Palestine Journal of Mathematics

Vol. 5(2) (2016) , 138–144

Complementary Nil Edge Vertex Domination

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Communicated by Ayman Badawi

MSC 2010 Classifications: 05C69.

Keywords and phrases: dominating set, edge dominating set, edge vertex dominating set, global neighbourhood dominating set.

Abstract An edge vertex dominating set(*evd*-set)$E^{'}$ of a connected graph G with edge set E is said to be a complementary nil edge vertex dominating set(*cnevd*-set) of G if and only if $E - E^{'}$ is not an *evd*-set of G. A *cnevd*-set of minimum cardinality is called a minimum *cnevd*-set(*mcnevd*-set)of G and this minimum cardinality is called the complementary nil edge vertex domination number of G and is denoted by $\gamma_{cnev}(G)$. We have given a characterization result for an *evd*-set to be a *cnevd*-set and also bounds for this parameter are obtained.

1 Introduction & Preliminaries

Domination is an active topic in graph theory and has numerous applications to distributed computing, the web graph and adhoc networks. Haynes *et al.*[4] gave a comprehensive introduction to the theoretical and applied facets of domination in graphs.

A subset D of the vertex set V of G is said to be a *dominating set* of G, if each vertex in $V - D$ is adjacent to some vertex of D. The *domination number* , $\gamma(G)$ is the minimum cardinality of the dominating set of G[5]. A subset $E^{'}$ of the edge set E of a graph G is said to be an *edge dominating set* of G if each edge in $E - E^{'}$ is adjacent to some edge in $E^{'}$. The *edge domination number* , $\gamma^{'}(G)$ is the minimum cardinality of the edge dominating sets of G[5]. A subset $E^{'}$ of E is said to be an *edge vertex dominating set* of G if and only if each vertex in G is either incident to an edge in $E^{'}$ (or) it is adjacent with an end vertex of some edge in $E^{'}$. The *edge vertex domination number* , $\gamma_{ev}(G)$ is the minimum cardinality of the edge vertex dominating set of G[7]. A subset S of the vertex set V of G is said to be a *global dominating set* of G if it is a dominating set for G as well as to its complement $\overline{G}$. The *global domination number* $\gamma_g(G)$ is the minimum cardinality of the *global dominating sets* of G[8].

Many variants of vertex - vertex, edge - edge, vertex - edge, edge - vertex dominating sets have been studied. In the present paper, we introduce a new variant in edge - vertex dominating set namely, complementary nil edge vertex dominating set. We have given the characterization result for edge vertex dominating set to be a complementary nil edge vertex dominating set and characterized the graphs having *cnevd* numbers $1, 2, 3, \epsilon - 1$, ϵ, ϵ being number of edges of G. Bounds for this parameter are also obtained.

All graphs considered in this paper are simple, finite, undirected and connected. For standard terminology and notation we refer Bondy & Murthy[1].

2 Characterization and other relevant results

In this section, we initially state characterization result for a proper subset of the edge set of G to be a *cnevd*-set of G. There after we have given the bounds for this parameter in terms of various other parameters.

Theorem 2.1. ***(Characterization Result)*** *An evd-set $E^{'}$ of a (connected)graph G is a cnevd - set if and only if there is a vertex v in G such that all the edges incident with v and the edges that are adjacent to v are in $E^{'}$.*

Proof. Trivially follows from the definition. completes the proof. □
Notation: For any graph G with a vertex v,

$$F_v = \{e \in E : v\ is\ an\ end\ vertex\ of\ e\ or\ v\ is\ adjacent\ to\ an\ end\ of\ e\}$$

.

We now give the bounds for *cnevd* number of connected graphs.

