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Using two dimensional differential transform method solve of third order complex differential equations.

Murat DÜZ and Uğur İLTER

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Abstract In this study, differential transforms of first, second and third derivatives of a complex function were given. Later, third order complex equations were solved using two dimensional differential transform.

1 Introduction

The differential transform method (DTM) is a method for to solve differential equation or differential equation systems. This method is a numerical method. One dimension DTM was first proposed and applied by Zahou[1]. Two dimensional DTM was proposed by C.K. Chen and Shing Huei Ho[3]. Many studies has been made with DTM recently. For example, by using one dimensional DTM was solved nonlinear differential equations in [2]. By using two dimensional DTM was solved partial differential equations , systems of partial differential equations, complex partial differential equations in[3],[4],[5],[6].

This method which is consist of computing coeffient of Taylor series of solution by using initial value ia a iterative method

In this paper we solved third order complex partial differential equations by using DTM. . Firstly we seperated real and imaginer parts equation. Thus from one unknown equation was obtained two unknown equation system. Later using DTM we obtained differenatial transforms of real and imaginer parts of solutions. In the latest using inverse differenatial transform we obtained real and imaginer parts of solution.

2 Derivatives of Complex Functions

Let $w = w(z,\overline{z})$ be a complex function. Here $z = x+iy, w(z,\overline{z}) = u(x,y)+iv(x,y)$. Derivative of according to z and $\overline{z}$ of $w(z,\overline{z})$ is defined as follows:

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \tag{2.1}$$

$$\frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \tag{2.2}$$

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{2.3}$$

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \tag{2.4}$$

Similarly second order derivative of $w(z,\overline{z})$ are defined as follows:

$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2} - 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right) \tag{2.5}$$

$$\frac{\partial^2 w}{\partial \overline{z}^2} = \frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2} + 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right) \tag{2.6}$$

$$\frac{\partial^2 w}{\partial \overline{z} \partial z} = \frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{1}{4}\Delta w \tag{2.7}$$

Theorem 2.1. *Third order derivative of $w(z, \overline{z})$ are defined as follows:*

$$\frac{\partial^3 w}{\partial z^3} = \frac{1}{8}\left(\frac{\partial^3 w}{\partial x^3} - 3\frac{\partial^3 w}{\partial x \partial y^2} - 3i\frac{\partial^3 w}{\partial x^2 \partial y} + i\frac{\partial^3 w}{\partial y^3}\right) \tag{2.8}$$

$$\frac{\partial^3 w}{\partial \overline{z}^3} = \frac{1}{8}\left(\frac{\partial^3 w}{\partial x^3} - 3\frac{\partial^3 w}{\partial x \partial y^2} + 3i\frac{\partial^3 w}{\partial x^2 \partial y} - i\frac{\partial^3 w}{\partial y^3}\right) \tag{2.9}$$

$$\frac{\partial^3 w}{\partial \overline{z}^2 \partial z} = \frac{1}{8}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} + i\frac{\partial^3 w}{\partial x^2 \partial y} + i\frac{\partial^3 w}{\partial y^3}\right) \tag{2.10}$$

$$\frac{\partial^3 w}{\partial \overline{z} \partial z^2} = \frac{1}{8}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} - i\frac{\partial^3 w}{\partial x^2 \partial y} - i\frac{\partial^3 w}{\partial y^3}\right) \tag{2.11}$$

Proof. (2) Definition of (2.1) and (2.5)

$$\begin{aligned}
\frac{\partial^3 w}{\partial z^3} &= \frac{\partial}{\partial z}\left(\frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial}{\partial z}\left(\frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2} - 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right)\right) \\
&= \frac{1}{8}\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} - 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right) \\
&\quad -i\frac{1}{8}\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} - 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right) \\
&= \frac{1}{8}\left(\frac{\partial^3 w}{\partial x^3} - 2i\frac{\partial^3 w}{\partial x^2 \partial y} - \frac{\partial^3 w}{\partial x \partial y^2}\right) \\
&\quad -\frac{i}{8}\left(\frac{\partial^3 w}{\partial x^2 \partial y} - 2i\frac{\partial^3 w}{\partial x \partial y^2} - \frac{\partial^3 w}{\partial y^3}\right)
\end{aligned}$$

$$\frac{\partial^3 w}{\partial z^3} = \frac{1}{8}\left(\frac{\partial^3 w}{\partial x^3} - 3\frac{\partial^3 w}{\partial x \partial y^2} - 3i\frac{\partial^3 w}{\partial x^2 \partial y} + i\frac{\partial^3 w}{\partial y^3}\right) \tag{2.12}$$

Thus, proof of (2.8) is completed. Similarly proofs of (2.9), (2.10) and (2.11) can be done. □ □

