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GENERATING FUNCTIONS FOR CERTAIN BALANCING AND LUCAS-BALANCING NUMBERS

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Abstract It is well known that the generating function for any sequence $\{a_n\}$, denoted by the function $g(x)$ and defined by $g(x) = \sum_{n=0}^{\infty} a_n x^n$. This function is used to solve both homogenous and non-homogenous recurrence relations. In this study, we find generating function of certain balancing and Lucas-balancing numbers.

1 Introduction

Balancing numbers n and the balancers r are the solutions of the Diophantine equation $1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$. The square roots of $8n^2 + 1$ also generate a sequence of numbers called as Lucas-balancing numbers. The balancing numbers and the Lucas-balancing numbers satisfy the same recurrence relation with different initial values, that is, $B_{n+1} = 6B_n - B_{n-1};\ B_0 = 0,\ B_1 = 1$ and $C_{n+1} = 6C_n - C_{n-1};\ C_0 = 1,\ C_1 = 3$, where $n \geq 1$ and B_n and C_n are the n^{th} balancing and Lucas-balancing numbers respectively [1, 3]. The details of balancing and Lucas-balancing numbers are available in [1–19].

It is well known that for the sequence $a_0, a_1, \ldots$ of real numbers, the function $g(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ is called the generating function for the sequence $\{a_n\}$. Also, by letting $a_i = 0$ for $i > n$; $g(x) = \sum_{i=0}^{n} a_n x^n$ represents the generating function for the finite sequence $\{a_n\}$. In this study, authors main aim is to establish some generating functions of certain balancing and Lucas-balancing numbers.

Generating functions are used to solve both homogenous and non-homogenous recurrence relations. The following example show how generating function is used to solve recurrence relation for balancing numbers and derive the famous Binet's formula for these numbers.

Example 1.1. Use generating functions to solve the balancing recurrence relation $B_{n+1} = 6B_n - B_{n-1}$, where $B_1 = 1, B_2 = 6$.

Solution. Let $g(x) = B_0 + B_1 x + B_2 x^2 + \ldots + B_n x^n + \ldots$. be the generating function of the balancing sequence. Using the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$, we can find $6xg(x)$ and $x^2 g(x)$ as follows:

$$6xg(x) = 6B_0 x + 6B_1 x^2 + 6B_2 x^3 + \ldots + 6B_{n-1} x^n + \ldots.$$
$$x^2 g(x) = B_0 x^2 + B_1 x^3 + B_2 x^4 + \ldots + B_{n-2} x^n + \ldots,$$

which follows that

$$g(x) - 6xg(x) + x^2 g(x) = B_0 + (B_1 - 6B_0)x + (B_2 - 6B_1 + B_0)x^2 + \ldots = x,$$

and therefore, we have

$$g(x) = \frac{x}{(1 - 6x + x^2)} = \frac{1}{2\sqrt{8}}\left[\frac{1}{1 - \lambda_1 x} - \frac{1}{1 - \lambda_2 x}\right].$$

Which implies that

$$g(x) = \sum_{n=0}^{\infty} B_n x^n = \sum_{n=0}^{\infty} \frac{(\lambda_1^n - \lambda_2^n)x^n}{2\sqrt{8}}.$$

From this expression, it follows that

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

which is nothing but the Binet's formula for balancing numbers.

2 Generating functions for certain balancing and Lucas-balancing numbers

In this section, we establish generating functions of certain balancing and Lucas-balancing numbers.

2.1 Generating functions for B_{3n} and B_n^3

Let $g_1(x)$ be the generating function for B_{3n}. Then

$$\begin{aligned} g_1(x) &= B_0 + B_3 x + B_6 x^2 + \ldots + B_{3n} x^n + \ldots, \\ 4x g_1(x) &= 4x B_0 + 4B_3 x^2 + 4B_6 x^3 + \ldots + 4B_{3n-3} x^n + \ldots, \\ x^2 g_1(x) &= B_0 x^2 + B_3 x^3 + B_6 x^4 + \ldots + B_{3n-6} x^n + \ldots. \end{aligned}$$

