If $F(z)$ and $G(z)$ share 1 CM, then one of the following cases holds:

(I) $f \equiv tg$ for a constant t such that $t^l = 1$, where $l = GCD\{\lambda_0 + \lambda, \lambda_1 + \lambda, \cdots, \lambda_n + \lambda\}$ and

$$\lambda_i = \begin{cases} i+1, & a_i \neq 0 \\ n+1, & a_i = 0 \end{cases} \quad i = 0, 1, \cdots, n.$$

(II) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(\omega_1, \omega_2) = P(\omega_1)\prod_{j=1}^{d} \omega_1(q_j z + c_j)^{s_j} - P(\omega_2)\prod_{j=1}^{d} \omega_2(q_j z + c_j)^{s_j}.$$

Theorem G.[11] Under the assumptions of theorem F, if

$$E_l(1; P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}) = E_l(1; P(g)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j})$$

and l, n, m, d are integers satisfying one of the following conditions:

(I) $l \geq 3, n > max\{2\Gamma_0 + 4d - \lambda, \lambda\}$;

(II) $l = 2, n > max\{2\Gamma_0 + 5d + m - \lambda - d\chi, \lambda\}$;

(III) $l = 1, n > max\{2\Gamma_0 + 6d + 2m - \lambda - 2d\chi, \lambda\}$;

(IV) $l = 0, n > max\{2\Gamma_0 + 7d + 3m - \lambda - 3d\chi, \lambda\}$.

Then the conclusions of Theorem F hold, where $\chi = min\{\Theta(0,f), \Theta(0,g)\}$.

In this paper, we assume $q_j \in \mathbb{C}\backslash\{0\}(j=1,\cdots,d)$ are constants, $c_j \in \mathbb{C}\backslash\{0\}(j=1,\cdots,d)$ are distinct constants, $n, d, s_j(j=1,\cdots,d) \in N_+$. $\lambda = s_1+\cdots+s_d$. Let

$$F(z) = P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j},\ G(z) = P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}. \tag{1.1}$$

where $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a nonzero polynomial of degree n, m_1 is the number of the simple zero of $P(z)$, m_2 is the number of multiple zeros of $P(z)$, d_1 is the number of elements of $A = \{s_j \mid s_j = 1, j = 1,\cdots,d\}$; d_2 is the number of elements of $B = \{s_j \mid s_j \geq 2, j = 1,\cdots,d\}$.

We consider the uniqueness problems of q-shift difference-differential polynomials $F^{(k)}(z)$ and obtain the following results, which improve the above theorems.

Theorem 1.1. *Let f and g be two transcendental entire functions with zero order. $F(z)$ and $G(z)$ are stated as in (1.1). Suppose that $n > 2m_1 + 2d_1 + (2k+2)(m_2+d_2) - \lambda$. If $F^{(k)}$ and $G^{(k)}$ share 1 CM, then one of the following cases holds:*

(I) $f \equiv tg$ for a constant t such that $t^l = 1$, where $l = GCD\{\lambda_0+\lambda, \lambda_1+\lambda, \cdots, \lambda_n+\lambda\}$ and

$$\lambda_i = \begin{cases} i, & a_i \neq 0 \\ n, & a_i = 0 \end{cases} \quad i = 0,1,\cdots,n.$$

(II) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(\omega_1,\omega_2) = P(\omega_1)\prod_{j=1}^{d}\omega_1(q_jz+c_j)^{s_j} - P(\omega_2)\prod_{j=1}^{d}\omega_2(q_jz+c_j)^{s_j}.$$

Remark 1.2. When $P(z) = z^n, d = 1, c_1 = 0, k = 0$, we know that $2m_1+2d_1+(2k+2)(m_2+d_2) - \lambda = 3$. Moreover, from $R(f,g) \equiv 0$, we have $f^n(z)f(qz) = g^n(z)g(qz)$. Proceeding similarly as the proof of Theorem 5.1 in [13], we get $f \equiv tg$. Therefore, Theorem 1.1 improves Theorem A.

Remark 1.3. When $P(z) = z^n(z-1), d = 1, c_1 = 0, k = 0$, we know that $2m_1 + 2d_1 + (2k+2)(m_2+d_2) - \lambda = 5$. Therefore, Theorem 1.1 improves Theorem B.

Remark 1.4. When $P(z) = z^n(z^m - a), d = 1$, we know that $2m_1+2d_1+(2k+2)(m_2+d_2)-\lambda = 2m+2k+3$. Therefore, Theorem 1.1 improves Theorem E.

Remark 1.5. Since $F^{(0)}(z) = F(z)$ and $2m_1+2d_1+2m_2+2d_2-\lambda$ is less than $max\{2(\Gamma_0 + 2d) - \lambda, \lambda\}$, Theorem 1.1 improves Theorem F.