3 Two Dimensional differential transform

Two dimensional differential transform of function $f(x, y)$ is defined as follows

$$F(k,h) = \frac{1}{k!.h!}\left[\frac{\partial^{k+h} f(x,y)}{\partial x^k \partial y^h}\right]_{x=0,y=0} \tag{3.1}$$

In Equation (3.1), $f(x, y)$ is original function and $F(k, h)$ is transformed function, which is called T - function is brief. Differential inverse transform of $F(k, h)$ is defined as follows

$$f(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} F(k,h) x^k y^h \tag{3.2}$$

From (3.1) and (3.2) can be concluded

$$f(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!.h!}\left[\frac{\partial^{k+h} f(x,y)}{\partial x^k \partial y^h}\right]_{x=0,y=0} x^k y^h \tag{3.3}$$

Equation (3.3) implies that the concept of two dimensional differential transform is derived from two dimensional Taylor series expansion.

Theorem 3.1. *[3, 4], If $w(x,y) = u(x,y) \mp v(x,y)$ then $W(k,h) = U(k,h) \mp V(k,h)$.*

Theorem 3.2. *[3, 4] If $w(x,y) = \lambda u(x,y)$ then $W(k,h) = \lambda U(k,h)$.*

Theorem 3.3. *[3, 4] If $w(x,y) = \frac{\partial u(x,y)}{\partial x}$ then $W(k,h) = (k+1)U(k+1,h)$.*

Theorem 3.4. *[3, 4] If $w(x,y) = \frac{\partial u(x,y)}{\partial y}$ then $W(k,h) = (h+1)U(k,h+1)$.*

Theorem 3.5. *[3, 4] If $w(x,y) = \frac{\partial^{r+s} u(x,y)}{\partial x^r \partial y^s}$ then*

$$W(k,h) = (k+1)(k+2)\dots(k+r)(h+1)(h+2)\dots(h+s)U(k+r,h+s). \tag{3.4}$$

Theorem 3.6. *[3, 4] If $w(x,y) = u(x,y).v(x,y)$ then*

$$W(k,h) = \sum_{k=0}^{r} \sum_{s=0}^{h} U(r,h-s).V(k-r,s) \tag{3.5}$$

Theorem 3.7. *[3, 4] If $w(x,y) = x^m y^n$ then*

$$W(k,h) = \delta(k-m,h-n) = \begin{cases} 1 & k=m, h=n \\ 0 & , otherwise \end{cases} \tag{3.6}$$

4 Differential Transforms Of Derivatives Of Complex Functions

Theorem 4.1. *If $w(x,y) = u(x,y) + iv(x,y)$, then*

$$\begin{aligned} W_z(k,h) &= \frac{1}{2}\left[(k+1)U(k+1,h) + (h+1)V(k,h+1)\right] \\ &+ \frac{i}{2}\left[(k+1)V(k+1,h) - (h+1)U(k,h+1)\right] \end{aligned} \tag{4.1}$$

where, $W_z(k,h)$ is differential transform of $\frac{\partial w}{\partial z}$.

Proof. Since the $w(x,y) = u(x,y) + iv(x,y)$ and by the equality (2.1), we have

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \tag{4.2}$$

It is not diffucult to see that

$$\begin{aligned} W_z(k,h) &= \frac{1}{2}[(k+1)U(k+1,h) + (h+1)V(k,h+1)] \\ &+\frac{i}{2}[(k+1)V(k+1,h) - (h+1)U(k,h+1)] \end{aligned} \tag{4.3}$$

is obtained. □

Theorem 4.2. *If $w(x,y) = u(x,y) + iv(x,y)$, then*

$$\begin{aligned} W_{\bar{z}}(k,h) &= \frac{1}{2}[(k+1)U(k+1,h) - (h+1)V(k,h+1)] \\ &+\frac{i}{2}[(k+1)V(k+1,h) + (h+1)U(k,h+1)] \end{aligned} \tag{4.4}$$

where, $W_{\bar{z}}(k,h)$ is differential transform of $\frac{\partial w}{\partial \bar{z}}$.

Proof. By the equality (2.2), the proof is similar to teorem 3.1 □

Theorem 4.3. *If* $w(x,y) = u(x,y) + iv(x,y)$, *then*

$$\begin{aligned} W_{zz}(k,h) &= \frac{1}{4}[(k+1)(k+2)U(k+2,h) + 2(k+1)(h+1)V(k+1,h+1) \\ &\quad -(h+1)(h+2)U(k,h+2)] + \frac{i}{4}[(k+1)(k+2)V(k+2,h) \\ &\quad -2(k+1)(h+1)U(k+1,h+1) - (h+1)(h+2)V(k,h+2)] \end{aligned} \quad (4.5)$$

where, $W_{zz}(k,h)$ *is differential transform of* $\frac{\partial^2 w}{\partial z^2}$.

