

COUPLED FIXED POINT THEOREM FOR RATIONAL CONTRACTION CONDITIONS IN DISLOCATED QUASI-METRIC SPACE

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Abstract In this article, we have establish a coupled fixed point theorem satisfying rational contraction conditions in dislocated quasi-metric space. In order to validate our establish theorem and corollaries we have provide an example.

1 Introduction and Preliminaries

The concept of dislocated metric space was introduced by Hitzler [1]. In such a space self-distance between points need not to be zero necessarily. They also generalized famous Banach contraction principle in dislocated metric space. Dislocated metric space play a vital role in topology, logical programming, computer science and electronic engineering etc. In 2005, Zeyada, Hassan and Ahmad [2] initiated the notion of complete dislocated quasi-metric space and generalized the result of Hitzler [1] in dislocated quasi-metric space. With the passage of time many papers have been published containing fixed point results for a single and a pair of mappings for different type of contractive conditions in dislocated quasi-metric space (see [3, 4, 5, 6]).

In 2006, Bhaskar and Lakshmikantham [7] initiated the concept of coupled fixed point for non-linear contractions in partially ordered metric spaces. Furthermore, after the work of Bhaskar and Lakshmikantham [7] coupled fixed point theorems have been established by many authors in different type of spaces (see [8, 9, 10]).

In this paper, we have establish a coupled fixed point theorem satisfying rational contraction conditions in the context of dislocated quasi-metric space. An example is given in the support of our main results. Throughout the paper \mathbb{R}^+ represent the set of non-negative real numbers.

Definition[2]. Let X be a non-empty set and let $d : X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the conditions

- $d_1) d(x, x) = 0;$
- $d_2) d(x, y) = d(y, x) = 0$ implies $x = y;$
- $d_3) d(x, y) = d(y, x);$
- $d_4) d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X.$

If d satisfy the conditions from d_1 to d_4 then it is called metric on X . If d satisfy conditions d_2 to d_4 then it is called dislocated metric (d -metric) on X and if d satisfy conditions d_2 and d_4 only then it is called dislocated quasi-metric (dq -metric) on X . The pair (X, d) is called dislocated quasi-metric space.

Clearly every metric space is a dislocated metric space but the converse is not necessarily true as clear form the following example:

Example. Let $X = \mathbb{R}^+$ define the distance function $d : X \times X \rightarrow \mathbb{R}^+$ by

$$d(x, y) = \max\{x, y\}$$

Clearly X is dislocated metric space but not a metric space.

Also every metric and dislocated metric spaces are dislocated quasi-metric spaces but the converse is not true.

Example. Let $X = \mathbb{R}$ define the distance function $d : X \times X \rightarrow \mathbb{R}^+$ by

$$d(x, y) = |x| \quad \text{for all } x, y \in X.$$

Evidently X is dq -metric space but not a metric space nor dislocated metric space.

In our main work we will use the following definitions which can be found in [2].

Definition. A sequence $\{x_n\}$ is called dislocated quasi convergent (dq -convergent) in X if for $n \in N$ we have

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0.$$

In this case x is called dislocated quasi limit (dq -limit) of the sequence $\{x_n\}$.

Definition. A sequence $\{x_n\}$ in dq -metric space is called Cauchy sequence if for $\epsilon > 0$ there exists a positive integer n_0 such that for $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$ i.e

$$\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0.$$

Definition. A dq -metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X .

Definition[7]. An element $(x, y) \in X^2$ is called coupled fixed point of the mapping $T : X \times X \rightarrow X$ if $T(x, y) = x$ and $T(y, x) = y$ for $x, y \in X$.

Example. Let $X = \mathbb{R}$ and $T : X \times X \rightarrow X$ defined by

$$T(x_1, x_2) = \frac{x_1 x_2}{2}.$$

Here $(0, 0)$ is the coupled fixed point of T .

The following well-known results can be seen in [2].

Lemma 1.1. *Limit of a convergent sequence in dq - metric space is unique.*

Theorem 1.2. *Let (X, d) be a complete dq -metric space $T : X \rightarrow X$ be a contraction. Then T has a unique fixed point.*

Remark.