Theorem 2.2. *If G is a connected graph, then*

$$\gamma_{ev}(G) + \delta(G) \leq \gamma_{cnev}(G)$$

Proof. Let $E^{'}$ be a minimum *cnevd*-set of G. By the Characterization Result there is a vertex v in G such that all the edges incident with v and the edges that are adjacent to v are in $E^{'}$. Clearly $E^{'} - \{e : v\ is\ incident\ to\ e\}$ is a *evd* - set of G whose cardinality is $\gamma_{cnev}(G) - d(v)$. Hence $\gamma_{ev}(G) \leq \gamma_{cnev}(G) - d(v) \Rightarrow \gamma_{ev}(G) + \delta(G) \leq \gamma_{cnev}(G)$.

Furthermore, the lower bound is attained in the case of House graph(i.e the graph C_5 together with an edge obtained by joining a pair of non adjacent vertices in C_5). Hence the bound is sharp.□

A lower bound for $\gamma_{cnev}(G)$ is obtained in terms of the number of vertices n and minimum degree $\delta(G)$ of the vertices of G.

Corollary 2.3. *For any graph G,*

$$\lceil \frac{n}{2\Delta(G) - 1} \rceil + \delta(G) \leq \gamma_{cnev}(G)$$

Proof. Let $E^{'}$ be an $\gamma_{ev}(G)$ - set. Any edge in $E^{'}$ dominates atmost $2\Delta(G) - 1$ vertices. Hence,

$$n \leq \gamma_{ev}(G)(2\Delta(G) - 1)$$

This implies,

$$\lceil \frac{n}{2\Delta(G) - 1} \rceil \leq \gamma_{ev}(G)$$

. Then by the above theorem the inequality follows.□

Note: The bound is sharp as it is attained in the case of C_6.

Corollary 2.4. *For any k - regular graph G with n vertices,*

$$\lceil \frac{n}{2k - 1} \rceil + k \leq \gamma_{cnev}(G).$$

Proof. The proof follows from the above theorem and the fact that $\Delta(G) = k = \delta(G)$.□

Theorem 2.5. *If G is a connected graph of order n and having ϵ edges, then*

$$\lceil \frac{6\epsilon\ -\ n^2\ +\ n}{4} \rceil \leq \gamma_{cnev}(G).$$

Proof. Let $E^{'}$ be a minimum *cnevd*-set of G. Since $E - E^{'}$ is not an *evd*-set of G, there is a vertex v non adjacent to all the edges in $E - E^{'}$. Hence,

$$\epsilon \leq n_{C_2} - 2(\epsilon - \gamma_{cnev}(G))$$

This implies,

$$\lceil \frac{6\epsilon\ -\ n^2\ +\ n}{4} \rceil \leq \gamma_{cnev}(G).$$

□

Note: The bound is sharp as it is attained in the case of $< v_1 v_2 v_3 v_1 > \bigcup \{v_1 v_4\}$.

Theorem 2.6. *For a connected graph G,*

$$\frac{\delta(G)}{2} < \gamma_{cnev}(G)$$

.

Proof. Let $E^{'}$ be a minimum *cnevd*-set of G. Since $E - E^{'}$ is not a *evd*-set of G, there is a vertex v of G which is non adjacent to any of the edges in $E - E^{'}$. Let $S = \{v \in V : v \text{ is an end point of an edge in } E^{'}\}$. Then, $N[v] \subset S \Rightarrow |N[v]| < |S| \Rightarrow |N[v]| < 2\gamma_{cnev}(G) \Rightarrow \frac{\delta(G)}{2} < \gamma_{cnev}(G)$.□

Theorem 2.7. *For a connected graph G of order $n(n \geq 4)$,*

$$\gamma_{cnev}(G) \leq \frac{\epsilon}{2} + \delta^{'}(G).$$

($\delta^{'}(G)$ being the minimum degree of the edges in G).

Proof. By Theorem.2.3.[1], $\gamma_{ev}(G) \leq \frac{\epsilon}{2}$. Now the $\gamma_{ev}(G)$ - set along with $N[e](e \in G)$ is a *cnevd* - set of G. Hence,

$$\gamma_{cnev}(G) \leq \frac{\epsilon}{2} + \delta^{'}(G).$$

□

Note: The bound is sharp as it is attained in the case of subdivision of star.