Theorem 1.6. *Under the assumptions of Theorem 1.1, if*

$$E_l(1;F^{(k)}) = E_l(1;G^{(k)})$$

and l, n, m_1, m_2, d_1, d_2 are integers satisfying one of the following conditions:

(I) $l \geq 3, n > 2m_1 + 2d_1 + (2k+4)(m_2+d_2) - \lambda - (2k+4)d_2\chi$;

(II) $l = 2, n > 3m_1 + 3d_1 + (3k+5)(m_2+d_2) - \lambda - (3k+5)d_2\chi$;

(III) $l = 1, n > 4m_1 + 4d_1 + (4k+6)(m_2+d_2) - \lambda - (4k+6)d_2\chi$;

(IV) $l = 0, n > 5m_1 + (3k+5)d_1 + (5k+7)(m_2+d_2) - \lambda - [(3k+3)d_1 + (5k+7)d_2]\chi$.

Then the conclusions of Theorem 1.1 hold, where $\chi = min\{\Theta(0,f), \Theta(0,g)\}$.

Remark 1.7. Since $F^{(0)}(z) = F(z)$ and the lower bound of n in Theorem 1.6 is not larger than those of n in Theorem G respectively, Theorem 1.6 improves Theorem G.

Remark 1.8. When $P(z) = z^n, d = 1, c_1 = 0, k = 0$, we know that $5m_1 + (3k+5)d_1 + (5k+7)(m_2+d_2) - \lambda - [(3k+3)d_1 + (5k+7)d_2]\chi = 11 - 3\chi < 12$. Proceeding similarly as Remark 1.2, we get $f \equiv tg$. Therefore, Theorem 1.6 improves Theorem C.

Remark 1.9. When $P(z) = z^n(z-1), d = 1, c_1 = 0, k = 0$, we know that $5m_1 + (3k+5)d_1 + (5k+7)(m_2+d_2) - \lambda - [(3k+3)d_1 + (5k+7)d_2]\chi = 16 - 3\chi \leq 16$. Therefore, Theorem 1.6 improves Theorem D.

2 some lemmas

Next, we give some lemmas to prove the main results of this paper.

Lemma 2.1. [11] *Let f be a transcendental meromorphic function with zero order and q, c be two nonzero constants. Then*

$$N(r, f(qz+c)) = N(r, f(z)) + S_1(r, f), \quad \overline{N}(r, f(qz+c)) = \overline{N}(r, f(z)) + S_1(r, f),$$
$$N(r, \frac{1}{f(qz+c)}) = N(r, \frac{1}{f(z)}) + S_1(r, f), \quad \overline{N}(r, \frac{1}{f(qz+c)}) = \overline{N}(r, \frac{1}{f(z)}) + S_1(r, f),$$
$$T(r, f(qz+c)) = T(r, f(z)) + S_1(r, f).$$

Lemma 2.2. [11] *Let f be a transcendental entire function with zero order. $F(z)$ is defined as in (1.1). Then*

$$T(r, F(z)) = (n+\lambda)T(r, f) + S_1(r, f),$$

where $\lambda = s_1 + \cdots + s_d$.

Lemma 2.3. [12] *Let f be a nonconstant meromorphic function and k be an integer. Then*

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.4. [9] *Let f be a nonconstant meromorphic function and p, k be positive integers. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f),$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.5. [5] *Let f and g be two meromorphic functions and let l be a positive integer. If $E_l(1; f) = E_l(1; g)$, then one of the following cases hods:*

$$\begin{aligned}(I)T(r, f) + T(r, g) \leq{}& N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + N_2(r, f) + N_2(r, g)\\ &+ \overline{N}(r, \frac{1}{f-1}) + \overline{N}(r, \frac{1}{g-1}) - N_{11}(r, \frac{1}{f-1})\\ &+ \overline{N}_{(l+1}(r, \frac{1}{f-1}) + \overline{N}_{(l+1}(r, \frac{1}{g-1}) + S(r, f) + S(r, g);\end{aligned}$$

(II)$f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, where $a(\neq 0), b$ are two constants.

Lemma 2.6. [12] *Let f and g be two nonconstant meromorphic functions. If f and g share 1 IM, $H = \frac{f''}{f'} - 2\frac{f'}{f-1} - \frac{g''}{g'} + 2\frac{g'}{g-1} \not\equiv 0$. Then*

$$\begin{aligned}T(r, f) + T(r, g) \leq{}& 2(N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + N_2(r, f) + N_2(r, g))\\ &+ 3(\overline{N}(r, f) + \overline{N}(r, g) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g})) + S(r, f) + S(r, g).\end{aligned}$$

From the proof of case 2 in Theorem 1.3[11], we can get the following lemma.

