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ON $g(x)$-f-CLEAN RING

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Abstract Let R be an associative ring with identity, $C(R)$ denote the center of R and $g(x)$ be a polynomial in $C(R)[x]$. We introduce the new notion of $g(x)$-f-clean rings, as a generalization of $g(x)$-clean rings. R is called $g(x)$-f-clean if every element $r \in R$ can be written as $r = s + w$ with $g(s) = 0$ and w a full element of R. In this paper, we study some general properties of $g(x)$-f-clean rings.

1 Introduction

Through this paper, all rings are associative with identity. We denote the set of all invertible elements in R by $U(R)$, $C(R)$ the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. A ring R is called clean if for every element $r \in R$, $r = e+u$ with $e^2 = e$ and $u \in U(R)$ [8]. A ring R is called $g(x)$-clean if for every element $r \in R$, $r = s + u$ with $g(s) = 0$ and $u \in U(R)$ [3]. In [5, 11] Fan, Yang, Wang and Chen completely determined the relation between clean rings and $g(x)$-clean rings independently. It's clear that, $x(x-1)$-clean rings are precisely the clean rings. If V is a vector space of countable infinite dimension over a division ring D, Camillo and Simon [3] proved that $\mathrm{End}_D(V)$ is $g(x)$-clean provided that $g(x)$ has two distinct roots in $C(D)$. Moreover, this result has been extended as the following:

Theorem 1.1. *(see [9]) Let R be a ring, ${}_RM$ be a semisimple module over R and $C = C(R)$. If $g(x) \in (x-a)(x-b)C[x]$ where $a, b \in C$ and $b, b-a$ are both units in R, then $\mathrm{End}_R M$ is $g(x)$-clean.*

An element $x \in R$ is said to be full element if there exist $s, t \in R$ such that $sxt = 1$. The set of all full elements of a ring R will be denoted by $K(R)$. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$. In [7], Li and Feng introduced f-clean rings. A ring R is said to be f-clean if every element of R is the sum of an idempotent and full element. Clearly every clean ring is f-clean. We know that, the notion of purely infinite simple rings was introduced by Ara, Goodearl and Pardo [1]. A simple unital ring R is purely infinite in case that it is not a division ring and for each non-zero element $x \in R$, there exist element $z, t \in R$ such that $zxt = 1$. The class of purely infinite simple rings is quite large, one can find various examples in [1]. We do not know whether every purely infinite simple ring is a clean ring. But for any x in a purely infinite simple ring, we have $x = 0$ or $x \in K(R)$. Hence, every purely infinite simple ring is a f-clean ring.

In this paper, we continue this topic. Thus we define $g(x)$-f-clean rings and determine some general properties of these rings.

Throughout this paper all rings are assumed to be associative with identity and modules are unitary. $M_n(R)$ denotes the $n \times n$ matrix ring over the ring R. $T_n(R)$ stands for $n \times n$ upper triangular matrix ring. The notation $R^{n\times 1}$ always stands for the set

$$\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \dots, x_n \in R \right\},$$

which is an $(M_n(R), R)$-bimodule. The notation $R^{1\times n}$ stands for the set $\{(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, which is an $(R, M_n(R))$-bimodule.

2 Main Results

Firstly, we define and get some basic properties of $g(x)$-f-clean rings.

Definition 2.1. Let $g(x)$ be a polynomial in $C(R)[x]$. An element $r \in R$ is $g(x)$-f-clean if $r = s + w$ with $g(s) = 0$ and $w \in K(R)$. R is $g(x)$-f-clean if every element of R is $g(x)$-f-clean.

It is clear that, f-clean rings are exactly $(x^2 - x)$-f-clean rings. However, there are $g(x)$-f-clean rings which are not f-clean.
Let $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} \mid \gcd(p, n) = 1, p \text{ is prime } \}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ and C_3 be the cyclic group of order 3.

Example 2.2. Let R be a commutative local or commutative semiperfect ring with $2 \in U(R)$. By [11, Theorem 2.7], RC_3 is $(x^6 - 1)$-f-clean. In particular, $\mathbb{Z}_{(7)}C_3$ is a $(x^6 - 1)$-f-clean. Furthermore, by [5, Example 1], $\mathbb{Z}_{(7)}C_3$ is $(x^4 - x)$-f-clean. However, $\mathbb{Z}_{(7)}C_3$ is not f-clean.

We will investigate the equivalence of $g(x)$-f-cleanness and f-cleaness.

