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A REPRESENTATION FOR DIFFUSION EQUATION WITH DISCONTINUOUS COEFFICIENT

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Abstract In this paper, an integral representation for the solution of a diffusion equation with discontinuous coefficients which are important equations of Quantum physics, has been obtained. Existence of transformation operator for equation has been proved, some important properties of kernel function have been taken.

1 Introduction

Diffusion equation has a place in quantum physics. It originated from the problem of describing interactions between colliding particles in physics [8]. Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. Some aspects of direct or inverse spectral problems for second order differential pencils were studied in ([2], [3], [4], [6], [7], [9], [13], [14], [15]).

Transformation operators have been played important role in spectral theory of operators of mathematical physics. Marchenko ([10], [11], [12]) first applied the transformation operator to the solution of the inverse problem. They were also used in the fundamental paper of Gelfand and Levitan ([5]) .

This paper deal with diffusion equation with discontinous coefficient

$$-y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \rho(x)\, y,\ \ 0 \le x \le \pi \tag{1.1}$$

$$e(0,\lambda) = 1,\ e'(0,\lambda) = w_\nu \lambda \tag{1.2}$$

where

$$p(x) \in W_2^1(0,\pi),\, q(x) \in L_2(0,\pi), \tag{1.3}$$

λ is the spectral parameter, $w_\nu = (-1)^{\nu+1} i,\ \nu = 1,2$ and

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a \\ \alpha^2, & a < x \le \pi \end{cases},\ 0 < \alpha < 1. \tag{1.4}$$

In this work, we obtain the integral representation for solution of equation (1.1) with discontinous coefficient and investigate some properties for the kernel of the integral representation.

2 Representation For The Solution

The main result of this section is the following theorem proved by using method in [1].

Theorem 2.1. If conditions (1.3), (1.4) are fulfilled, then equation (1.1) has a unique solution $e_\upsilon(x,\lambda)$ satisfying initial condition (1.2) for all λ, such that

$$e_\upsilon(x,\lambda) = e_{\upsilon 0}(x,\lambda) + \int_{-\mu^+(x)}^{\mu^+(x)} K_\nu(x,t)\, e^{w_\nu \lambda t} dt \tag{2.1}$$

where

$$\mu^{\pm}(x) = \pm\sqrt{\rho(x)}x + a\left(1 \mp \sqrt{\rho(x)}\right), \quad \alpha^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\alpha}\right),$$

$$e_{\upsilon,0}(x,\lambda) = \begin{cases} R_{\nu,1}(x)\, e^{w_\nu \lambda x}, & 0 \le x \le a \\ \alpha^{+} R_{\nu,2}(x)\, e^{w_\nu \lambda(\alpha x - \alpha a + a)} + \alpha^{-} R_{\nu,3}(x)\, e^{w_\nu \lambda(-\alpha x + \alpha a + a)}, & a < x \le \pi. \end{cases}$$

Moreover, the function $K_\nu(x,\cdot) \in L_1(-\mu^{+}(x), \mu^{+}(x))$, the function $K_\nu(x,t)$ is countinous and has partial derivatives $\frac{\partial K_\nu(x,t)}{\partial x}$, $\frac{\partial K_\nu(x,t)}{\partial t}$ for $t \ne \mu^{-}(x)$ and it processes the following properties:

$i)$ $\int\limits_{-\mu^{+}(x)}^{\mu^{+}(x)} |K_\nu(x,t)|\, dt \le C\left(e^{\int\limits_0^{\mu^{+}(x)}[(x-t)|q(t)|+2|p(t)|]dt} - 1\right),\ 0 < C$-constant

$ii)$ $\frac{d}{dx}K_\nu(x,\mu^{+}(x)) + \frac{w_\nu}{\sqrt{\rho(x)}}p(x)K_\nu(x,\mu^{+}(x))$

$= \frac{1}{4\sqrt{\rho(x)}}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)\left[\frac{w_\nu}{\sqrt{\rho(x)}}p'(x) + q(x) + \frac{1}{\rho(x)}p^2(x)\right] e^{\frac{w_\nu}{\sqrt{\rho(x)}}\int\limits_0^x p(t)dt}$

$iii)$ $\frac{d}{dx}\left\{K_\nu(x,\mu^{-}(x)-0) - K_\nu(x,\mu^{-}(x)+0)\right\} -$