Lemma 2.7. [11] *Let f and g be two transcendental entire functions with zero order. $F(z), G(z)$ are defined as in Theorem 1.1. If $F(z) \equiv G(z)$, then the conclusions of Theorem 1.1 hold.*

Lemma 2.8. *Let f and g be two transcendental entire functions with zero order. $F(z), G(z)$ are defined as in Theorem 1.1. Suppose that $n > 2(m_1+m_2+d_1+d_2)-\lambda-2(d_1+d_2)\chi$. If*

$$(F(z))^{(k)} \equiv (G(z))^{(k)},$$

then the conclusions of Theorem 1.1 hold, where $\chi = min\{\Theta(0,f), \Theta(0,g)\}$.

Proof. By $(F(z))^{(k)} = (G(z))^{(k)}$, we get $F(z) = G(z) + Q(z)$, where $Q(z)$ is a polynomial of degree at most $k-1$. If $Q(z) \not\equiv 0$, then

$$\frac{P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}}{Q(z)} = \frac{P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}}{Q(z)} + 1.$$

By the second fundamental theorem and Lemma 2.2, we deduce that

$$\begin{aligned}
(n+\lambda)T(r,f) &= T(r, \frac{P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}}{Q(z)}) + S(r,f)\\
&\le \overline{N}(r, \frac{P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}}{Q(z)}) + \overline{N}(r, \frac{Q(z)}{P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}})\\
&+ \overline{N}(r, \frac{Q(z)}{P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}}) + S(r,f)\\
&\le \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}}) + \overline{N}(r, \frac{1}{P(g)})\\
&+ \overline{N}(r, \frac{1}{\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}}) + S_1(r,f) + S_1(r,g)\\
&\le (m_1+m_2)[T(r,f)+T(r,g)]\\
&+ (d_1+d_2)[\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g})] + S_1(r,f) + S_1(r,g).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
(n+\lambda)T(r,g) &\le (m_1+m_2)[T(r,f)+T(r,g)]\\
&+ (d_1+d_2)[\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g})] + S_1(r,f) + S_1(r,g).
\end{aligned}$$

So

$$\begin{aligned}
(n+\lambda)[T(r,f)+T(r,g)] &\le 2(m_1+m_2)[T(r,f)+T(r,g)]\\
&+ 2(d_1+d_2)[\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g})] + S_1(r,f) + S_1(r,g).
\end{aligned}$$

which contradicts with the assumption that $n > 2(m_1+m_2+d_1+d_2)-\lambda-2(d_1+d_2)\chi$. Hence $Q(z) \equiv 0$. Then

$$P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j} = P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}.$$

By Lemma 2.7, we get the conclusions of Lemma 2.8. □

3 Proof of theorem

3.1 Proof of Theorem 1.1.
By Lemma 2.3, we have

$$T(r,F^{(k)})\le T(r,P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{s_j})+S(r,P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{s_j}).$$

By Lemma 2.2, we get $S(r,F^{(k)})=S(r,f)$, similarly $S(r,G^{(k)})=S(r,g), S_1(r,F^{(k)})=S_1(r,f), S_1(r,G^{(k)})=S_1(r,g)$.
Since $f,\ g$ are two transcendental entire functions with zero order, $F^{(k)}$ and $G^{(k)}$ share 1 CM, there exists a nonzero constant c such that

$$\frac{F^{(k)}-1}{G^{(k)}-1}=c.$$

Rewriting the above equation, we have

$$cG^{(k)}=F^{(k)}-1+c.$$

Assume that $c\ne 1$. By the second fundamental theorem and Lemma 2.4, we get

$$\begin{aligned}T(r,F^{(k)})&\le \overline{N}(r,F^{(k)})+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-1+c})+S_1(r,f)\\&\le \overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)\\&\le T(r,F^{(k)})-T(r,F)+N_{k+1}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g)\\&\le T(r,F^{(k)})-T(r,F)+N_{k+1}(r,\frac{1}{F})+N_{k+1}(r,\frac{1}{G})+S_1(r,f)+S_1(r,g).\end{aligned}$$

So

$$T(r,F)\le N_{k+1}(r,\frac{1}{F})+N_{k+1}(r,\frac{1}{G})+S_1(r,f)+S_1(r,g).$$

By the definitions of $F,\ G$ and Lemma 2.2, we have

$$(n+\lambda)T(r,f)\le [m_1+d_1+(k+1)(m_2+d_2)][T(r,f)+T(r,g)]+S_1(r,f)+S_1(r,g).$$

Similarly, we obtain

$$(n+\lambda)T(r,g)\le [m_1+d_1+(k+1)(m_2+d_2)][T(r,f)+T(r,g)]+S_1(r,f)+S_1(r,g).$$