Therefore, we have

$$(1 - 4x - x^2) g_1(x) = B_0 + (B_3 - 4B_0)x + (B_6 - 4B_3 - B_0)x^2 + \ldots,$$

which follows that

$$g_1(x) = \frac{35x + 6790x^2 + \ldots}{1 - 4x - x^2}.$$

Similarly, if $g_2(x)$ be the generating function for B_n^3, then we have

$$\begin{aligned} g_2(x) &= B_0^3 + B_1^3 x + B_2^3 x^2 + \ldots + B_n^3 x^n + \ldots, \\ 3x g_2(x) &= 3x B_0^3 + 3B_1^3 x^2 + 3B_2^3 x^3 + \ldots + 3B_{n-1}^3 x^n + \ldots, \\ 6x^2 g_2(x) &= 6B_0^3 x^2 + 6B_1^3 x^3 + 6B_2^3 x^4 + \ldots + 6B_{n-2}^3 x^n + \ldots, \\ 3x^3 g_2(x) &= 3B_0^3 x^3 + 3B_1^3 x^4 + \ldots + 3B_{n-3}^3 x^n + \ldots, \\ x^4 g_2(x) &= B_0^3 x^4 + B_1^3 x^5 + B_2^3 x^6 + \ldots + B_{n-4}^3 x^n + \ldots. \end{aligned}$$

Therefore, we have

$$(1 - 3x - 6x^2 + 3x^3 + x^4) g_2(x) = x + 213x^2 + 42221x^3 + \ldots.$$

Which follows that

$$g_2(x) = \frac{x + 213x^2 + 42221x^3 + \ldots}{(1 - 3x - 6x^2 + 3x^3 + x^4)}.$$

2.2 Generating function for B_{2n+1} and C_{2n+2}

Let $g_3(x)$ be the generating function for B_{2n+1}, then

$$\begin{aligned} g_3(x) &= B_1 + B_3 x + B_5 x^2 + \ldots + B_{2n+1} x^n + \ldots, \\ 3x g_3(x) &= 3x B_1 + 3B_3 x^2 + 3B_5 x^3 + \ldots + 3B_{2n-1} x^n + \ldots, \\ x^2 g_3(x) &= B_1 x^2 + B_3 x^3 + B_5 x^4 + \ldots + B_{2n-3} x^n + \ldots. \end{aligned}$$

Therefore, we have

$$(1-3x-x^2)g_3(x) = B_1+(B_3-3B_1)x+(B_5-3B_3-B_1)x^2+\ldots = 1+32x+1083x^2+\ldots,$$

which follows that

$$g_3(x) = \frac{1+32x+1083x^2+\ldots}{(1-3x-x^2)}.$$

In a similar manner, we can find the generating function $g_4(x)$ of C_{2n+2} as

$$g_4(x) = \frac{C_2+(C_4-3C_2)x+\ldots}{(1-3x+x^2)}.$$

2.3 Generating functions for B_{m+n} and C_{m+n}

The generating function of B_{m+n} is given by;

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{m+n}x^n &= \sum_{n=0}^{\infty} \frac{({\lambda_1}^{m+n}-{\lambda_2}^{m+n})x^n}{\lambda_1-\lambda_2} \\
&= \frac{1}{\lambda_1-\lambda_2}\left[\lambda_1^m\sum_{n=0}^{\infty}\lambda_1^n x^n - \lambda_2^m\sum_{n=0}^{\infty}\lambda_2^n x^n\right] \\
&= \frac{1}{\lambda_1-\lambda_2}\left[\frac{\lambda_1^m}{1-\lambda_1 x} - \frac{\lambda_2^m}{1-\lambda_2 x}\right] \\
&= \frac{B_m - B_{m-1}x}{x^2-6x+1}.
\end{aligned}$$

Likewise, it can be shown that

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{m+n}x^n &= \sum_{n=0}^{\infty} \frac{({\lambda_1}^{m+n}+{\lambda_2}^{m+n})x^n}{2} \\
&= \frac{1}{2}\left[\lambda_1^m\sum_{n=0}^{\infty}\lambda_1^n x^n + \lambda_2^m\sum_{n=0}^{\infty}\lambda_2^n x^n\right] \\
&= \frac{1}{2}\left[\frac{\lambda_1^m}{1-\lambda_1 x} + \frac{\lambda_2^m}{1-\lambda_2 x}\right] \\
&= \frac{C_m - C_{m-1}x}{x^2-6x+1}.
\end{aligned}$$