- It is obvious that the the following statement hold.
for real numbers a, b and c ,
if $a < b$ and $c > 0$. Then $ac < bc$.

2 Main Results

Theorem 2.1. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \times X \rightarrow X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha \cdot [d(x, u) + d(y, v)] + \beta \cdot \frac{d(x, T(x, y)) \cdot d(x, T(u, v))}{1 + d(x, u) + d(y, v)} + \gamma \cdot \frac{d(x, T(x, y)) \cdot d(u, T(u, v))}{1 + d(x, u)} \quad (2.1)$$

for all $x, y, u, v \in X$ and α, β and γ are non-negative constants with $2(\alpha + \beta) + \gamma < 1$. Then T has a unique coupled fixed point in $X \times X$.

proof. Let x_0 and y_0 are arbitrary in X , we define the sequences $\{x_n\}$ and $\{y_n\}$ as following,

$$x_{n+1} = T(x_n, y_n) \quad \text{and} \quad y_{n+1} = T(y_n, x_n) \quad \text{for } n \in N.$$

Consider

$$d(x_n, x_{n+1}) = d(T(x_{n-1}, y_{n-1}), T(x_n, y_n))$$

Now by (2.1) we have

$$d(x_n, x_{n+1}) \leq \alpha \cdot [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \beta \cdot \frac{d(x_{n-1}, T(x_{n-1}, y_{n-1})).d(x_{n-1}, T(x_n, y_n))}{1 + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)} \\ + \gamma \cdot \frac{d(x_{n-1}, T(x_{n-1}, y_{n-1})).d(x_n, T(x_n, y_n))}{1 + d(x_{n-1}, x_n)}.$$

Using the definition of the sequences $\{x_n\}$ and $\{y_n\}$ we have

$$\leq \alpha \cdot [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \beta \cdot \frac{d(x_{n-1}, x_n).d(x_{n-1}, x_{n+1})}{1 + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)} \\ + \gamma \cdot \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}.$$

Simplifying and using Remark 1 we have

$$< \alpha \cdot [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \beta \cdot d(x_{n-1}, x_{n+1}) + \gamma \cdot d(x_n, x_{n+1}) \\ \leq \alpha \cdot [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \beta \cdot [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \gamma \cdot d(x_n, x_{n+1}).$$

Simplification yeilds

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \beta}{1 - (\beta + \gamma)} d(x_{n-1}, x_n) + \frac{\alpha}{1 - (\beta + \gamma)} d(y_{n-1}, y_n). \quad (2.2)$$

Similarly we can show that

$$d(y_n, y_{n+1}) \leq \frac{\alpha + \beta}{1 - (\beta + \gamma)} d(y_{n-1}, y_n) + \frac{\alpha}{1 - (\beta + \gamma)} d(x_{n-1}, x_n). \quad (2.3)$$

Adding (2.2) and (2.3) we have

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \leq \frac{2\alpha + \beta}{1 - (\beta + \gamma)} [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)].$$

Since $2\alpha + 2\beta + \gamma < 1$, so $h = \frac{2\alpha + \beta}{1 - (\beta + \gamma)} < 1$. Therefore the above inequality becomes,

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \leq h \cdot [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)].$$

Also

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \leq h^2 \cdot [d(x_{n-2}, x_{n-1}) + d(y_{n-2}, y_{n-1})].$$

Similarly proceeding we have

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \leq h^n \cdot [d(x_0, x_1) + d(y_0, y_1)].$$

Since $h < 1$ taking limit $n \rightarrow \infty$ we have

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \rightarrow 0.$$

Implies

$$d(x_n, x_{n+1}) \rightarrow 0 \text{ and } d(y_n, y_{n+1}) \rightarrow 0.$$

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in complete dislocated quasi-metric space X . So there must exist $w, z \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = w \text{ and } \lim_{n \rightarrow \infty} y_n = z.$$

Also since T is continuous and $T(x_n, y_n) = x_{n+1}$ so taking limit $n \rightarrow \infty$. We have

$$\lim_{n \rightarrow \infty} T(x_n, y_n) = \lim_{n \rightarrow \infty} x_{n+1}$$

$$T(\lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n) = \lim_{n \rightarrow \infty} x_{n+1}$$

$$T(w, z) = w.$$

Also from $T(y_n, x_n) = y_{n+1}$. We can show that

$$d(z, w) = z.$$

Thus $(w, z) \in X \times X$ is the coupled fixed point of T in X .