Therefore

$$\begin{aligned}&(n+\lambda)[T(r,f)+T(r,g)]\\&\le 2[m_1+d_1+(k+1)(m_2+d_2)][T(r,f)+T(r,g)]+S_1(r,f)+S_1(r,g),\end{aligned}$$

which contradicts with the assumption that $n>2m_1+2d_1+(2k+2)(m_2+d_2)-\lambda$. Hence $F^{(k)}\equiv G^{(k)}$.
By Lemma 2.8, we can get the conclusions of Theorem 1.1.
This completes the proof of Theorem 1.1. □

3.2 Proof of Theorem 1.6.
Similarly as the proof of Theorem 1.1, we have $S(r,F^{(k)})=S(r,f), S(r,G^{(k)})=S(r,g), S_1(r,F^{(k)})=S_1(r,f), S_1(r,G^{(k)})=S_1(r,g)$.
By Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned}(n+\lambda)T(r,f)&=T(r,F)+S_1(r,f)\\&\le T(r,F^{(k)})-N_2(r,\frac{1}{F^{(k)}})+N_{k+2}(r,\frac{1}{F})+S_1(r,f).\end{aligned}\tag{3.1}$$

By Lemma 2.4, we get

$$\begin{aligned}N_2(r,\frac{1}{F^{(k)}}) &\leq N_{k+2}(r,\frac{1}{F})+S_1(r,f)\\ &\leq N_{k+2}(r,\frac{1}{P(f)})+N_{k+2}(r,\frac{1}{\prod_{j=1}^{d}f(q_jz+c_j)^{s_j}})+S_1(r,f)\\ &\leq [m_1+d_1+(k+2)m_2]T(r,f)+(k+2)d_2\overline{N}(r,\frac{1}{f})+S_1(r,f). \qquad (3.2)\end{aligned}$$

Similarly, we obtain

$$(n+\lambda)T(r,g)\leq T(r,G^{(k)})-N_2(r,\frac{1}{G^{(k)}})+N_{k+2}(r,\frac{1}{G})+S_1(r,g), \qquad (3.3)$$

$$\begin{aligned}N_2(r,\frac{1}{G^{(k)}}) &\leq N_{k+2}(r,\frac{1}{G})+S_1(r,g)\\ &\leq [m_1+d_1+(k+2)m_2]T(r,g)+(k+2)d_2\overline{N}(r,\frac{1}{g})+S_1(r,g), \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}\overline{N}(r,\frac{1}{F^{(k)}}) &\leq N_{k+1}(r,\frac{1}{F})+S_1(r,f)\\ &\leq [m_1+d_1+(k+1)m_2]T(r,f)+(k+1)d_2\overline{N}(r,\frac{1}{f})+S_1(r,f), \qquad (3.5)\end{aligned}$$

$$\begin{aligned}\overline{N}(r,\frac{1}{G^{(k)}}) &\leq N_{k+1}(r,\frac{1}{G})+S_1(r,g)\\ &\leq [m_1+d_1+(k+1)m_2]T(r,g)+(k+1)d_2\overline{N}(r,\frac{1}{g})+S_1(r,g). \qquad (3.6)\end{aligned}$$

Next, we shall prove the theorem under the following four various conditions that $l\geq 3,\ l=2,\ l=1$ and $l=0$ respectively.

(I) $l\geq 3$. Since

$$\begin{aligned}&\overline{N}(r,\frac{1}{F^{(k)}-1})+\overline{N}(r,\frac{1}{G^{(k)}-1})-N_{11}(r,\frac{1}{F^{(k)}-1})+\overline{N}_{(l+1}(r,\frac{1}{F^{(k)}-1})+\overline{N}_{(l+1}(r,\frac{1}{G^{(k)}-1})\\ &\leq \frac{1}{2}N(r,\frac{1}{F^{(k)}-1})+\frac{1}{2}N(r,\frac{1}{G^{(k)}-1})+S_1(r,f)+S_1(r,g)\\ &\leq \frac{1}{2}T(r,F^{(k)})+\frac{1}{2}T(r,G^{(k)})+S_1(r,f)+S_1(r,g). \qquad (3.7)\end{aligned}$$

We distinguish the following two cases to prove.