Theorem 2.8. *If G is a connected graph, then*

$$\gamma_{cnev}(G) \leq \gamma_{ve}(G) + \Delta(G)(\Delta(G) - 1)$$

.

Proof. Let $E^{'}$ be a minimum *evd* - set for G. Then for any vertex v in G, $E^{'} \bigcup F_v$ is a *cnevd* - set of G. Hence

$$\begin{aligned}\gamma_{cnev}(G) &\leq |E^{'} \bigcup F_v| \\ &\leq |E^{'}| + \Delta(G)(\Delta(G) - 1) \\ &\leq \gamma_{ev}(G) + \Delta(G)(\Delta(G) - 1)\end{aligned}$$

□

Note: The bound is sharp as it is attained in the case of C_5.

Remark: Since $\gamma_{ev}(G) \leq \gamma^{'}(G)$, follows that

$$\gamma_{cnev}(G) \leq \gamma^{'}(G) + \Delta(G)(\Delta(G) - 1).$$

Theorem 2.9. *If G is a connected graph having a pendant vertex, then*

$$\gamma_{cnev}(G) \leq \gamma_{ev}(G) + (\Delta(G) - 1).$$

Proof. Suppose that $E^{'}$ is a minimum *evd* - set for G. Let v be a pendant vertex in G. Clearly $E^{'} \bigcup F_v$ is a *cnevd* - set for G. Hence,

$$\begin{aligned}\gamma_{cnev}(G) &\leq |E^{'} \bigcup F_v| \\ &\leq \gamma_{ev}(G) + (\Delta(G) - 1)\end{aligned}$$

□

Note: The bound is sharp as it is attained in the case of P_4.

Corollary 2.10. *If $G = P_n$, then*

$$\gamma_{cnev}(G) \leq \gamma_{ve}(G) + 1.$$

Proof. Since $\Delta(G) = 2$, the proof follows from the above theorem. □

Theorem 2.11. *If G is a connected graph such that $\overline{G}$(the complement of G) is connected, then*

$$\gamma_{cnev}(G) + \gamma_{cnev}(\overline{G}) \leq \frac{n(n-1)}{2}.$$

Proof. For any graph G, $\gamma_{cnev}(G) \leq \epsilon$. Similarly, $\gamma_{cnev}(\overline{G}) \leq \epsilon^{'}(\epsilon^{'}$ is the number of edges in $\overline{G}$). So,

$$\gamma_{cnev}(G) + \gamma_{cnev}(\overline{G}) \leq \epsilon + \epsilon^{'} = \frac{n(n-1)}{2}.$$

Hence,

$$\gamma_{cnev}(G) + \gamma_{cnev}(\overline{G}) \leq \frac{n(n-1)}{2}.$$

□

Note: If G is a connected graph with ϵ edges, then $\gamma_{cneve}(G) \leq \epsilon$.

Now, we characterize the graphs for which $\gamma_{cnev}(G) = \epsilon$.

Theorem 2.12. *For a connected graph G with ϵ edges, $\gamma_{cnev}(G) = \epsilon$ if and only if for each vertex v in G, $F_v = E$.*

Proof. Assume that $\gamma_{cnev}(G) = \epsilon$. Suppose that there is a vertex v in G such that $F_v \neq E$. Consider the set $E - F_v$.
Let $S = \{u : u\ is\ an\ end\ point\ of\ an\ edge\ in\ E\ - F_v\ \}$.
If $S = \phi$, then $F_v(\subset E)$ is a *cnevd*-set of G whose cardinality is less than ϵ.
Suppose $S \neq \phi$.
If there is an edge e between two vertices of S which is not incident or adjacent to v, then $E - \{e\}$ is a *cnevd* - set of G of cardinality less than ϵ.
If there is no edge $e(in\ E - F_v)$ between two vertices of S, then
$E - \{e = v_1v_2\}(v_1\ or\ v_2 \notin S)$ is a *cnevd* - set of cardinality less than ϵ. Hence our supposition is false.