Theorem 2.3. *Let R be a ring, $g(x) = (x - a)(x - b) \in C(R)[x]$ with $a, b \in C(R)$ and $(b - a) \in U(R)$. Then R is f-clean if and only if R is $g(x)$-f-clean.*

Proof. $(\Rightarrow)$ Let $r \in R$. Since R is f-clean and $(b - a) \in U(R)$, $\frac{(r - a)}{b - a} = e + w$ where $e^2 = e \in R$ and $w \in K(R)$. Thus, $r = [e(b - a) + a] + w(b - a)$ where $w(b - a) \in k(R)$ by [7, Lemma 3.1]. Also

$$[e(b - a) + a - a][e(b - a) + a - b] = 0.$$

Hence, R is $(x-a)(x-b)$-f-clean. $(\Leftarrow)$ Let $r \in R$. Since R is $(x-a)(x-b)$-f-clean, $r(b-a)+a = s + w$ where $(s - a)(s - b) = 0$ and $w \in K(R)$. Thus, $r = \frac{s - a}{b - a} + \frac{w}{b - a}$ where $\frac{w}{b - a} \in K(R)$ by [7, Lemma 3.1]. Moreover

$$(\frac{s - a}{b - a})^2 = \frac{(s - a)(s - b + b - a)}{(b - a)^2} = \frac{(s - a)(b - a)}{(b - a)^2} = \frac{s - a}{b - a}.$$

Therefore R is f-clean. □

Theorem 2.4. *Let R be a $(x - a)(x - b)$-f-clean ring with $a, b \in C(R)$ and $b - a \in U(R)$. Then for any central idempotent e in R, eRe is $(x - ea)(x - eb)$-f-clean.*

Proof. By Theorem 2.3, R is f-clean. Therefore, eRe is f-clean by [7, Proposition 2.12]. Since $eb - ea \in U(eRe)$, then eRe is $(x - ea)(x - eb)$-f-clean by Theorem 2.3. □

A ring R is called left quasi-duo ring if every maximal left ideal of R is a two-sided ideal. Commutative rings, local rings, rings in which every non-unit has a power that is central are all belong to this class of rings [12]. A ring R is said to be Dedekind finite if $xy = 1$ always implies $yx = 1$ for any $x, y \in R$. A ring R is called abelian if all idempotents are central.

Proposition 2.5. *Let R be a left quasi-duo ring, then R is clean if and only if R is $g(x)$-f-clean.*

Proof. It's clear by [7, Theorem 2.9]. □

Corollary 2.6. *Every abelian $g(x)$-f-clean ring is $g(x)$-f-clean.*

Proof. Note that every abelian ring is Dedekind finite and so the proof is done by the proof of [7, Theorem 2.9]. □

Let R and S be rings and $\theta : C(R) \to C(S)$ be a ring homomorphism with $\theta(1) = 1$. For $g(x) = \sum a_i x^i \in C(R)[x]$, let $\theta'(g(x)) = \sum \theta(a_i)x^i \in C(S)[x]$. Then θ induces a map θ' from $C(R)[x]$ to $C(S)[x]$. If $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta'(g(x)) = g(x)$. Now, we have the following:

Theorem 2.7. *Let $\theta : R \to S$ be a ring epimorphism. If R is $g(x)$-f-clean, then S is $\theta'(g(x))$-f-clean.*

Proof. Let $g(x) = a_0 + a_1x + \cdots + a_nx^n \in C(R)[x]$. Then

$$\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + \cdots + \theta(a_n)x^n \in C(S)[x].$$

For any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since R is $g(x)$-f-clean, there exist $t \in R$ and $w \in K(R)$ such that $r = t + w$ with $g(t) = 0$. Then $s = \theta(r) = \theta(t) + \theta(w)$ with $\theta(w) \in K(S)$, $\theta'(g(x)) \mid_{x=\theta(t)} = 0$. Thus S is $\theta'(g(x))$-f-clean. □

Now by Theorem 2.7, the following holds:

Corollary 2.8. *If R is $g(x)$-f-clean, then for any ideal I of R, R/I is $\bar{g}(x)$-f-clean with $\bar{g}(x) \in C(R/I)[x]$.*

Corollary 2.9. *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is $g(x)$-f-clean if and only if R_i is $g(x)$-f-clean for each $i \in I$.*

Recall that for a ring R with a ring endomorphism $\alpha : R \to R$, the skew power series ring $R[[t;\alpha]]$ of R is the ring obtained by giving the formal power series ring over R with the new multiplication $tr = \alpha(r)t$ for all $r \in R$.