$-\frac{w_\nu}{\sqrt{\rho(x)}}p(x)\left\{K_\nu(x,\mu^{-}(x)-0) - K_\nu(x,\mu^{-}(x)+0)\right\}$

$= \frac{1}{4\sqrt{\rho(x)}}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)\left[-\frac{w_\nu}{\sqrt{\rho(x)}}p'(x) + q(x) + \frac{1}{\rho(x)}p^2(x)\right] e^{-\frac{w_\nu}{\sqrt{\rho(x)}}\int\limits_0^x p(t)dt}$

$iv)$ $\frac{d}{dx}K_\nu(x,-\mu^{+}(x)) - \frac{w_\nu}{2\sqrt{\rho(x)}}p(x)K_\nu(x,-\mu^{+}(x)) = 0$

$v)$ for $0 \le x \le a$;

$$R_{\nu,1}(x) = 1 - i\int\limits_0^x p(t)R_{\nu,1}(t)\, dt$$

and for $a < x \le \pi$;

$$R_{\nu,2}(x) = 1 - i\int\limits_0^a p(t)R_{\nu,1}(t)\, dt - \frac{i}{\alpha}\int\limits_a^x p(t)R_{\nu,2}(t)\, dt$$

$$R_{\nu,2}(x) = 1 - i\int\limits_0^a p(t)R_{\nu,1}(t)\, dt + \frac{i}{\alpha}\int\limits_a^x p(t)R_{\nu,3}(t)\, dt$$

$vi)$ If $p(x) \in W_2^1(0,\pi)$ and $q(x) \in W_2^0(0,\pi)$ then the function $K_\nu(x,t)$ satisfies the following equation almost everywhere:

$$\frac{\partial^2 K_\nu(x,t)}{\partial x^2} - \rho(x)\frac{\partial^2 K_\nu(x,t)}{\partial t^2} - 2w_\nu p(x)\frac{\partial K_\nu(x,t)}{\partial t} - q(x)K_\nu(x,t) = 0$$

Consider an integral equation for the solutions $e_\upsilon(x,\lambda)$,

$$e_\upsilon(x,\lambda) = e_{\upsilon,0}(x,\lambda) + \int\limits_0^x \Phi(x,t,\lambda)\left(2\lambda p(t) + q(t)\right) e_\upsilon(t,\lambda)\, dt, \tag{2.2}$$

where $\Phi(x,t,\lambda) = \Phi_\nu(x,t,\lambda)$ and

$$\Phi_\nu(x,t,\lambda) = (-1)^{\nu+1}\left[s_{\nu,0}(x,\lambda)\, c_{\nu,0}(t,\lambda) - s_{\nu,0}(t,\lambda)\, c_{\nu,0}(x,\lambda)\right]$$

for $\nu = 1, 2$.

It is clear from (1.2) that the solutions $s_{\nu,0}(x,\lambda)$ and $c_{\nu,0}(x,\lambda)$ satisfy the initial conditions $s_{\nu,0}(0,\lambda) = c'_{\nu,0}(0,\lambda) = 0$, $s'_{\nu,0}(0,\lambda) = (-1)^{\nu+1}$, $c_{\nu,0}(0,\lambda) = 1$ while $p(x) \equiv q(x) \equiv 0$.

The functions $s_{\nu,0}(x,\lambda)$ and $c_{\nu,0}(x,\lambda)$ satisfing the following equations.

$$c_{\nu,0}(x,\lambda) = \\ = \begin{cases} \cos \lambda x, & 0 \le x \le a \\ \alpha^+ \cos \lambda(\alpha x - \alpha a + a) + \alpha^- \cos \lambda(-\alpha x + \alpha a + a), & a < x \le \pi \end{cases}$$

$$s_{\nu,0}(x,\lambda) = \\ = \begin{cases} (-1)^{\nu+1} \dfrac{\sin \lambda x}{\lambda}, & 0 \le x \le a \\ (-1)^{\nu+1}\left[\alpha^+ \dfrac{\sin \lambda(\alpha x - \alpha a + a)}{\lambda} + \alpha^- \dfrac{\sin \lambda(-\alpha x + \alpha a + a)}{\lambda}\right], & a < x \le \pi \end{cases}$$

or

$$s_{\nu,0}(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin \lambda\mu^+(x)}{\lambda} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin \lambda\mu^-(x)}{\lambda},$$

$$c_{\nu,0}(x,\lambda) = (-1)^{\nu+1}\left[\frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)\cos \lambda\mu^+(x) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)\cos \lambda\mu^-(x)\right]$$