Assume that the converse holds. Let $E^{'}$ be a minimum *cnevd*-set of G. By the Characterization Result for *cnevd*-set there is a vertex v such that
$F_v = E$. Hence, $\gamma_{cnev}(G) = \epsilon$.□

Corollary 2.13. *If $G = C_n$,then $\gamma_{cnev}(G) = \epsilon$ if and only if $n = 3, 4$.*

Corollary 2.14. *If $G = P_n$,then $\gamma_{cnev}(G) = \epsilon$ if and only if $n = 2, 3$.*

Corollary 2.15. *(i)* $\gamma_{cnev}(S_n) = n, n \geq 3$

(ii) $\gamma_{cnev}(K_n) = n, n \geq 3$

(iii) $\gamma_{cnev}(K_{m,n}) = m + n$

Theorem 2.16. *Let G be a connected graph with $diam(G) = 2$ and $c(G) = 3$, then $\gamma_{cnev}(G) \neq \epsilon$ if and only if there is a triangle T in G, for which there is a vertex in $V - V(T)$ adjacent to exactly one vertex in T.*

Proof. Assume that $\gamma_{cnve}(G) \neq \epsilon$.
By Theorem.2.12. there is a vertex v in G such that $F_v \neq E$. Since $diam(G) = 2$ and $c(G) = 3$, for $e(= v_1v_2) \in E - F$ there is an edge $e^{'}(= v_2v_3)$ in F, such that $< v_1v_2v_3 > (= T)$ is a triangle in G. Clearly v is the vertex in $V - V(T)$ which is adjacent to exactly one vertex v_3 in T.

The converse part is clear.□

Corollary 2.17. *G be a tree with ϵ edges, then $\gamma_{cnev}(G) = \epsilon$ if and only if $G \cong S_n$.*

Proof. Assume that $\gamma_{cnev}(G) = \epsilon$. Then by the Characterization Result for each vertex v in G, $F_v = E$. So for any pendant edge uv in G, $N(u)\bigcup N(v) = V$, which implies one of u, v is adjacent to all the vertices in G. W.l.g assume that v is the vertex adjacent to all the vertices in G(i.e u is a pendant vertex). Since G is a tree, there is no edge between $v_1, v_2 \in V - \{v\}$(i.e. all the vertices in $V - \{v\}$ are pendant). Hence $G \cong S_n$.

For the converse part, any vertex v in S_n has the property that $F_v = E$. Hence by the Characterization Result the claimant holds.□

Theorem 2.18. *For any tree G of order $n(\neq 3)$ and $\delta^{'}(G) = 1$, $\gamma_{cnev}(G) \leq \frac{\epsilon}{2} + 1$; Equality holds if and only if G is isomorphic to subdivision of a star.*

Proof. By Theorem.2.3.[1], $\gamma_{ev}(G) \leq \frac{\epsilon}{2}$. Now a $\gamma_{ev}(G)$ - set along with an edge of minimum degree is a *cnevd* - set of G. Hence $\gamma_{cnev}(G) \leq \frac{\epsilon}{2} + 1$.□

Theorem 2.19. *G be a connected graph, then $\gamma_{cnev}(G) = 1$ if and only if G = P_2.*

Now we characterize the graphs for which $\gamma_{cnev}(G) = 2$.