Proof. By the equality (2.5) we have

$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} - 2i(\frac{\partial^2 u}{\partial x \partial y}) + i\frac{\partial^2 v}{\partial x \partial y} - \left(\frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 v}{\partial y^2}\right)\right] \quad (4.6)$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + i\left(\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial y^2}\right)\right] \quad (4.7)$$

Hence,

$$\begin{aligned} W_{zz}(k,h) &= \frac{1}{4}[(k+1)(k+2)U(k+2,h) + 2(k+1)(h+1)V(k+1,h+1) \\ &\quad -(h+1)(h+2)U(k,h+2)] + \frac{i}{4}[(k+1)(k+2)V(k+2,h) \\ &\quad -2(k+1)(h+1)U(k+1,h+1) - (h+1)(h+2)V(k,h+2)] \end{aligned} \quad (4.8)$$

is obtained. □

Theorem 4.4. *If* $w(x,y) = u(x,y) + iv(x,y)$, *then*

$$\begin{aligned} W_{\bar{z}\bar{z}}(k,h) &= \frac{1}{4}[(k+1)(k+2)U(k+2,h) - 2(k+1)(h+1)V(k+1,h+1) \\ &\quad -(h+1)(h+2)U(k,h+2)] + \frac{i}{4}[(k+1)(k+2)V(k+2,h) \\ &\quad +2(k+1)(h+1)U(k+1,h+1) - (h+1)(h+2)V(k,h+2)] \end{aligned} \quad (4.9)$$

where, $W_{\bar{z}\bar{z}}(k,h)$ *is differential transform of* $\frac{\partial^2 w}{\partial \bar{z}^2}$.

Proof. Proof can be done similar to proof of the teorem 4.3 by using equality (2.6). □

Theorem 4.5. *If* $w(x,y) = u(x,y) + iv(x,y)$, *then*

$$\begin{aligned} W_{\bar{z}z}(k,h) = &\frac{1}{4}[(k+1)(k+2)U(k+2,h) + (h+1)(h+2)U(k,h+2) \\ &+i(k+1)(k+2)V(k+2,h) + i(h+1)(h+2)V(k,h+2)] \end{aligned} \quad (4.10)$$

where, $W_{\bar{z}z}(k,h)$ *is differential transform of* $\frac{\partial^2 w}{\partial \bar{z} \partial z}$

Proof. Proof can be done similar to proof of the teorem 4.3 by using equality (2.7). □

Theorem 4.6. *If* $w(x,y) = u(x,y) + iv(x,y)$, *then*

$$\begin{aligned} W_{zzz}(k,h) &= \frac{1}{8}[(k+1)(k+2)(k+3)U(k+3,h) - 3(k+1)(h+1)(h+2)U(k+1,h+2) \\ &+ 3(k+1)(k+2)(h+1)V(k+2,h+1) - (h+1)(h+2)(h+3)V(k,h+3) \\ &+ i(k+1)(k+2)(k+3)V(k+3,h) - 3i(k+1)(h+1)(h+2)V(k+1,h+2) \\ &- 3i(k+1)(k+2)(h+1)U(k+2,h+1) \\ &+ i(h+1)(h+2)(h+3)U(k,h+3)] \end{aligned} \quad (4.11)$$

where, $W_{zzz}(k,h)$ *is differential transform of* $\frac{\partial^3 w}{\partial z^3}$.

Proof. Since $w(x,y) = u(x,y) + iv(x,y)$ and equality (2.8) we have:

$$\frac{\partial^3 w}{\partial z^3} = \frac{1}{8}[\frac{\partial^3}{\partial x^3}(u+iv) - 3\frac{\partial^3}{\partial x \partial y^2}(u+iv) - 3i\frac{\partial^3}{\partial y \partial x^2})(u+iv) + i\frac{\partial^3}{\partial y^3}(u+iv)] \tag{4.12}$$

$$\frac{\partial^3 w}{\partial z^3} = \frac{1}{8}[\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x \partial y^2} + 3\frac{\partial^3 v}{\partial y \partial x^2} - \frac{\partial^3 v}{\partial y^3} + i(\frac{\partial^3 v}{\partial x^3} - 3\frac{\partial^3 v}{\partial x \partial y^2} - 3\frac{\partial^3 u}{\partial y \partial x^2} + \frac{\partial^3 u}{\partial y^3})]. \tag{4.13}$$

Hence it is get that:

$$\begin{aligned} W_{zzz}(k,h) &= \frac{1}{8}[(k+1)(k+2)(k+3)U(k+3,h) - 3(k+1)(h+1)(h+2)U(k+1,h+2) \\ &\quad +3(k+1)(k+2)(h+1)V(k+2,h+1) - (h+1)(h+2)(h+3)V(k,h+3) \\ &\quad +i(k+1)(k+2)(k+3)V(k+3,h) - 3i(k+1)(h+1)(h+2)V(k+1,h+2) \\ &- 3i(k+1)(k+2)(h+1)U(k+2,h+1) \\ &+ i(h+1)(h+2)(h+3)U(k,h+3)] \end{aligned} \tag{4.14}$$