Then the function $\Phi(x,t,\lambda)$ has been identified as follows:

$$\Phi(x,t,\lambda) = \\ = \begin{cases} \dfrac{\sin \lambda(x-t)}{\lambda}, & t \le x \le a \\ \alpha^+ \dfrac{\sin \lambda[\alpha x - \alpha a + a - t]}{\lambda} + \alpha^- \dfrac{\sin \lambda[-\alpha x + \alpha a + a - t]}{\lambda}, & t \le a < x \\ \dfrac{\sin \lambda\alpha(x-t)}{\lambda\alpha}, & a < t \le x \end{cases}$$

From the equation (2.1) and (2.2), we obtain the following integral equation for the kernel $K_\nu(x,t)$ (for the simplicity it's assumed that $\alpha x - \alpha a - a < 0$)

if $0 \le x \le a$, $|t| < x$;

$$K_\nu(x,t) = \frac{w_\nu}{2} p\left(\frac{x+t}{2}\right) R_{\nu,1}\left(\frac{x+t}{2}\right) + \frac{1}{2}\int_0^{(x+t)/2} q(u)\, R_{\nu,1}(u)\, du$$

$$-w_\nu \int_{(x-t)/2}^{x} p(u)\, K_\nu(u, t-x+u)\, du + w_\nu \int_{(x+t)/2}^{x} p(u)\, K_\nu(u, t+x-u)\, du$$

$$+\frac{1}{2}\int_0^x q(u) \int_{t-x+u}^{t+x-u} K_\nu(u,\xi)\, d\xi du.$$

if $a < x \le \pi$, $-\alpha x + \alpha a - a < t < \alpha x - \alpha a - a$;

$$K_\nu(x,t) = \frac{w_\nu \alpha^+}{2} p\left(\frac{t+\alpha x-\alpha a+a}{2}\right) R_{\nu,1}\left(\frac{t+\alpha x-\alpha a+a}{2}\right)$$

$$+\frac{\alpha^+}{2}\int_0^{(t+\alpha x-\alpha a+a)/2} q(u) R_{\nu,1}(u)\,du$$

$$+w_\nu\alpha^+\int_{(t+\alpha x-\alpha a+a)/2}^{a} p(u) K_\nu(u,t+\alpha x-\alpha a+a-u)\,du$$

$$-w_\nu\alpha^-\int_{(-t-\alpha x+\alpha a+a)/2}^{a} p(u) K_\nu(u,t+\alpha x-\alpha a-a+u)\,du$$

$$+\frac{\alpha^+}{2}\int_0^a q(u)\int_{t-\alpha x+\alpha a-a+u}^{t+\alpha x-\alpha a+a-u} K_\nu(u,\xi)\,d\xi du+\frac{\alpha^-}{2}\int_0^a q(u)\int_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u} K_\nu(u,\xi)\,d\xi du$$

$$-\frac{w_\nu}{\alpha}\int_{(-t+\alpha x+\alpha a-a)/2\alpha}^{x} p(u) K_\nu(u,t-\alpha x+\alpha u)\,du+\frac{w_\nu}{\alpha}\int_a^x p(u) K_\nu(u,t+\alpha x-\alpha u)\,du$$

$$+\frac{1}{2\alpha}\int_a^x q(u)\int_{t-\alpha x+\alpha u}^{t+\alpha x-\alpha u} K_\nu(u,\xi)\,d\xi du.$$

if $a < x \le \pi$, $\alpha x - \alpha a - a < t < -\alpha x + \alpha a + a$;

$$K_\nu(x,t) = \frac{w_\nu \alpha^+}{2} p\left(\frac{t+\alpha x-\alpha a+a}{2}\right) R_{\nu,1}\left(\frac{t+\alpha x-\alpha a+a}{2}\right)$$

$$+\frac{w_\nu \alpha^-}{2} p\left(\frac{t-\alpha x+\alpha a+a}{2}\right) R_{\nu,1}\left(\frac{t-\alpha x+\alpha a+a}{2}\right)$$