Theorem 2.20. *G be a connected graph of order $n \geq 4$, then $\gamma_{cnev}(G) = 2$ if and only if $\delta(G) = \delta^{'}(G) = 1, diam(G) = 3$.*

Proof. Assume that $\gamma_{cnev}(G) = 2$. Then by the Characterization Result, there is a vertex v_1 in G for which $F_{v_1} \subseteq E$. By hypothesis and our assumption $|F_{v_1}| = 2$ and F_{v_1} is a *cnevd* - set of G. Since $|F_{v_1}| = 2$, v_1 is a pendant vertex adjacent to $v_2(say)$ and there is exactly one edge(v_2v_3) adjacent to v_1. So $\delta(G) = \delta^{'}(G) = 1$.
Clearly any diammetral path in G is from v_1 to some other vertex in $V - \{v_2\}$ adjacent to v_3. Thus v_2v_3 is the edge which dominates the vertices in $V - \{v_1, v_2, v_3\}$(i.e.v_3 dominates the vertices in $V - \{v_1, v_2, v_3\}$). Hence $diam(G) = 3$.

Conversely if G is acyclic, then $G = P_3$ or G is obtained by adding zero or more leaves to exactly one support vertex of P_3. So $\gamma_{cnev}(G) = 2$.
Suppose G is cyclic. By our assumption there is a path $< v_1v_2v_3 >$ in which v_1 is a pendant vertex and v_1v_2 is adjacent to exactly one edge i.e.v_2v_3, the vertices in $V - \{v_1, v_2, v_3\}$ are adjacent to $v_3(only)$. By the Characterization Result $\{v_1v_2, v_2v_3\}$ is a *cnevd* - set of G of cardinality two. By the above theorem $\{v_1v_2, v_2v_3\}$ is a $\gamma_{cnev}(G)$ - set. Hence the result.□

Corollary 2.21. *G be a connected unicyclic graph with $n(\geq 5)$ vertices, then $\gamma_{cnev}(G) = 2$ if and only if G is obtained by adding exactly one edge between the adjacent pendant vertices of $S_{1,q}(q \geq 2)$.*

Proof. Assume that $\gamma_{cnev}(G) = 2$. Then by the above theorem, $\delta(G) = \delta^{'}(G) = 1, diam(G) = 3$. Since $\delta(G) = \delta^{'}(G) = 1$ and $diam(G) = 3$, there is a path $< v_1v_2v_3 >$ in G with $d(v_1) = 1, d(v_1v_2) = 1$(i.e.$v_1v_2$ is pendant edge adjacent to v_2v_3(only)); any vertex in $V - \{v_1, v_2, v_3\}$ is adjacent to $v_3(only)$(in $\{v_1, v_2, v_3\}$). By hypothesis there is exactly one cycle C_p(say). Clearly v_3 lies on C_p. Since $diam(G) = 3$, vertices in $C_p - \{v_3\}$ are adjacent to v_3. Since G is unicyclic, $p = 3$. Therefore G is obtained by adding exactly one edge between the adjacent pendant vertices of $S_{1,q}(q \geq 2)$.
The converse part is clear.□

Corollary 2.22. *G be a tree with n vertices, then $\gamma_{cnev}(G) = 2$ if and only if G = P_3 or G = P_4 or G is obtained by adding zero or more leaves to exactly one support vertex of P_4.*

Proof. Assume that $\gamma_{cnev}(G) = 2$.
Clearly $n \geq 3$. If $n = 3$, then $G = P_3$.
Suppose $n > 3$. By the above theorem G is a tree with diameter three. Let $< v_1v_2v_3v_4 >$ be the diammetral path in G. Since $diam(G) = 3$ any vertex in $V - \{v_1, v_2, v_3, v_4\}$ is adjacent to v_2 or v_3. By our assumption both v_2, v_3 cannot have neighbours from $V - \{v_1, v_4\}$. Hence $G = P_4$ or G is obtained by adding zero or more leaves to exactly one support vertex of P_4. □

Corollary 2.23. *G be a connected graph with $n(\geq 4)$ vertices. If $\delta(G) = \delta^{'}(G) = 1, diam(G) = 3$, then $\gamma_{cnev}(G) = \gamma_g(G) = 2$.*

Theorem 2.24. *G be a connected graph of order $n(\geq 4)$, then $\gamma_{cnev}(G) = 3$ if and only if $\delta^{'}(G) = 2, 1 < diam(G) \leq 3$.*