□

Theorem 4.7. *If $w(x,y) = u(x,y) + iv(x,y)$, then*

$$\begin{aligned} W_{\bar{z}\bar{z}\bar{z}}(k,h) &= \frac{1}{8}[(k+1)(k+2)(k+3)U(k+3,h) + 3(k+1)(h+1)(h+2)U(k+1,h+2) \\ &\quad -3(k+1)(k+2)(h+1)V(k+2,h+1) + (h+1)(h+2)(h+3)V(k,h+3) \\ &\quad +i(k+1)(k+2)(k+3)V(k+3,h) - 3i(k+1)(h+1)(h+2)V(k+1,h+2) \\ &\quad +3i(k+1)(k+2)(h+1)U(k+2,h+1) \\ &\quad -i(h+1)(h+2)(h+3)U(k,h+3)] \end{aligned} \tag{4.15}$$

where, $W_{\bar{z}\bar{z}\bar{z}}(k,h)$ is differential transform of $\frac{\partial^3 w}{\partial \bar{z}^3}$.

Proof. Proof is similar to the theorem 4.6 using equality (2.9). □

Theorem 4.8. *If $w(x,y) = u(x,y) + iv(x,y)$, then*

$$\begin{aligned} W_{\bar{z}\bar{z}z}(k,h) &= \frac{1}{8}[(k+1)(k+2)(k+3)U(k+3,h) + (k+1)(h+1)(h+2)U(k+1,h+2) \\ &\quad -(k+1)(k+2)(h+1)V(k+2,h+1) - (h+1)(h+2)(h+3)V(k,h+3) \\ &\quad +i(k+1)(k+2)(k+3)V(k+3,h) + i(k+1)(h+1)(h+2)V(k+1,h+2) \\ &\quad +i(k+1)(k+2)(h+1)U(k+2,h+1) \\ &\quad +i(h+1)(h+2)(h+3)U(k,h+3)] \end{aligned} \tag{4.16}$$

where, $W_{\bar{z}\bar{z}z}(k,h)$ is differential transform of $\frac{\partial^3 w}{\partial \bar{z}^2 \partial z}$.

Proof. Proof is similar to the theorem 4.6 using equality (2.10). □

Theorem 4.9. *If $w(x,y) = u(x,y) + iv(x,y)$, then*

$$\begin{aligned} W_{zzz}(k,h) &= \frac{1}{8}[(k+1)(k+2)(k+3)U(k+3,h) + (k+1)(h+1)(h+2)U(k+1,h+2) \\ &\quad +(k+1)(k+2)(h+1)V(k+2,h+1) + (h+1)(h+2)(h+3)V(k,h+3) \\ &\quad +i(k+1)(k+2)(k+3)V(k+3,h) + i(k+1)(h+1)(h+2)V(k+1,h+2) \\ &\quad -i(k+1)(k+2)(h+1)U(k+2,h+1) \\ &\quad -i(h+1)(h+2)(h+3)U(k,h+3)] \end{aligned} \tag{4.17}$$

where, $W_{zzz}(k,h)$ is differential transform of $\frac{\partial^3 w}{\partial \bar{z} \partial z^2}$.

Proof. Proof is similar to the theorem 4.6 using equality (2.11). □

5 Using two-dimensional differential tansform solve of Third Order Complex Partial Differential Equations.

In this section, to demonstrate how to use two-dimensional transform to solve complex partial differential equations are solved.

Example 5.1. Solve the following initial value problem

$$\frac{\partial^3 w}{\partial z^3} + 2\frac{\partial^2 w}{\partial \overline{z}^2} = 18 \tag{5.1}$$

$$w(x,0) = x^3 + 3x^2 \tag{5.2}$$

$$\frac{\partial w}{\partial y}(x,0) = i(3x^2 - 6x) \tag{5.3}$$

$$\frac{\partial^2 w}{\partial z^2}(x,0) = -6x - 6 \tag{5.4}$$

Since $w = u + iv$, equation (5.1) is equivalent following to equation system (5.5)-(5.6).

$$\frac{1}{8}\left(\frac{\partial^3 u}{\partial x^3} + 3\frac{\partial^3 v}{\partial x^2 \partial y} - 3\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 v}{\partial y^3}\right) + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}\right) = 18 \tag{5.5}$$

$$\frac{1}{8}\left(\frac{\partial^3 v}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} - 3\frac{\partial^3 v}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial y^3}\right) + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial y^2}\right) = 0 \tag{5.6}$$