Uniqueness. Let (w, z) and (w', z') are two distinct coupled fixed points of T in $X \times X$. Then by use of (2.1) we have

$$\begin{aligned} d(w, w) &= d(T(w, z), T(w, z)) \leq \alpha \cdot [d(w, w) + d(z, z)] + \\ &\beta \cdot \frac{d(w, T(w, z)) \cdot d(w, T(w, z))}{1 + d(w, w) + d(z, z)} + \gamma \cdot \frac{d(w, T(w, z)) \cdot d(w, T(w, z))}{1 + d(w, w)} \\ &\leq \alpha \cdot [d(w, w) + d(z, z)] + \beta \cdot \frac{d(w, w) \cdot d(w, w)}{1 + d(w, w) + d(z, z)} + \gamma \cdot \frac{d(w, w) \cdot d(w, w)}{1 + d(w, w)}. \end{aligned}$$

Using Remark 2.1 and then simplifying we have

$$d(w, w) \leq (\alpha + 2\beta + \gamma)d(w, w) + \alpha d(z, z). \tag{2.4}$$

Similarly we can show that

$$d(z, z) \leq (\alpha + 2\beta + \gamma)d(z, z) + \alpha d(w, w). \tag{2.5}$$

Adding (2.4) and (2.5) we have

$$[d(w, w) + d(z, z)] \leq (2\alpha + 2\beta + \gamma)[d(w, w) + d(z, z)].$$

Since $2\alpha + 2\beta + \gamma < 1$ so the above inequality is possible only if

$$[d(w, w) + d(z, z)] = 0.$$

Implies

$$d(w, w) = d(z, z) = 0. \tag{2.6}$$

Now consider

$$\begin{aligned} d(w, w') &= d(T(w, z), T(w', z')) \leq \alpha \cdot [d(w, w') + d(z, z')] \\ &+ \beta \cdot \frac{d(w, T(w, z)) \cdot d(w, T(w', z'))}{1 + d(w, w') + d(z, z')} + \gamma \cdot \frac{d(w, T(w, z)) \cdot d(w', T(w', z'))}{1 + d(w, w')} \\ &\leq \alpha \cdot [d(w, w') + d(z, z')] + \beta \cdot \frac{d(w, w) \cdot d(w, w')}{1 + d(w, w') + d(z, z')} + \gamma \cdot \frac{d(w, w) \cdot d(w', w')}{1 + d(w, w')}. \end{aligned}$$

Now using (2.6) we have the following

$$d(w, w') \leq \alpha \cdot [d(w, w') + d(z, z')]. \tag{2.7}$$

By following similar procedure we can get

$$d(z, z') \leq \alpha \cdot [d(z, z') + d(w, w')]. \tag{2.8}$$

Adding (2.7) and (2.8) we have

$$d(w, w') + d(z, z') \leq 2\alpha \cdot [d(w, w') + d(z, z')].$$

Since $2\alpha < 1$ so the above inequality is possible only if

$$d(w, w') + d(z, z') = 0.$$

Which implies that

$$d(w, w') = d(z, z') = 0.$$

Implies

$$w = w' \text{ and } z = z'.$$

Hence

$$(w, z) = (w', z').$$

Thus coupled fixed point of T in $X \times X$ is unique.

We deduce the following corollaries from Theorem 2.1.

Corollary 2.2. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \times X \rightarrow X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha \cdot [d(x, u) + d(y, v)] + \beta \cdot \frac{d(x, T(x, y)) \cdot d(x, T(u, v))}{1 + d(x, u) + d(y, v)}$$

for all $x, y, u, v \in X$ and α, β are non-negative constants with $2(\alpha + \beta) < 1$. Then T has a unique coupled fixed point in $X \times X$.