Case 1. Suppose that $F^{(k)},G^{(k)}$ satisfy Lemma 2.5(i). By (3.7), we have

$$\begin{aligned}T(r,F^{(k)})+T(r,G^{(k)}) &\leq N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_2(r,F^{(k)})+N_2(r,G^{(k)})\\ &\quad+\overline{N}(r,\frac{1}{F^{(k)}-1})+\overline{N}(r,\frac{1}{G^{(k)}-1})-N_{11}(r,\frac{1}{F^{(k)}-1})\\ &\quad+\overline{N}_{(l+1}(r,\frac{1}{F^{(k)}-1})+\overline{N}_{(l+1}(r,\frac{1}{G^{(k)}-1})+S_1(r,f)+S_1(r,g)\\ &\leq N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+\frac{1}{2}T(r,F^{(k)})\\ &\quad+\frac{1}{2}T(r,G^{(k)})+S_1(r,f)+S_1(r,g),\end{aligned}$$

which means,

$$T(r,F^{(k)})+T(r,G^{(k)})\le 2N_2(r,\frac{1}{F^{(k)}})+2N_2(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g). \tag{3.8}$$

From (3.1) and (3.3), we have

$$T(r,F^{(k)})+T(r,G^{(k)})\ge (n+\lambda)[T(r,f)+T(r,g)]+N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})$$
$$-N_{k+2}(r,\frac{1}{F})-N_{k+2}(r,\frac{1}{G})+S_1(r,f)+S_1(r,g). \tag{3.9}$$

By (3.2),(3.4),(3.8) and (3.9), we obtain

$$\begin{aligned}(n+\lambda)[T(r,f)+T(r,g)]&\le N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_{k+2}(r,\frac{1}{F})\\&+N_{k+2}(r,\frac{1}{G})+S_1(r,f)+S_1(r,g)\\&\le[2m_1+2d_1+(2k+4)m_2][T(r,f)+T(r,g)]\\&+(2k+4)d_2[\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})]+S_1(r,f)+S_1(r,g),\end{aligned}$$

which contradicts with the assumption that $n>2m_1+2d_1+(2k+4)(m_2+d_2)-\lambda-(2k+4)d_2\chi$.

Case 2. Suppose that $F^{(k)},G^{(k)}$ satisfy Lemma 2.5(ii), then

$$F^{(k)}=\frac{(b+1)G^{(k)}+(a-b-1)}{bG^{(k)}+(a-b)}, \tag{3.10}$$

where $a(\neq 0),b$ are two constants.
We now consider three subcases as follows.

Subcase 2.1. $b\neq 0,-1$.
If $a-b-1\neq 0$, then $\overline{N}(r,\frac{1}{F^{(k)}})=\overline{N}(r,\frac{1}{G^{(k)}+\frac{a-b-1}{b+1}})$.
Using the second fundamental theorem, by Lemma 2.4 and (3.4), (3.5), we have

$$\begin{aligned}T(r,G)&\le T(r,G^{(k)})+N_{k+2}(r,\frac{1}{G})-N_2(r,\frac{1}{G^{(k)}})+S_1(r,g)\\&\le\overline{N}(r,G^{(k)})+\overline{N}(r,\frac{1}{G^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}+\frac{a-b-1}{b+1}})\\&+N_{k+2}(r,\frac{1}{G})-N_2(r,\frac{1}{G^{(k)}})+S_1(r,g)\\&\le\overline{N}(r,\frac{1}{F^{(k)}})+N_{k+2}(r,\frac{1}{G})+S_1(r,f)+S_1(r,g)\\&\le[m_1+d_1+(k+1)m_2]T(r,f)+(k+1)d_2\overline{N}(r,\frac{1}{f})\\&+[m_1+d_1+(k+2)m_2]T(r,g)+(k+2)d_2\overline{N}(r,\frac{1}{g})+S_1(r,f)+S_1(r,g),\end{aligned}$$

which means

$$(n+\lambda)T(r,g)\le[m_1+d_1+(k+1)m_2]T(r,f)+(k+1)d_2\overline{N}(r,\frac{1}{f})$$
$$[m_1+d_1+(k+2)m_2]T(r,g)+(k+2)d_2\overline{N}(r,\frac{1}{g})+S_1(r,f)+S_1(r,g). \tag{3.11}$$

Similarly, we have

$$(n+\lambda)T(r,f) \le [m_1+d_1+(k+1)m_2]T(r,g)+(k+1)d_2\overline{N}(r,\frac{1}{g})$$
$$[m_1+d_1+(k+2)m_2]T(r,f)+(k+2)d_2\overline{N}(r,\frac{1}{f})+S_1(r,f)+S_1(r,g). \tag{3.12}$$