As a result, we get equalities (5.7)-(5.8) from differential transforms of (5.5)-(5.6).

$$\begin{aligned} &(k+1)(k+2)(k+3)U(k+3,h) + 3(k+1)(k+2)(h+1)V(k+2,h+1) \\ &-3(k+1)(h+1)(h+2)U(k+1,h+2) - (h+1)(h+2)(h+3)V(k,h+3) \\ &+4(k+1)(k+2)U(k+2,h) - 8(k+1)(h+1)V(k+1,h+1) - 4(h+1)(h+2)U(k,h+2) = 144 \end{aligned} \tag{5.7}$$

$$\begin{aligned} &(k+1)(k+2)(k+3)V(k+3,h) - 3(k+1)(k+2)(h+1)U(k+2,h+1) \\ &-3(k+1)(h+1)(h+2)V(k+1,h+2) + (h+1)(h+2)(h+3)U(k,h+3) \\ &+4(k+1)(k+2)V(k+2,h) + 8(k+1)(h+1)U(k+1,h+1) \\ &-4(h+1)(h+2)V(k,h+2) = 0 \end{aligned} \tag{5.8}$$

Since

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h \tag{5.9}$$

and the equality (5.2), following results are obtained.

$$\begin{aligned} U(0,0) &= 0, U(1,0) = 0, U(2,0) = 3, U(3,0) = 1, U(i,0) = 0 (i = 4,5,\ldots), \\ V(i,0) &= 0 (i = 0,1,2,3,\ldots) \end{aligned} \tag{5.10}$$

Similarly,

$$\frac{\partial w}{\partial y} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} h.W(k,h)x^k y^{h-1} \tag{5.11}$$

and the equality (5.3), following results are obtained.

$$\begin{aligned} V(0,1) &= 0, V(1,1) = -6, V(2,1) = 3, V(i,1) = 0 (i = 3,4,5,\ldots), \\ U(i,1) &= 0 (i = 0,1,2,\ldots) \end{aligned} \tag{5.12}$$

Finally, by

$$\frac{\partial^2 w}{\partial y^2} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} h.(h-1).W(k,h)x^k y^{h-2} \tag{5.13}$$

and the equality (5.4), following results are obtained

$$U(0,2) = -3, U(1,2) = -3, U(i,2) = 0(i = 2,3,4,\ldots), V(i,2) = 0(i = 0,1,2,\ldots) \tag{5.14}$$

If $k = 0, h = 0$ are written in the equation (5.7),

$$6U(3,0) + 6V(2,1) - 6U(1,2) - 6V(0,3) + 8U(2,0) - 8V(1,1) - 8U(0,2) = 144 \tag{5.15}$$

$$6.1 + 6.3 - 6.(-3) - 6V(0,3) + 8.3 - 8.(-6) - 8.(-3) = 144 \tag{5.16}$$

are obtained. So, we have

$$V(0,3) = -1. \tag{5.17}$$

When $h = 0$ is written in the equality (5.8)

$$(k+1)(k+2)(k+3)V(k+3,0) - 3(k+1)(k+2)U(k+2,1) - 6(k+1)V(k+1,2)$$
$$+6U(k,3) + 4(k+1)(k+2)V(k+2,0) + 8(k+1)U(k+1,1) - 8V(k,2) = 0 \tag{5.18}$$

is obtained. It is clear that we have $V(k,0) = 0, U(k,1) = 0, V(k,2) = 0$, from the equalities (5.10),(5.12) and (5.14), respectively. As a result we get by (5.18)

$$U(k,3) = 0 \tag{5.19}$$

for every $k \geq 0$. When $h = 0$ is written in the equality (5.7)

$$(k+1)(k+2)(k+3)U(k+3,0) + 3(k+1)(k+2)V(k+2,1)$$

$$-6(k+1)U(k+1,2) - 6V(k,3) + 4(k+1)(k+2)U(k+2,0)$$

$$-8(k+1)V(k+1,1) - 8U(k,2) = 144 \tag{5.21}$$

Using (5.10), (5.12) and (5.14) in the equality (5.21) for $k = 1$ we get that:

$$-6V(1,3) + 24U(3,0) - 16V(2,1) - 8U(1,2) = 0 \tag{5.22}$$

$$-6V(1,3) + 24 - 48 + 24 = 0 \tag{5.23}$$

$$V(1,3) = 0 \tag{5.24}$$

Similarly using (5.10), (5.12) and (5.14) in the equality (5.21) for $k \geq 2$

$$V(k,3) = 0 \tag{5.25}$$

are obtained. When $h = 1$ is written in the equality (5.7) , we have that

$$(k+1)(k+2)(k+3)U(k+3,1) + 6(k+1)(k+2)V(k+2,2) - 18(k+1)U(k+1,3)$$
$$-24V(k,4) + 4(k+1)(k+2)U(k+2,1) - 16(k+1)V(k+1,2) - 24U(k,3) = 0 \tag{5.26}$$