Corollary 2.10. *Let α be an endomorphism of R and $g(x) = f_0 + f_1x + \cdots + f_nx^n \in C(R[[t,\alpha]])[x]$ where $f_i = a_{0i} + a_{01}t + \cdots \in C(R[[t,\alpha]])$. If $R[[t,\alpha]]$ is a $g(x)$-f-clean ring then R is $a_{00} + a_{01}x + \cdots + a_{0n}x^n$-f-clean.*

Proposition 2.11. *Let α be an endomorphism of R. If R is $g(x)$-f-clean ring, then the skew power series ring $R[[t,\alpha]]$ of R is a $g(x)$-f-clean ring.*

Proof. For any $h = a_0 + a_1t + \cdots \in R[[t,\alpha]]$, write $a_0 = s_0 + w_0$ with $g(s_0) = 0$ and $w_0 \in k(R)$. Assume that $l_0w_0k_0 = 1$ for some $l_0, k_0 \in R$ and let $h' = h - s_0 = w_0 + a_1t + \cdots$. The equation $w = (l_0 + 0 + \cdots)h'(k_0 + 0 + \cdots) = 1 + l_0a_1\alpha(k_0)x + \cdots$ shows that $w \in U(R[[t,\alpha]])$, since $U(R[[t,\alpha]]) = \{a_0 + a_1x + \cdots \;\; a_0 \in U(R)\}$ without any assumption on the endomorphism α. Hence $h' \in k(R[[t,\alpha]])$ and $h = s_0 + h'$ with $g(s_0) = 0$. □

Corollary 2.12. *Let α be an endomorphism of R and $g(x) \in C(R)[x]$. Then R is $g(x)$-f-clean if and only if $R[[t,\alpha]]$ is $g(x)$-f-clean ring.*

Li and Feng [7] show that every (finite) matrix over a f-clean ring is f-clean. We recall that for a ring R, $C(M_n(R)) = \{aI_n | a \in C(R)\}$ where I_n is $n \times n$ identity matrix. Now we have the following:

Theorem 2.13. *If R is a $a_0 + a_1x + \cdots + a_mx^m$-f-clean ring, then $M_n(R)$ is $a_0I_n + a_1I_nx + \cdots + a_mI_nx^m$-f-clean ring for $n \geq 1$, where I_n is $n \times n$ identity matrix.*

Proof. Let $g(x) = a_0 + a_1x + \cdots + a_mx^m$ and R be $g(x)$-f-clean ring. Given any $r \in R$, we have some $l \in R$ and $w \in k(R)$ such that $r = l + w$. We write $swt = 1$ for some $s, t \in R$ and $g(l) = 0$. Assume that theorem holds for the matrix ring $M_k(R)$, $k \geq 1$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{k+1}(R)$$

with $a_{11} \in R, a_{12} \in R^{1\times k}, a_{21} \in R^{k\times 1}$ and $a_{22} \in M_k(R)$. We have $a_{11} = l + w$ with $g(l) = 0$ and $swt = 1$ for any $s, t \in R$. There also exist a matrix L and a full matrix W such that $a_{22} = a_{21}tsa_{12} = L + W$, $a_0I_k + a_1I_kL + \cdots + a_mI_kL^m = 0$ by hypothesis. We write $SWT = I_k$ for some $S, T \in M_k(R)$. Therefore, we have

$$A = diag(l, L) + \begin{pmatrix} w & a_{12} \\ a_{21} & W + a_{21}tsa_{12} \end{pmatrix}.$$

Obviously, $a_0I_{k+1} + a_1I_{k+1}diag(l, L) + \cdots + a_mI_{k+1}(diag(l, L))^m = 0$. Let

$$P = \begin{pmatrix} s & 0 \\ -Sa_{21}ts & S \end{pmatrix}, Q = \begin{pmatrix} t & -tsa_{12}T \\ 0 & T \end{pmatrix} \in M_{k+1}(R)$$

and the equation

$$P \begin{pmatrix} w & a_{12} \\ a_{21} & W + a_{21}tsa_{12} \end{pmatrix} Q = \begin{pmatrix} 1 & 0 \\ 0 & I_k \end{pmatrix} = I_{k+1}$$

shows that $\left(\begin{smallmatrix} w & a_{12} \\ a_{21} & W+a_{21}tsa_{12} \end{smallmatrix}\right)$ is a full matrix, hence A is $a_0 + a_1I_{k+1}x + \cdots + a_mI_{k+1}x^m$-f-clean, as desired. □

A Morita Context (A, B, V, W, ψ, ϕ) consists two rings A, B, two bimodules ${}_AV_B, {}_BW_A$ and a pair of bimodule homomorphisms $\psi : V \otimes_B W \to A, \phi : W \otimes_A V \to B$, such that $\psi(v\otimes w)v' = v\phi(w \otimes v')$, $\phi(w \otimes v)w' = w\psi(v \otimes w')$. we can form

$$M = \left\{ \begin{pmatrix} a & v \\ w & b \end{pmatrix} \mid a \in A, b \in B, v \in V, w \in W \right\}$$

and define a multiplication on M as follows:

$$\begin{pmatrix} a & v \\ w & b \end{pmatrix} \begin{pmatrix} a' & v' \\ w' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(v \otimes w') & av' + vb' \\ wa' + bw' & \phi(w \otimes v') + bb' \end{pmatrix}.$$