By (3.11) and (3.12), we get

$$(n+\lambda)[T(r,f)+T(r,g)] \le [2m_1+2d_1+(2k+3)m_2][T(r,f)+T(r,g)]$$
$$+(2k+3)d_2[\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})]+S_1(r,f)+S_1(r,g),$$

which contradicts with the assumption that $n > 2m_1+2d_1+(2k+4)(m_2+d_2)-\lambda-(2k+4)d_2\chi$. Hence $a-b-1=0$, from(3.10), we get

$$F^{(k)} = \frac{(b+1)G^{(k)}}{bG^{(k)}+1}.$$

Since f is an entire function, we have $\overline{N}(r,\frac{1}{G^{(k)}+\frac{1}{b}})=0$.
Using the same method as above, we get

$$(n+\lambda)T(r,g) \le T(r,G^{(k)})+N_{k+2}(r,\frac{1}{G})-N_2(r,\frac{1}{G^{(k)}})+S_1(r,g)$$
$$\le N_{k+2}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G^{(k)}+\frac{1}{b}})+S_1(r,g)$$
$$\le N_{k+2}(r,\frac{1}{G})+S_1(r,f)+S_1(r,g)$$
$$\le [m_1+d_1+(k+2)m_2]T(r,g)+(k+2)d_2\overline{N}(r,\frac{1}{g})+S_1(r,f)+S_1(r,g),$$

which contradicts with the assumption that $n > 2m_1+2d_1+(2k+4)(m_2+d_2)-\lambda-(2k+4)d_2\chi$.

Subcase 2.2. $b=0$.
From (3.10), we have

$$F^{(k)} = \frac{G^{(k)}+a-1}{a}.$$

If $a \neq 1$, then $\overline{N}(r,\frac{1}{F^{(k)}}) = \overline{N}(r,\frac{1}{G^{(k)}+a-1})$. Similarly, we can also get a contradiction. Then $a=1$, thus, we have $F^{(k)} \equiv G^{(k)}$. By Lemma 2.8, we get the conclusions of Theorem 1.6.

Subcase 2.3. $b=-1$.
From (3.10), we have

$$F^{(k)} = \frac{a}{a+1-G^{(k)}}.$$

If $a \neq -1$, then $\overline{N}(r,\frac{1}{G^{(k)}-(a+1)}) = \overline{N}(r,F^{(k)}) = 0$. Similarly, we can also get a contradiction. Then $a=-1$, thus, we have $F^{(k)}G^{(k)}=1$.
Since $f,\ g$ be transcendental entire functions, we get $F^{(k)}$ and $G^{(k)}$ have no zeros. Then $F^{(k)} = e^{s(z)}, G^{(k)} = e^{t(z)}$, where $s(z),t(z)$ are nonzero polynomials. Since the order of $f,\ g$ be zero, we get $s(z),t(z)$ are constants. So $F(z),G(z)$ be polynomials of degree at most $k-1$, which contradicts with the assumption that $f,\ g$ be transcendental entire functions.

(II) $l=2$. Since

$$\overline{N}(r,\frac{1}{F^{(k)}-1})+\overline{N}(r,\frac{1}{G^{(k)}-1})-N_{11}(r,\frac{1}{F^{(k)}-1})$$
$$\le \frac{1}{2}T(r,F^{(k)})+\frac{1}{2}T(r,G^{(k)})+S_1(r,f)+S_1(r,g), \tag{3.13}$$

and

$$\overline{N}_{(l+1}(r,\frac{1}{F^{(k)}-1})\leq\frac{1}{2}N(r,\frac{F^{(k)}}{F^{(k+1)}})=\frac{1}{2}N(r,\frac{F^{(k+1)}}{F^{(k)}})+S_1(r,f)$$
$$\leq\frac{1}{2}\overline{N}(r,\frac{1}{F^{(k)}})+S_1(r,f). \tag{3.14}$$

Similarly, we have

$$\overline{N}_{(l+1}(r,\frac{1}{G^{(k)}-1})\leq\frac{1}{2}\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,g). \tag{3.15}$$

We distinguish the following two cases to prove.

Case 1. Suppose that $F^{(k)},G^{(k)}$ satisfy Lemma 2.5(i). By (3.13),(3.14) and (3.15), we have

$$\begin{aligned}T(r,F^{(k)})+T(r,G^{(k)})\leq{}& N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_2(r,F^{(k)})+N_2(r,G^{(k)})\\&+\overline{N}(r,\frac{1}{F^{(k)}-1})+\overline{N}(r,\frac{1}{G^{(k)}-1})-N_{11}(r,\frac{1}{F^{(k)}-1})\\&+\overline{N}_{(l+1}(r,\frac{1}{F^{(k)}-1})+\overline{N}_{(l+1}(r,\frac{1}{G^{(k)}-1})+S_1(r,f)+S_1(r,g)\\&\leq N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+\frac{1}{2}T(r,F^{(k)})+\frac{1}{2}T(r,G^{(k)})\\&+\frac{1}{2}\overline{N}(r,\frac{1}{F^{(k)}})+\frac{1}{2}\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g),\end{aligned}$$

which means,

$$\begin{aligned}T(r,F^{(k)})+T(r,G^{(k)})\leq{}& 2N_2(r,\frac{1}{F^{(k)}})+2N_2(r,\frac{1}{G^{(k)}})\\&+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g).\end{aligned} \tag{3.16}$$