If $k = 0$ is written in the equality (5.26), we get that

$$6U(3,1) + 12V(2,2) - 18U(1,3) - 24V(0,4) + 8U(2,1) - 16V(1,2) - 24U(0,3) = 0 \tag{5.27}$$

Therefore using equalities (5.10),(5.12), (5.14) and (5.19) we obtain

$$V(0,4) = 0 \tag{5.28}$$

If $k=1$ is written in the equality (5.26), we get that

$$24U(4,1)+36V(3,2)-36U(2,3)-24V(1,4)+24U(3,1)-32V(2,2)-24U(1,3)=0 \quad (5.29)$$

Therefore,

$$V(1,4)=0 \quad (5.30)$$

is obtained. If $k=n$ is written in the equality (5.26), then $V(n,4)=0$ is obtained. As a result,

$$U(n,m)=V(n,m)=0 \quad (5.31)$$

are obtained for every $n\geq 0, m>3$. If these finding values writing as follows

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k y^h \quad (5.32)$$

So, following solution

$$w(x,y)=x^3-3xy^2+3x^2-3y^2+i(3x^2y-y^3-6xy)=z^3+3(\overline{z})^2 \quad (5.33)$$

is obtained.

Example 5.2. Solve the following initial value problem

$$2\frac{\partial^3 w}{\partial \overline{z}^3}+\frac{\partial^2 w}{\partial z^2}-\frac{\partial w}{\partial z}-2w=0 \quad (5.34)$$

$$w(x,0)=2e^{2x}+3e^{x} \quad (5.35)$$

$$\frac{\partial w}{\partial y}(x,0)=i(4e^{2x}-3e^{x}) \quad (5.36)$$

$$\frac{\partial^2 w}{\partial y^2}(x,0)=-8e^{2x}-3e^{x} \quad (5.37)$$

Since $w=u+iv$, equation (5.34) is equivalent following system of equation (5.38) - (5.39).

$$\begin{aligned}&\left(\frac{\partial^3 u}{\partial x^3}-3\frac{\partial^3 u}{\partial x\partial y^2}-3\frac{\partial^3 v}{\partial x^2\partial y}+\frac{\partial^3 v}{\partial y^3}\right)\\&+\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial x}-2\frac{\partial v}{\partial y}-8u=0\end{aligned} \quad (5.38)$$

$$\begin{aligned}&\left(\frac{\partial^3 v}{\partial x^3}-3\frac{\partial^3 v}{\partial x\partial y^2}+3\frac{\partial^3 u}{\partial x^2\partial y}-\frac{\partial^3 u}{\partial y^3}\right)\\&+\frac{\partial^2 v}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial y^2}-2\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial y}-8v=0\end{aligned} \quad (5.39)$$

We get from differential transform of (5.38), (5.39) that:

$$\begin{aligned}&(k+1)(k+2)(k+3)U(k+3,h)-3(k+1)(h+1)(h+2)U(k+1,h+2)\\&-3(k+1)(k+2)(h+1)V(k+2,h+1)+(h+1)(h+2)(h+3)V(k,h+3)\\&+(k+1)(k+2)U(k+2,h)+2(k+1)(h+1)V(k+1,h+1)\\&-(h+1)(h+2)U(k,h+2)-2(k+1)U(k+1,h)\\&-2(h+1)V(k,h+1)-8U(k,h)=0\end{aligned} \quad (5.40)$$

$$\begin{aligned}
&(k+1)(k+2)(k+3)V(k+3,h) - 3(k+1)(h+1)(h+2)V(k+1,h+2)\\
&+3(k+1)(k+2)(h+1)U(k+2,h+1) - (h+1)(h+2)(h+3)U(k,h+3)\\
&+(k+1)(k+2)V(k+2,h) - 2(k+1)(h+1)U(k+1,h+1)\\
&-(h+1)(h+2)V(k,h+2) - 2(k+1)V(k+1,h)\\
&+2(h+1)U(k,h+1) - 8V(k,h) = 0
\end{aligned} \tag{5.41}$$

Since

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h \tag{5.42}$$

and by the equality (5.35), following results are obtained.

$$U(0,0) = 5, U(1,0) = 7, U(2,0) = \frac{11}{2}, \ldots, U(n,0) = \frac{2^{n+1}+3}{n!}, \ldots, V(i,0) = 0(i = 0,1,2,3,\ldots) \tag{5.43}$$