$$+\frac{\alpha^+}{2}\int_0^{(t+\alpha x-\alpha a+a)/2} q(u) R_{\nu,1}(u)\,du+\frac{\alpha^-}{2}\int_0^{(t-\alpha x+\alpha a+a)/2} q(u) R_{\nu,1}(u)\,du$$

$$-w_\nu\alpha^+\int_{(-t+\alpha x-\alpha a+a)/2}^{a} p(u) K_\nu(u,t-\alpha x+\alpha a-a+u)\,du$$

$$+w_\nu\alpha^+\int_{(t+\alpha x-\alpha a+a)/2}^{a} p(u) K_\nu(u,t+\alpha x-\alpha a+a-u)\,du$$

$$-\frac{w_\nu}{\alpha}\int_a^x p(u) K_\nu(u,t-\alpha x+\alpha u)\,du$$

$$-w_\nu\alpha^-\int_{(-t-\alpha x+\alpha a+a)/2}^{a} p(u) K_\nu(u,t+\alpha x-\alpha a-a+u)\,du$$

$$+w_\nu\alpha^-\int_{(t-\alpha x+\alpha a+a)/2}^{a} p(u) K_\nu(u,t-\alpha x+\alpha a+a-u)\,du$$

$$+\frac{\alpha^-}{2}\int_0^a q(u)\int_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u} K_\nu(u,\xi)\,d\xi du+\frac{\alpha^+}{2}\int_0^a q(u)\int_{t-\alpha x+\alpha a-a+u}^{t+\alpha x-\alpha a+a-u} K_\nu(u,\xi)\,d\xi du$$

$$+\frac{w_\nu}{\alpha}\int_a^x p(u)\,K_\nu(u,t+\alpha x-\alpha u)\,du+\frac{1}{2\alpha}\int_a^x q(u)\int_{t-\alpha x+\alpha u}^{t+\alpha x-\alpha u} K_\nu(u,\xi)\,d\xi du.$$

if $a<x\le\pi$, $-\alpha x+\alpha a+a<t<\alpha x-\alpha a+a$

$$K_\nu(x,t)=\frac{\alpha^+}{2}\int_0^a q(u)\,R_{\nu,1}(u)\,du+\frac{w_\nu\alpha^-}{2}p\left(\frac{t-\alpha x+\alpha a+a}{2}\right)R_{\nu,1}\left(\frac{t-\alpha x+\alpha a+a}{2}\right)$$

$$+\frac{w_\nu\alpha^+}{2\alpha^2}p\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)R_{\nu,2}\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)-\frac{\alpha^-}{2}\int_0^a q(u)\,R_{\nu,1}(u)\,du$$

$$-\frac{w_\nu\alpha^-}{2\alpha^2}p\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)R_{\nu,3}\left(\frac{-t+\alpha x+\alpha a+a}{2\alpha}\right)$$

$$+\frac{\alpha^+}{2\alpha}\int_a^{(t+\alpha x+\alpha a-a)/2\alpha} q(u)\,R_{\nu,2}(u)\,du+\frac{\alpha^-}{2\alpha}\int_a^{(-t+\alpha x+\alpha a+a)/2\alpha} q(u)\,R_{\nu,3}(u)\,du$$

$$-w_\nu\alpha^+\int_{(-t+\alpha x-\alpha a+a)/2}^{a} p(u)\,K_\nu(u,t-\alpha x+\alpha a-a+u)\,du$$

$$+\frac{\alpha^+}{2}\int_0^a q(u)\int_{t-\alpha x+\alpha a-a+u}^{t+\alpha x-\alpha a+a-u} K_\nu(u,\xi)\,d\xi du-\frac{\alpha^-}{2}\int_{(t-\alpha x+\alpha a+a)/2}^{a} q(u)\,R_{\nu,1}(u)\,du$$

$$+w_\nu\alpha^-\int_{(t-\alpha x+\alpha a+a)/2}^{a} p(u)\,K_\nu(u,t-\alpha x+\alpha a+a-u)\,du$$

$$+\frac{\alpha^-}{2}\int_0^a q(u)\int_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u} K_\nu(u,\xi)\,d\xi du-\frac{w_\nu}{\alpha}\int_a^x p(u)\,K_\nu(u,t-\alpha x+\alpha u)\,du$$

$$+\frac{w_\nu}{\alpha}\int_{(t+\alpha x+\alpha a-a)/2\alpha}^{x} p(u)\,K_\nu(u,t+\alpha x-\alpha u)\,du$$

$$+\frac{1}{2\alpha}\int_a^x q(u)\int_{t-\alpha x+\alpha u}^{t+\alpha x-\alpha u} K_\nu(u,\xi)\,d\xi du.$$