By (3.1)—(3.6) and (3.16), we obtain

$$\begin{aligned}(n+\lambda)[T(r,f)+T(r,g)]\leq{}& N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_{k+2}(r,\frac{1}{F})\\&+N_{k+2}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g)\\&\leq[3m_1+3d_1+(3k+5)m_2][T(r,f)+T(r,g)]\\&+(3k+5)d_2[\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})]+S_1(r,f)+S_1(r,g),\end{aligned}$$

which contradicts with the assumption that $n>3m_1+3d_1+(3k+5)(m_2+d_2)-\lambda-(3k+5)d_2\chi$.

Case 2. Suppose that $F^{(k)},G^{(k)}$ satisfy Lemma 2.5(ii), similar to the proof of Case 2 in (I), we get the conclusions of Theorem 1.6.

(III) $l=1$. Since

$$\begin{aligned}&\overline{N}(r,\frac{1}{F^{(k)}-1})+\overline{N}(r,\frac{1}{G^{(k)}-1})-N_{11}(r,\frac{1}{F^{(k)}-1})\\&\leq\frac{1}{2}N(r,F^{(k)})+\frac{1}{2}N(r,G^{(k)})+S_1(r,f)+S_1(r,g)\\&\leq\frac{1}{2}T(r,F^{(k)})+\frac{1}{2}T(r,G^{(k)})+S_1(r,f)+S_1(r,g),\end{aligned} \tag{3.17}$$

and

$$\overline{N}_{(2}(r,\frac{1}{F^{(k)}})\leq\overline{N}(r,\frac{1}{F^{(k)}})+S_1(r,f), \tag{3.18}$$

$$\overline{N}_{(2}(r,\frac{1}{F^{(k)}})\leq\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,g). \tag{3.19}$$

We distinguish the following two cases to prove.

Case 1. Suppose that $F^{(k)}, G^{(k)}$ satisfy Lemma 2.5(i). By (3.17),(3.18) and (3.19), we have

$$\begin{aligned} T(r,F^{(k)})+T(r,G^{(k)}) &\le N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_2(r,F^{(k)})+N_2(r,G^{(k)}) \\ &+\overline{N}(r,\frac{1}{F^{(k)}-1})+\overline{N}(r,\frac{1}{G^{(k)}-1})-N_{11}(r,\frac{1}{F^{(k)}-1}) \\ &+\overline{N}_{(l+1}(r,\frac{1}{F^{(k)}-1})+\overline{N}_{(l+1}(r,\frac{1}{G^{(k)}-1})+S_1(r,f)+S_1(r,g) \\ &\le N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+\frac{1}{2}T(r,F^{(k)})+\frac{1}{2}T(r,G^{(k)}) \\ &+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g), \end{aligned}$$

which means,

$$\begin{aligned} T(r,F^{(k)})+T(r,G^{(k)}) &\le 2N_2(r,\frac{1}{F^{(k)}})+2N_2(r,\frac{1}{G^{(k)}}) \\ &+2\overline{N}(r,\frac{1}{F^{(k)}})+2\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g). \end{aligned} \tag{3.20}$$

By (3.1)—(3.6) and (3.20), we obtain

$$\begin{aligned} (n+\lambda)[T(r,f)+T(r,g)] &\le N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_{k+2}(r,\frac{1}{F}) \\ &+N_{k+2}(r,\frac{1}{G})+2\overline{N}(r,\frac{1}{F^{(k)}})+2\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g) \\ &\le [4m_1+4d_1+(4k+6)m_2][T(r,f)+T(r,g)] \\ &+(4k+6)d_2][\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})]+S_1(r,f)+S_1(r,g), \end{aligned}$$

which contradicts with the assumption that $n>4m_1+4d_1+(4k+6)(m_2+d_2)-\lambda-(4k+6)d_2]\chi$.

Case 2. Suppose that $F^{(k)}, G^{(k)}$ satisfy Lemma 2.5(ii), similar to the proof of Case 2 in (I), we get the conclusions of Theorem 1.6.