Because,

$$\begin{aligned}
w(x,0) &= 2e^{2x} + 3e^x\\
&= 2\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} + 3\sum_{n=0}^{\infty}\frac{x^n}{n!}\\
&= 2(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots) + 3(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)\\
&= 5 + 7x + \frac{11x^2}{2} + \frac{19x^3}{6} + \cdots + \frac{2.(2^n+3)x^n}{n!} + \cdots
\end{aligned} \tag{5.44}$$

Similarly,

$$\frac{\partial w}{\partial y} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} h.W(k,h)x^k y^{h-1} \tag{5.45}$$

and by the equality (5.36), following results are obtained.

$$V(0,1) = 1, V(1,1) = 5, V(2,1) = 13/2, \ldots, V(n,1) = (4.2^n - 3)/n!, \ldots, U(i,1) = 0(i = 0,1,2,\ldots) \tag{5.46}$$

Because,

$$\begin{aligned}
\frac{\partial w}{\partial y}(x,0) &= i(4e^{2x} - 3e^x)\\
&= 4i\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} - 3i\sum_{n=0}^{\infty}\frac{x^n}{n!}\\
&= 4i(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots) - 3i(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)\\
&= i(1 + 5x + \frac{13x^2}{2} + \frac{29x^3}{6} + \cdots + \frac{4.(2^n-3)x^n}{n!} + \cdots)
\end{aligned} \tag{5.47}$$

Finally, from

$$\frac{\partial^2 w}{\partial y^2} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} h.(h-1).W(k,h)x^k y^{(h-2)} \tag{5.48}$$

and the equality (5.37), following results are obtained

$$U(0,2) = -\frac{11}{2}, U(1,2) = -19/2, U(2,2) = -35/4, \ldots, U(n,2) = -(8.2^n+3)/2n!, \ldots$$

$$V(i,2) = 0(i = 0,1,2,\ldots) \tag{5.49}$$

Because,

$$\begin{aligned}\frac{\partial^2 w}{\partial y^2}(x,0) &= -8e^{2x}-3e^{x}\\ &= -8\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}-3\sum_{n=0}^{\infty}\frac{x^n}{n!}\\ &= -8(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\cdots)-3(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)\\ &= -11-19x-\frac{35x^2}{2}-\frac{67x^3}{6}-\ldots-\frac{8.(2^n+3)x^n}{n!}-\ldots \end{aligned} \tag{5.50}$$

When $h=0$ is written in the equality (5.40) , we have that

$$\begin{aligned}&(k+1)(k+2)(k+3)U(k+3,0)-6(k+1)U(k+1,2)\\ &-3(k+1)(k+2)V(k+2,1)+6V(k,3)+(k+1)(k+2)U(k+2,0)\\ &+2(k+1)V(k+1,1)-2U(k,2)-2(k+1)U(k+1,0)-2V(k,1)-8U(k,0)=0\end{aligned} \tag{5.51}$$

We get the following equality by using the equalities (5.43),(5.46) and (5.49) in the equality (5.51).

$$\begin{aligned}&(k+1)(k+2)(k+3)(2^{(k+4)}+3)/(k+3)!+6(k+1)(8.2^{(k+1)}+3)/2(k+1)!\\ &-3(k+1)(k+2)(4.2^{(k+2)}-3)/(k+2)!+6V(k,3)+(k+1)(k+2)(2^{(k+3)}+3)/(k+2)!\\ &+2(k+1)(4.2^{(k+1)}-3)/(k+1)!+2(8.2^{(k+3)}/(2.k!)-2(k+1)(2^{(k+2)}+3)/(k+1)!\\ &-2(4.2^{k}-3)/k!-8(2^(k+1)+3)/k!=0\end{aligned} \tag{5.52}$$

$$\begin{aligned}&(2^{(k+4)}+3)/k!+3(8.2^{(k+1)}+3)/2k!-3(4.2^{(k+2)}-3)/k!+6V(k,3)\\ &+(2^{(k+3)}+3)/k!+2(4.2^{(k+1)}-3)/k!+(8.2^{k}+3)/k!\\ &-2(2^{(k+2)}+3)/k!-2(4.2^{k}-3)/k!-8(2^{(k+1)}+3)/k!=0\end{aligned} \tag{5.53}$$

Therefore we get that:

$$V(k,3)=(16.2^{k}-3)/(-6.k!). \tag{5.54}$$

Similarly, when $h=0$ is written in the equality (5.40) , we have that

$$\begin{aligned}&(k+1)(k+2)(k+3)V(k+3,0)-6(k+1)V(k+1,2)+3(k+1)(k+2)U(k+2,1)\\ &-6U(k,3)+(k+1)(k+2)V(k+2,0)-2(k+1)U(k+1,1)\\ &-2V(k,2)-2(k+1)V(k+1,0)+2U(k,1)-8V(k,0)=0\end{aligned} \tag{5.55}$$

We get the following equality by using the equalities (5.43),(5.46) and (5.49) in the equality (5.55).