By the method of successive approximations,
in case of $0\le x\le a,\ |t|<x$;

$$K_\nu^{(0)}(x,t)=\frac{w_\nu}{2}p\left(\frac{x+t}{2}\right)R_{\nu,1}\left(\frac{x+t}{2}\right)+\frac{1}{2}\int_0^{(x+t)/2} q(u)\,R_{\nu,1}(u)\,du,$$

$$K_\nu^{(n)}(x,t)=-w_\nu\int_{(x-t)/2}^{x} p(u)\,K_\nu^{(n-1)}(u,t-x+u)\,du$$

$$+w_{\nu}\int\limits_{(x+t)/2}^{x} p\left(u\right) K_{\nu}^{(n-1)}\left(u,t+x-u\right) du+\frac{1}{2}\int\limits_{0}^{x} q\left(u\right) \int\limits_{t-x+u}^{t+x-u} K_{\nu}^{(n-1)}\left(u,\xi\right) d\xi du,$$

we get

$$\int\limits_{-x}^{x}\left|K_{\nu}^{(0)}\left(x,t\right)\right| dt\leq\sigma\left(x\right),$$

where

$$\sigma\left(x\right)=\int\limits_{0}^{\mu^{+}(x)}\left[\left(x-t\right)\left|q\left(t\right)\right|+2\left|p\left(t\right)\right|\right] dt.$$

By the mathematical induction method, one can easily verify that the inequality

$$\int\limits_{-x}^{x}\left|K_{\nu}^{(n)}\left(x,t\right)\right| dt\leq\frac{\sigma^{n+1}\left(x\right)}{\left(n+1\right)!}$$

is valid.

In case of $a<x\leq\pi$, $-\alpha x+\alpha a-a<t<\alpha x-\alpha a-a$;

$$K_{\nu}^{(0)}\left(x,t\right)=\frac{w_{\nu}\alpha^{+}}{2}p\left(\frac{t+\alpha x-\alpha a+a}{2}\right) R_{\nu,1}\left(\frac{t+\alpha x-\alpha a+a}{2}\right)$$
$$+\frac{\alpha^{+}}{2}\int\limits_{0}^{(t+\alpha x-\alpha a+a)/2} q\left(u\right) R_{\nu,1}\left(u\right) du,$$

$$K_{\nu}^{(n)}\left(x,t\right)=w_{\nu}\alpha^{+}\int\limits_{(t+\alpha x-\alpha a+a)/2}^{a} p\left(u\right) K_{\nu}^{(n-1)}\left(u,t+\alpha x-\alpha a+a-u\right) du$$
$$-w_{\nu}\alpha^{-}\int\limits_{(-t-\alpha x+\alpha a+a)/2}^{a} p\left(u\right) K_{\nu}^{(n-1)}\left(u,t+\alpha x-\alpha a-a+u\right) du$$
$$+\frac{\alpha^{+}}{2}\int\limits_{0}^{a} q\left(u\right)\int\limits_{t-\alpha x+\alpha a-a+u}^{t+\alpha x-\alpha a+a-u} K_{\nu}^{(n-1)}\left(u,\xi\right) d\xi du$$
$$+\frac{\alpha^{-}}{2}\int\limits_{0}^{a} q\left(u\right)\int\limits_{t+\alpha x-\alpha a-a+u}^{t-\alpha x+\alpha a+a-u} K_{\nu}^{(n-1)}\left(u,\xi\right) d\xi du$$
$$-\frac{w_{\nu}}{\alpha}\int\limits_{(-t+\alpha x+\alpha a-a)/2\alpha}^{x} p\left(u\right) K_{\nu}^{(n-1)}\left(u,t-\alpha x+\alpha u\right) du$$
$$+\frac{w_{\nu}}{\alpha}\int\limits_{a}^{x} p\left(u\right) K_{\nu}^{(n-1)}\left(u,t+\alpha x-\alpha u\right) du$$
$$+\frac{1}{2\alpha}\int\limits_{a}^{x} q\left(u\right)\int\limits_{t-\alpha x+\alpha u}^{t+\alpha x-\alpha u} K_{\nu}^{(n-1)}\left(u,\xi\right) d\xi du.$$