Corollary 2.25. *G be a tree with $n(\geq 4)$ vertices, then $\gamma_{cnev}(G) = 3$ if and only if $G = S_3$ or $G = S_{2,q}(q \geq 2)$.*

Theorem 2.26. *G be a tree and $E^{'}$ be the set of all pendant edges in G. Then $E^{'}$ is a cnevd - set if and only if $G = S_n$.*

Proof. Suppose that $E^{'}$ is a *cnevd*-set of G. Then by the Characterization Result for any *cnevd*-set, there is a vertex $v \in V$ in G such that $F_v = E$. Clearly $diam(G) = 2$(otherwise there is a non pendant edge in G which is a member of $E^{'}$, a contradiction). Hence $G = S_n$. The converse part is clear.

Theorem 2.27. *G be a connected graph of order $n(\geq 4)$. Then $\gamma_{cnev}(G) = \epsilon - 1$ if and only if the following conditions are satisfied:*
(i) $|E - F_v| \leq 2$, $|E - F_v| \neq 0$(for all $v \in V$).
(ii) Whenever $|E - F_v| = 2$ for some $v \in V$, $< E - F_v >$ is connected and adjacent to exactly one edge in F_v, for some $v \in V$.

Proof. Assume that $\gamma_{cnev}(G) = \epsilon - 1$.
Suppose that $|E - F_v| = k(k \geq 3)$ for some $v \in V$.
If all the edges in $E - F_v$ are adjacent with the edges in F_v,then F_v is a *cnevd* - set of cardinality less than $\epsilon - 1$ which is a contradiction to our assumption.
If atleast one of the edges in $E - F_v$ is not adjacent to an edge in F_v, then $\gamma_{cnev}(G) < \epsilon - 2$ again a contradiction to our assumption. Hence $|E - F_v| \leq 2$ for all $v \in V$.

By our assumption $|E - F_v| \neq 0$ for all $v \in V$. Hence $|E - F_v| \leq 2$ for some $v \in V$. Suppose that $|E - F_v| = 2$ for some $v \in V$.
If all the edges in $E - F_v$ are adjacent with the edges in F_v, then $\gamma_{cnev}(G) = \epsilon - 2$ which is a contradiction to our assumption. So there is an edge in $E - F_v$ which is not adjacent to edges in F_v. Hence $< E - F_v >$ is connected and adjacent to exactly one edge in F_v.

The converse part is clear.□

Corollary 2.28. *If $G = C_n$,then $\gamma_{cnev}(G) = \epsilon - 1$ if and only if $n = 5$.*

Corollary 2.29. *If $G = P_n$,then $\gamma_{cnev}(G) = \epsilon - 1$ if and only if $n = 4, 5$.*

Corollary 2.30. *If G is a tree, then $\gamma_{cnev}(G) = \epsilon - 1$ if and only if $G = P_4$ or $G = P_5$.*

Proof. Assume that $\gamma_{cnev}(G) = \epsilon - 1$.
Suppose that $diam(G) = k(k \geq 5)$. Let $P =< v_1v_2...v_{k-1}v_k > \ (k \geq 6)$ be a diammetral path in G.
If $G = P$, then $E(P) - \{v_3v_4, v_4v_5\}$ is a *cnved* - set of G of cardinality less than $\epsilon - 1$, which is a contradiction to our assumption.
Suppose $G \neq P$. If $min\{d(v_3), d(v_4), d(v_5)\} \geq 3$ (or) $d(v_3) = d(v_4) = d(v_5) = 2$, then also $E(P) - \{v_3v_4, v_4v_5\}$ is a *cnevd* - set of G of cardinality less than $\epsilon - 1$ which is a contradiction to our assumption.

Hence $diam(G) \leq 4$.
If $diam(G) \leq 4$. Clearly $G = P_4$ or $G = P_5$. The converse part is clear.□

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Received: March 23, 2015.

Accepted: September 14, 2015