These two generating functions can be applied to derive identities. For example,

$$\sum_{n=0}^{\infty} B_{n+1}x^n = \frac{1}{D}, \sum_{n=0}^{\infty} B_{n-1}x^n = \frac{6x-1}{D} \ and \ \sum_{n=0}^{\infty} C_n x^n = \frac{1-3x}{D},$$

where $D = x^2-6x+1$. Since $2\left(\frac{1-3x}{D}\right) = \frac{1}{D} - \frac{6x-1}{D}$, we have

$$2\sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} B_{n+1}x^n - \sum_{n=0}^{\infty} B_{n-1}x^n = \sum_{n=0}^{\infty}(B_{n+1}-B_{n-1})x^n,$$

which implies that $B_{n+1} - B_{n-1} = 2C_n$. To prove the identity $B_m C_n - B_{m-1} C_{n-1} = C_{m+n-1}$, we proceed as follows:

$$\begin{aligned}\sum_{m=0}^{\infty}(B_m C_n - B_{m-1} C_{n-1}) x^m &= C_n \sum_{m=0}^{\infty} B_m x^m - C_{n-1} \sum_{m=0}^{\infty} B_{m-1} x^m \\ &= C_n \frac{x}{D} - C_{n-1} \frac{6x-1}{D} \\ &= \frac{C_{n-1} + (C_n - 6C_{n-1})x}{D} \\ &= \frac{C_{n-1} - C_{n-2}x}{D} \\ &= \sum_{m=0}^{\infty} C_{m+n-1} x^n,\end{aligned}$$

which follows that $B_m C_n - B_{m-1} C_{n-1} = C_{m+n-1}$.

3 Exponential generating functions

In this section, we develop the generating functions for $\frac{B_n}{n!}$ and $\frac{C_n}{n!}$. As $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, it follows that

$$e^{\lambda_1 x} = \sum_{n=0}^{\infty} \frac{{\lambda_1}^n x^n}{n!} \quad \text{and} \quad e^{\lambda_2 x} = \sum_{n=0}^{\infty} \frac{{\lambda_2}^n x^n}{n!}.$$

Therefore, we have

$$\frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} = \sum_{n=0}^{\infty} \frac{({\lambda_1}^n - {\lambda_2}^n)}{\lambda_1 - \lambda_2} \frac{x^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Thus, the exponential function $\frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2}$ generates the numbers $\frac{B_n}{n!}$. More generally, we can show that $\frac{e^{{\lambda_1}^k x} - e^{{\lambda_2}^k x}}{\lambda_1 - \lambda_2} = \sum_{n=0}^{\infty} B_{kn} \frac{x^n}{n!}$.

Likewise, the generating function $e^{\lambda_1 x} + e^{\lambda_2 x} = \sum_{n=0}^{\infty} 2C_n \frac{x^n}{n!}$ can derive the formula $e^{3x} \cosh(\sqrt{8}x) = \sum_{n=0}^{\infty} C_n \frac{x^n}{n!}$. Let us consider the functions $A(t)$ and $B(t)$ defined by $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ and $B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$, so that their products $A(t)B(t)$ and $A(t)B(-t)$ are given by

$$A(t).B(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!}, \tag{3.1}$$

and

$$A(t).B(-t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k b_{n-k} \right] \frac{t^n}{n!}.$$

In particular for $A(t) = \frac{e^{-6\lambda_1 t} - e^{-6\lambda_2 t}}{\lambda_1 - \lambda_2}$ and $B(t) = e^t$, we have

$$\frac{e^t(e^{-6\lambda_1 t} - e^{-6\lambda_2 t})}{\lambda_1 - \lambda_2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=0}^{\infty} B_n \frac{(-6t)^n}{n!}.$$

Which follows that

$$\frac{e^{(1-6\lambda_1)t}-e^{(1-6\lambda_2)t}}{\lambda_1-\lambda_2}=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k\right]\frac{t^n}{n!}.$$

Using the characteristic equation $\lambda^2=6\lambda-1$, we obtain

$$\frac{e^{-\lambda_1^2 t}-e^{-\lambda_2^2 t}}{\lambda_1-\lambda_2}=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k\right]\frac{t^n}{n!},$$

which implies that

$$\frac{1}{\lambda_1-\lambda_2}\left[\sum_{n=0}^{\infty}\frac{(-\lambda_1{}^2 t)^n}{n!}-\sum_{n=0}^{\infty}\frac{(-\lambda_2{}^2 t)^n}{n!}\right]=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k\right]\frac{t^n}{n!}.$$

That is,

$$\sum_{n=0}^{\infty}B_{2n}(-1)^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k\right]\frac{t^n}{n!}.$$

Equating the coefficients of $t^n/n!$, we have

$$B_{2n}(-1)^n=\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k.$$

Replacing $B(t)$ by e^{-t} and proceeding similarly, we get

$$\begin{aligned}\frac{1}{\lambda_1-\lambda_2}\sum_{n=0}^{\infty}\frac{(-2t)^n}{n!}\left[\sum_{n=0}^{\infty}\frac{(-\lambda_1{}^2 t)^n}{n!}-\sum_{n=0}^{\infty}\frac{(-\lambda_2{}^2 t)^n}{n!}\right]&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k\right](-1)^n\frac{t^n}{n!}\\ &\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}(-2)^k\right](-1)^n B_{2n}\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k\right](-1)^n\frac{t^n}{n!},\end{aligned}$$

which follows a new combinatorial identity

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^k B_{2n}=\sum_{k=0}^{n}\binom{n}{k}B_k(-6)^k.$$