Moreover, for in cases $a<x\leq\pi$, $\alpha x-\alpha a-a<t<-\alpha x+\alpha a+a$ and $a<x\leq\pi$, $-\alpha x+\alpha a+a<t<\alpha x-\alpha a+a$ by the mathematical induction method we obtained the following in equalities:

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| K_\nu^{(0)}(x,t) \right| dt \le C\sigma(x)$$

and

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| K_\nu^{(n)}(x,t) \right| dt \le C\frac{\sigma^{n+1}(x)}{(n+1)!},$$

where $0 < C =$constant. From the validity of the obtained estimations, we get

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| K_\nu(x,t) \right| dt \le C\left(e^{\sigma(x)} - 1\right) \le C\left(e^{\int_0^{\mu^+(x)} [(x-t)|q(t)|+2|p(t)|]dt} - 1 \right)$$

for the function $K_\nu(x,\cdot) \in L_1(-\mu^+(x), \mu^+(x))$.

Under condition (1.2), uniqueness of the solution of the equation (1.1) is proved.

Substituting $e(x,\lambda)$ and $e''(x,\lambda)$ instead of y and y'' in equation (1.1) respectively, we directly get the properties $ii)$, $iii)$ and $iv)$. Using integral equation for $K_\nu(x,t)$, the property $v)$ is taken for functions $R_{\nu,1}(x)$, $R_{\nu,2}(x)$, $R_{\nu,3}(x)$.

References

[1] E. N. Akmedova, On representation of solution of Sturm-Liouville Equation with discontinious coefficient, Proceeding of IMM of NAS of Azerbaijan XVI(XXIV), 5-9 (2002).

[2] W. O. Amrein, A. B. Hinz, D. B. Pearson, Sturm Liouville Theory Past and Present, Birkhäuser Verlag, Basel-Boston-Berlin, (2000).

[3] R. Bellman, K.L. Kuk, Difference-differential Equations. M: Mir, 548p (1967), (Russian).

[4] M. G. Gasymov, G. Sh. Guseinov, Determining of the diffusion operator from spectral data, Dokl. Akad. Nauk Azerb. SSR., 37(2), 19-23 (1981).

[5] I. M. Gelfand, B. M. Levitan, On The Determination of a Differential Equation From Its Spectral Function, Izv. Akad. Nauk SSSR, SEr. Mat. 15, 309-360 (1951); English transl. In Amer. Math. Soc. Transl. (2) 1 (1955).

[6] G.Sh. Guseinov, Asymptotic Formulas for Solutions and Eigenvalues of Quadratic Pencil of Sturm-Liouville Equations., Preprint No. 113, Inst. Phys. Akad. Nauk Azerb SSR Baku, 49p (1984).

[7] G.Sh. Guseinov, Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval in spectral theory of operators and its applications, Elm., Baku, Azerb, 51-101 (1986).

[8] M. Jaulent, C. Jean, The inverse s-wave scattering problem for a class of potentials depending on energy, Comm. Math. Phys., 28, 177-220 (1972).

[9] M. Yamamoto, Inverse eigenvalue problem for a vibration of a string with viscous drag, J. Math. Anal. Appl., 152, 20-34 (1990).

[10] V. A. Marchenko, Some Problems In The Theory of Second-Order Differential Operator, Dock. Akad. Nauk SSSR, 72, 457-460 (1950).

[11] V. A. Marchenko, Some Problems In The Theory of Linear Differential Operators, Trudy Moskov. Mat. Obshch. 1, 327-420 (1952).

[12] V. A. Marchenko, Sturm-Liouville Problems and Their Applications, Naukova Dumkas, Kiev, (1977); English Trans. Birkhauser, (1986).

[13] S. A. Buterin, V. A. Yurko, Inverse spectral problem for pencils of differential operators on a finite interval, Vestnik Bashkir. Univ., 4, 8-12 (2006).

[14] V. A. Yurko, An inverse problem for pencils of differential operators, Matem. Sbornik, 191, 137-160 (2000); English transl., Sbornik: Mathematics, 191, 561-1586 (2000).

[15] V. A. Yurko, Inverse Sturm-Liouville Problems and Their Applications, New York, 1, 98 (2008).

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