$$U(k,3)=0 \tag{5.56}$$

It is clear that we obtain following equalities for every $k,h\in N$ $U(k,2h+1)=0, V(k,2h)=0$,

$$U(k,2h)=\begin{pmatrix}k+2h\\2h\end{pmatrix}\frac{(-1)^h}{(k+2h)!}(2^{k+2h+1}+3) \tag{5.57}$$

$$V(k,2h+1)=\begin{pmatrix}k+2h+1\\2h+1\end{pmatrix}\frac{(-1)^h}{(k+2h+1)!}(2^{k+2h+2}-3) \tag{5.58}$$

If these finding values write in following solution

$$
\begin{aligned}
w(x,y) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h \\
&= U(0,0)+iV(0,0)+[U(1,0)+iV(1,0)]x+[U(2,0)+iV(2,0)]x^2\cdots+[U(n,0)+iV(n,0)] \\
&+ [U(0,1)+iV(0,1)]y+[U(1,1)+iV(1,1)]xy+\cdots+[U(n,1)+iV(n,1)]x^n+\cdots \\
&+ [U(0,2)+iV(0,2)]y^2+[U(1,2)+iV(1,2)]xy^2+\cdots+[U(n,2)+iV(n,2)]x^n y^2+\cdots \\
&= 5+7x+\frac{11}{2}x^2+(19x^3)/6+\cdots+(2.(2^n+3))x^n/n!+\cdots \\
&+ i(1+5x+(13x^2)/2+(29x^3)/6+\cdots+(4.(2^n-3))x^n/n!+\cdots) \\
&+ \frac{1}{2}(-11y^2-19xy^2-35/2x^2y^2-67/6x^3y^2-\ldots-(8.(2^n+3))x^n/n!y^2-\ldots) \\
&+ i(-13/6y^3-29/6xy^3-61/6x^2y^3-\ldots-(16.2^k-3)/(6.k!)-\ldots) \\
&= 5+7x+iy+\frac{11}{2}x^2+5ixy-1\frac{1}{2}y^2+19/6x^3 \\
&+ 13i/2x^2y-19/2xy^2-13i/6y^3+35/24x^4+116i/24x^3y \\
&- 210/24x^2y^2-116i/24xy^3+35/24y^4+\cdots \\
&= 5+7((z+\overline{z})/2)+i((z-\overline{z})/2i)+\frac{11}{2}((z+\overline{z})/2)^2+5i((z+\overline{z})/2)((z-\overline{z})/2i) \\
&- \frac{11}{2}((z-\overline{z})/2i)^2+19/6((z+\overline{z})/2)^3+13i/2((z+\overline{z})/2)^2((z-\overline{z})/2i) \\
&- 19/2((z+\overline{z})/2)((z-\overline{z})/2i)^2-13i/6((z-\overline{z})/2i)^3 \\
&+ 35/24((z+\overline{z})/2)^4+116i/24((z+\overline{z})/2)^3((z-\overline{z})/2i)-210/24((z+\overline{z})/2)^2((z-\overline{z})/2i)^2 \\
&- 116i/24((z+\overline{z})/2)((z-\overline{z})/2i)^3+35/24((z-\overline{z})/2i)^4+\cdots \\
&= 5+7z/2+(7\overline{z})/2+z/2-\overline{z}/2+\frac{11}{2}(z^2+2z\overline{z}+\overline{z}^2)+5/4(z^2-\overline{z}^2)+\frac{11}{8}(z^2-2z\overline{z}+\overline{z}^2) \\
&+ 19/48(z^3+3z^2\overline{z}+3z\overline{z}^2+\overline{z}^3)+13/8(z^2+2z\overline{z}+\overline{z}^2)(z-\overline{z})+19/16(z+\overline{z})(z^2-2z\overline{z}+\overline{z}^2) \\
&+ 13/48(z^3-3z^2\overline{z}+3z\overline{z}^2-\overline{z}^3)+\cdots \\
&= (2+4z+4z^2+(8z^3)/3+(4z^4)/3+\cdots)+(3+3z+(3z^2)/2+z^3/2+\frac{1}{8}z^4+\cdots) \\
&= 2(1+2z+(2z)^2/2!+(2z)^3/3!+(2z)^4/4!+\cdots) \\
&+ 3(1+\overline{z}+(\overline{z})^2/2!+(\overline{z})^3/3!+(\overline{z})^4/4!+\cdots) \\
&= 2e^{2z}+3e^{\overline{z}}
\end{aligned}
$$

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Author information

Murat DÜZ, Department of Mathematics, University of Karabuk,78050 Karabuk, TURKEY.
E-mail: mduz@karabuk.edu.tr

Uğur İLTER, Department of Mathematics, University of Karabuk,78050 Karabuk, TURKEY.
E-mail: ugurilter78@hotmail.com

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