A routine check shows that, with this multiplication (and entry-wise addition), M becomes an associative ring. We call M a Morita Context ring. Obviously, the class of the rings of Morita Contexts includes all 2×2 matrix rings and all formal triangular matrix rings. Note that if $A = B$ then $C(M) = \{aI_2|a \in C(A)\}$ where I_2 is 2×2 matrix identity. Our concern here is the Morita Context rings with zero homomorphisms.

Theorem 2.14. *Let $M = \left(\begin{smallmatrix} A & V \\ W & A \end{smallmatrix}\right)$ be the Morita Context with $\psi, \phi = 0$. Then M is $a_0I_2 + a_1I_2x + \cdots + a_mI_2x^m$-f-clean if and only if A is $a_0 + a_1x + \cdots + a_mx^m$-f-clean.*

Proof. Let $g(x) = a_0 + a_1x + \cdots + a_mx^m$ and A is $g(x)$-f-clean. For any $r = \left(\begin{smallmatrix} a & v \\ w & b \end{smallmatrix}\right) \in M$, we have $a = l_1 + w_1$ and $b = l_2 + w_2$ with $g(l_1) = g(l_2) = 0$ and $w_1, w_2 \in K(A)$. Assume that $s_1w_1t_1 = 1$, $s_2w_2t_2 = 1$ for some $s_1, t_1, s_2, t_2 \in R$. Let $r = diag(l_1, l_2) + \left(\begin{smallmatrix} w_1 & v \\ w & w_2 \end{smallmatrix}\right) = diag(l_1, l_2) + W$. Obviously,

$$a_0I_2 + a_1I_2diag(l_1, l_2) + \cdots + a_mI_2(diag(l_1, l_2))^m = 0$$

and the equation

$$\begin{pmatrix} s_1 & 0 \\ -s_2wt_1s_1 & s_2 \end{pmatrix} \begin{pmatrix} w_1 & v \\ w & w_2 \end{pmatrix} \begin{pmatrix} t_1 & -t_1s_1vt_2 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

implies that W is a full matrix. Hence r is $g(x)$-f-clean, as required. Conversely, let $g'(x) = a_0I_2 + a_1I_2x + \cdots + a_mI_2x^m$ and M is $g'(x)$-f-clean. For any $r \in A$, we have $\left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right) = L + W$ where $L = \left(\begin{smallmatrix} a & v \\ w & b \end{smallmatrix}\right)$, $W = \left(\begin{smallmatrix} a' & v' \\ w' & b' \end{smallmatrix}\right)$, $g'(L) = 0$ and $W \in K(M)$. Assume that $SWT = I_2$ for some $S = \left(\begin{smallmatrix} a_1 & v_1 \\ w_1 & b_1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} a_2 & v_2 \\ w_2 & b_2 \end{smallmatrix}\right) \in M$. Therefore, $r = a + a'$ where $g(a) = 0$ and $a_1a'a_2 = 1$, i.e $a' \in K(A)$. Hence A is $g(x)$-f-clean as required.

Corollary 2.15. *For any $n \geq 1$, R is $a_0 + a_1 x + \cdots + a_m x^m$-f-clean ring if and only if the $n \times n$ upper triangular matrix ring $T_n(R)$ is $a_0 I_n + a_1 I_n x + \cdots + a_m I_n x^m$-f-clean.*

Proof. Let $E, A \in T_n(R)$. It is straightforward to calculate that $a_0 I_n + a_1 I_n E + \cdots + a_m I_n E^m = 0$ if and only if $a_0 + a_1 E_{ii} + \cdots + a_m E_{ii}^m = 0$ and $A \in K(T_n(R))$ if and only if $A_{ii} \in K(R)$. Hence the corollary is straightforward. □

Finally, we give a property which has related to $(ax^{2n} - bx)$-f-clean rings.

Proposition 2.16. *Let R be a ring and $n \in \mathbb{N}$. Then R is $(ax^{2n} - bx)$-f-clean if and only if R is $(ax^{2n} + bx)$-f-clean.*

Proof. Note that, for any $r \in R$, $-r = s + w$ with $as^{2n} - bs = 0$ and $w \in K(R)$ if and only if $r = (-s) + (-w)$ with $a(-s)^{2n} + b(-s) = 0$ and $-w \in K(R)$. Therefore the proof is complete. □

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