4 Some hybrid identities containing both balancing and Lucas-balancing numbers

In this section, once again we consider the functions $A(t)$ and $B(t)$ to develop some hybrid identities that contain both balancing and Lucas- balancing numbers. Let $A(t)=\dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}$ and $B(t)=e^{\lambda_1 t}+e^{\lambda_2 t}$, where $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$. Then we have the following results.

Lemma 4.1. *If B_n and C_n respectively denote the n^{th} balancing and Lucas-balancing numbers, then* $\sum_{k=0}^{n}\binom{n}{k}B_kC_{n-k}=2^{n-1}B_n$.

Proof. Using (3.1), we have

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_kC_{n-k}\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\sum_{n=0}^{\infty}C_n\frac{t^n}{n!}$$
$$=\frac{e^{2\lambda_1 t}-e^{2\lambda_2 t}}{2(\lambda_1-\lambda_2)}$$
$$=\sum_{n=0}^{\infty}2^{n-1}B_n\frac{t^n}{n!},$$

and hence the result follows. □

Lemma 4.2. *If B_n and C_n respectively denote the n^{th} balancing and Lucas-balancing numbers, then* $\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}=2^{n-4}(C_n-3^n)$ *and* $\sum_{k=0}^{n}\binom{n}{k}C_kC_{n-k}=2^{n-1}(C_n+3^n)$.

Proof. Again using (3.1), we have

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}$$
$$=\frac{(e^{\lambda_1 t}-e^{\lambda_2 t})^2}{(\lambda_1-\lambda_2)^2}$$
$$=\sum_{n=0}^{\infty}2^{n-4}(C_n-3^n)\frac{t^n}{n!},$$

and the first result follows. The second result follows analogously. □

Lemma 4.3. *For n^{th} balancing number B_n and n^{th} Lucas-balancing number C_n, the following identities are valid:*

$$\sum_{k=0}^{n}\binom{n}{k}B_{mk}C_{mn-mk}=2^{n-1}B_{mn},$$
$$\sum_{k=0}^{n}\binom{n}{k}B_{mk}B_{mn-mk}=2^{n-4}(C_{mn}-C_m^n),$$
$$\sum_{k=0}^{n}\binom{n}{k}C_{mk}C_{mn-mk}=2^{n-1}(C_{mn}+C_m^n).$$

Proof. In view of (3.1), we have

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}B_{mk}C_{mn-mk}\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}B_{mn}\frac{t^n}{n!}\sum_{n=0}^{\infty}C_{mn}\frac{t^n}{n!}$$
$$=\frac{(e^{2\lambda_1{}^m t}-e^{2\lambda_2{}^m t})}{2(\lambda_1-\lambda_2)}$$
$$=\sum_{n=0}^{\infty}2^{n-1}B_{mn}\frac{t^n}{n!},$$

which follows the first result. The other results can be proved similarly. □

The following result can be obtained by using the differential operator d/dt. Since $A(t)=\sum_{n=0}^{\infty}(-6)^n a_n\frac{t^n}{n!}$, we have

$$\frac{d^r}{dt^r}A(t)=\sum_{n=0}^{\infty}(-6)^r a_{n+r}\frac{(-6t)^n}{n!}.$$

Setting $A(t) = \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}$ and $B(t) = e^{\lambda_1 t}$, then by using (3.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}(-6)^k B_{k+r}\right]\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\sum_{n=0}^{\infty} B_{n+r}\frac{(-6t)^n}{n!}\right]\\
&= \frac{e^t}{(-6)^r}\left[\frac{d^r}{dt^r}\sum_{n=0}^{\infty} B_n \frac{(-6t)^n}{n!}\right]\\
&= \frac{e^t}{(-6)^r}\frac{d^r}{dt^r}\left(\frac{e^{-6\lambda_1 t} - e^{-6\lambda_2 t}}{\lambda_1 - \lambda_2}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n B_{2n+r}\frac{t^n}{n!},
\end{aligned}$$

which yields the identity

$$\sum_{k=0}^{n}\binom{n}{k}(-6)^k B_{k+r} = (-1)^n B_{2n+r}.$$

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