(IV) $l=0$, that is $F^{(k)}, G^{(k)}$ share 1 IM. Suppose that $H=\frac{F^{(k+2)}}{F^{(k+1)}}-2\frac{F^{(k+1)}}{F^{(k)}-1}-\frac{G^{(k+2)}}{G^{(k+1)}}+2\frac{G^{(k+1)}}{G^{(k)}-1}\not\equiv 0$, by Lemma 2.6, we get

$$\begin{aligned} T(r,F^{(k)})+T(r,G^{(k)}) &\le 2(N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_2(r,F^{(k)})+N_2(r,G^{(k)})) \\ &+3(\overline{N}(r,F^{(k)})+\overline{N}(r,G^{(k)})+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{G^{(k)}})) \\ &+S_1(r,f)+S_1(r,g). \end{aligned} \tag{3.21}$$

By (3.1)—(3.4) and (3.21), we obtain

$$\begin{aligned} (n+\lambda)[T(r,f)+T(r,g)] &\le N_2(r,\frac{1}{F^{(k)}})+N_2(r,\frac{1}{G^{(k)}})+N_{k+2}(r,\frac{1}{F})+N_{k+2}(r,\frac{1}{G}) \\ &+3\overline{N}(r,\frac{1}{F^{(k)}})+3\overline{N}(r,\frac{1}{G^{(k)}})+S_1(r,f)+S_1(r,g) \\ &\le 2[m_1+d_1+(k+2)m_2][T(r,f)+T(r,g)] \\ &+(2k+4)d_2[\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})] \\ &+3[m_1+(k+1)m_2][T(r,f)+T(r,g)] \\ &+3(k+1)(d_1+d_2)[\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})]+S_1(r,f)+S_1(r,g), \end{aligned}$$

which contradicts with the assumption that $n > 5m_1 + (3k+5)d_1 + (5k+7)(m_2+d_2) - \lambda - [(3k+3)d_1 + (5k+7)d_2]\chi$, then $H \equiv 0$.
By integration for H twice, we can get (3.10).
Proceeding similarly as the proof of Case 2 in (I), we get the conclusions of Theorem 1.6.
Thus, the proof of Theorem 1.6 is completed. □

4 Remarks

In Theorem1.1 and Theorem1.6, we mainly discuss the q-shift difference-differential polynomial of entire functions. It is natural to propose the following question: What happens to Theorem1.1 and Theorem1.6 if f is meromorphic? In this paper, we get the result related to Theorem 1.1 as follows.

Theorem 4.1. *Let f and g be two transcendental meromorphic functions with zero order. $F(z)$ and $G(z)$ are defined as in Theorem 1.1. Suppose that $n > 2m_1 + (2k+2)m_2 + 3d_1 + (2k+3)d_2 + 1 - \lambda$. If $F^{(k)}$ and $G^{(k)}$ share $1, \infty$ CM, then $F^{(k)} = G^{(k)}$.*

Proof. Since $f,\ g$ are two transcendental meromorphic functions with zero order, $F^{(k)}$ and $G^{(k)}$ share $1, \infty$ CM, there exists a nonzero constant c such that

$$\frac{F^{(k)} - 1}{G^{(k)} - 1} = c.$$

Rewriting the above equation, we have

$$cG^{(k)} = F^{(k)} - 1 + c.$$

Assume that $c \neq 1$. Using the second fundamental theorem, by Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned}
T(r, F^{(k)}) &\le \overline{N}(r, F^{(k)}) + \overline{N}(r, \frac{1}{F^{(k)}}) + \overline{N}(r, \frac{1}{F^{(k)} - 1 + c}) + S_1(r, f) \\
&\le \overline{N}(r, F) + \overline{N}(r, \frac{1}{F^{(k)}}) + \overline{N}(r, \frac{1}{G^{(k)}}) + S_1(r, f) \\
&\le (d_1 + d_2 + 1)T(r, f) + T(r, F^{(k)}) - T(r, F) \\
&+ N_{k+1}(r, \frac{1}{F}) + N_{k+1}(r, \frac{1}{G}) + S_1(r, f) + S_1(r, g).
\end{aligned}$$

So

$$\begin{aligned}
(n+\lambda)T(r, f) &\le (d_1 + d_2 + 1)T(r, f) + \\
&[m_1 + d_1 + (k+1)(m_2 + d_2)][T(r, f) + T(r, g)] + S_1(r, f) + S_1(r, g).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
(n+\lambda)T(r, g) &\le (d_1 + d_2 + 1)T(r, g) + \\
&[m_1 + d_1 + (k+1)(m_2 + d_2)][T(r, f) + T(r, g)] + S_1(r, f) + S_1(r, g).
\end{aligned}$$

So

$$\begin{aligned}
&(n+\lambda)[T(r, f) + T(r, g)] \\
&\le [2m_1 + 3d_1 + (2k+2)m_2 + (2k+3)d_2 + 1][T(r, f) + T(r, g)] + S_1(r, f) + S_1(r, g),
\end{aligned}$$

which contradicts with the assumption that $n > 2m_1 + 3d_1 + (2k+2)m_2 + (2k+3)d_2 + 1 - \lambda$.
Then $c = 1$, thus, we have $F^{(k)} \equiv G^{(k)}$.
This completes the proof of Theorem 4.1. □

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