Corollary 2.3. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \times X \rightarrow X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha \cdot [d(x, u) + d(y, v)] + \beta \cdot \frac{d(x, T(x, y)) \cdot d(u, T(u, v))}{1 + d(x, u)}$$

for all $x, y, u, v \in X$ and α, β are non-negative constants with $2\alpha + \beta < 1$. Then T has a unique coupled fixed point in $X \times X$.

Corollary 2.4. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \times X \rightarrow X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x, y), T(u, v)) \leq \alpha [d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ and $\alpha > 0$ with $2\alpha < 1$. Then T has a unique coupled fixed point in $X \times X$.

Example. Let $X = [0, 1]$. Define $d : X \times X \rightarrow \mathbb{R}^+$ by

$$d(x, y) = |x - y| + |x|$$

for all $x, y \in X$. Then (X, d) is a complete dislocated quasi-metric space. Define a continuous self-map $T : X \times X \rightarrow X$ by $T(x, y) = \frac{1}{6}xy$ for all $x, y \in X$. Since

$$|xy - uv| \leq |x - u| + |y - v| \text{ and } |xy| \leq |x| + |y|.$$

Hold for all $x, y, u, v \in X$. Then

$$\begin{aligned} d(T(x, y), T(u, v)) &= \left| \frac{1}{6}xy - \frac{1}{6}uv \right| + \left| \frac{1}{6}xy \right| \\ &\leq \frac{1}{6}(|x - u| + |y - v|) + \frac{1}{6}(|x| + |y|) \\ &\leq \frac{1}{6}(|x - u| + |y - v| + |x| + |y|) \\ &\leq \frac{1}{3}[(|x - u| + |x|) + (|y - v| + |y|)] \\ d(T(x, y), T(u, v)) &\leq \alpha \cdot [d(x, u) + d(y, v)]. \end{aligned}$$

So for $\alpha = \frac{1}{3}$ and $\beta = \gamma = 0$ all the conditions of Theorem 2.1 are satisfied having $(0, 0) \in X \times X$ is the unique coupled fixed point of T in $X \times X$.

References

- [1] P. Hitzler, Generalized metrics and topology in logic programming semantics, Ph.D Thesis, National Univeristy of Ireland, University College Cork, (2001).
- [2] F.M. Zeyada, G.F. Hassan and M.A. Ahmad, A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space, *Arabian J. Sci. Engg.*, **31**(2005), 111-114.
- [3] C.T. Aage and J.N. Salunke, Some results of fixed point theorem in dislocated quasi-metric space, *Bulletin of Marathadawa Mathematical Society*, **9**(2008), 1-5.
- [4] M. Sarwar, M.U. Rahman and G. Ali, Some fixed point results in dislocated quasi-metric (*dq*-metric) spaces, *Journal of Inequalities and Applications*, **278**:2014(2014), 1–11.
- [5] M.U. Rahman and M. Sarwar, Fixed point results in dislocated quasi-metric spaces, *International Mathematical Forum*, **9**(2014), 677-682.
- [6] M.U. Rahman and M. Sarwar, Fixed point results for some new type of contraction conditions in dislocated quasi-metric space, *International Journal of Mathematics and Scientific Computing*, **4**(2014), 68-71.
- [7] TG. Bhaskar and V. Lakshmikantham, Fixed point Ttheorem in partially ordered metric spaces and applications, *Non-linear Analysis Theory and Applications*, **65**(2006), 1379-1393.
- [8] D. Akcay and C. Alaca, Coupled fixed point theorem in dislocated quasi-metric space, *Journal of Advance Studies in Topology*, **4**(2012), 66-72.
- [9] Z.M. Fadil and A.G. Bin Ahmad, Coupled Ccincidence point and common coupled fixed point results in cone *b*-metric spaces, *Fixed Point Theorey and Applications*, (2010) **2013:177**.
- [10] J. Kumar and S. Vashistha, Coupled fixed point theorem for generalized contraction in complex valued metric spaces, *Int. Journal of Computer Applications*, **83**(2013), 36-40.

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