Palestine Journal of Mathematics

Vol. 5(2) (2016) , 175–182

THE PSEUDO DIFFERENTIAL-TYPE OPERATOR $(-x^{-1}D)^{(\alpha-\beta)}$ ASSOCIATED WITH THE FOURIER-BESSEL TYPE SERIES REPRESENTATION

B.B.Waphare and S.G.Gajbhiv

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 47 G 30, 26 A 33; Secondary 46 F 12.

Keywords and phrases: Pseudo differential-type operator, Hankel-type transform, Open mapping theorem, almost inverse.

Abstract. For certain Frechet space F consisting of complex valued C^{∞} functions defined on $I = (0, \infty)$ and characterized by their asymptotic behavior near the boundaries, we show that :

(i) The Pseudo differential operator $(-x^{-1}D)^{(\alpha-\beta)}$, $(\alpha-\beta) \in R, D = \frac{d}{dx}$, is an automorphism (in the topological sense) on F.

(ii) $(-x^{-1}D)^{(\alpha-\beta)}$ is almost an inverse of the Hankel type transform $h_{\alpha,\beta}$ in the sense that $h_{\alpha,\beta} \circ (x^{-1}D)^{(\alpha-\beta)}(\phi) = h_{\circ}(\phi)$,for all $\phi \in F$ and $(\alpha - \beta) \in R$

(iii) $(-x^{-1}D)^{(\alpha-\beta)}$ has a Fourier-Bessel type series representation on a subspace $F_b \subset F$ and also on its dual F'_b.

1 Introduction

The theory of pseudo differential operators has been developed by many researchers in India and abroad. In recent years pseudo differential operators involving Hankel transform, Hankel convolution, Bessel operators etc. has been studied by many mathematicians. It is the purpose of this paper to give the Fourier-Bessel type series representation of the pseudo differential type operator $(-x^{-1}D)^{(\alpha-\beta)}$.

We denote by F the space of all C^{∞}- complex valued functions $\phi(x)$ defined on $I = (0, \infty)$, such that

$$\phi(x) = \sum_{i=0}^{k} a_i x^{2i} + O(x^{2k}) \tag{1.1}$$

near the origin and is rapidly decreasing as $x \to \infty$.

For $(\alpha - \beta) > -1/2$, we define a $(\alpha - \beta)^{th}$ order Hankel-type transform $h_{\alpha,\beta}$ on F by

$$\Phi(y) = [h_{\alpha,\beta}\phi(x)](y) = \int_0^{\infty} \phi(x)\mathcal{J}_{\alpha,\beta}(xy)dm(x) \tag{1.2}$$

where

$$dm(x) = m'(x)dx = \left[2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right]^{-1} x^{4\alpha}dx,$$

$$\mathcal{J}_{\alpha,\beta}(x) = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)x^{-(\alpha-\beta)}J_{\alpha-\beta}(x),$$

and $J_{\alpha-\beta}(x)$ is the Bessel-type function of order $(\alpha-\beta)$. The inversion formula for (1.2) is given by [1, 3, 4],

$$\phi(x) = \int_0^\infty \Phi(y)\mathcal{J}_{\alpha,\beta}(xy)dm(y). \tag{1.3}$$

In the present paper we will show that for every real $(\alpha-\beta)$:

(i) The pseudo differentialâĂŞtype operator $(-x^{-1}D)^{(\alpha-\beta)}$ is a topological automorphism on F.

(ii) The Hankel-type transform $h_{\alpha,\beta}$ is also an automorphism on F.

(iii) On F,$(-x^{-1}D)^{(\alpha-\beta)}$ is almost an inverse of $h_{\alpha,\beta}$ in the sense that $[h_{\alpha,\beta}\circ(-x^{-1}D)^{(\alpha-\beta)}](\phi) = h\circ(\phi), \phi\in F$.

(iv) On a certain subspace $F_b \subset F$ and on its dual F_b', $(-x^{-1}D)^{(\alpha-\beta)}$ has Fourier-Bessel type series representation.

All automorphisms are topological automorphisms in the sequel.

2 Notations and Terminology

For any real number $(\alpha-\beta) \neq -1/2, F_{\alpha,\beta}$ is the space of all C^∞- complex valued functions $\phi(x)$ defined on I such that

$$\rho_{m,k}^{\alpha,\beta}(\phi) = Sup_{x\in I}\left|x^m \bigtriangleup_{\alpha,\beta,x}^k \phi(x)\right| < \infty \tag{2.1}$$

For each $m,k = 0,1,2\dots$ where $\bigtriangleup_{\alpha,\beta,x}= D^2 + x^{-1}(4\alpha)D$.
We can easily note that $F_{\alpha,\beta}$ is a Frechet space. Its topology is generated by the countable family of separating seminorms $\left\{\rho_{m,k}^{\alpha,\beta}\right\}_{m,k=0,1,2}$.
By Lee [3, Theorem 2.1(i), page 429], we have $F_{\alpha,\beta} = F_{a,b} = F$ (as a set) for each $(\alpha-\beta),(a-b)(\neq -1/2) \in R$. Thus for each $(\alpha-\beta)(\neq -1/2)$, we have a topology $T_{\alpha-\beta}$ on F generated by the countable family of seminorms $\rho_{m,k}^{\alpha,\beta}$. Hence $(F,T_{\alpha-\beta})$ is a Frechet space. When $(\alpha-\beta) = -1/2, F_{-1/2} \neq F$, since the factor $x^{-1}(4\alpha)D$ in $\bigtriangleup_{\alpha,\beta,x}$ responsible for the even nature of $\phi(x)\in F_{\alpha,\beta}(x)$ near the origin, vanishes. For example $e^{-x}\in F_{-1/2}$ but $e^{-x}\notin F_{\alpha,\beta}$.
Following Zemanian [7, 8] we define Hankel-type transform $\overline{h}_{\alpha,\beta}$ with $(\alpha-\beta)\geq -1/2$ by

$$\Psi(y) = \left[\overline{h}_{\alpha,\beta}\psi(x)\right](y) = \int_0^\infty \psi(x)(xy)^{\alpha+\beta}J_{\alpha-\beta}(xy)dx \tag{2.2}$$

It can be easily proved that $\overline{h}_{\alpha,\beta}$ is an automorphism on the space $H_{\alpha,\beta}$ that consists of complex valued C^∞ functions defined on I and satisfies the relation

$$\overline{\rho}_{m,k}^{\alpha,\beta}(\psi) = Sup_{x\in I}\left|x^m(x^{-1}D)^k\left[x^{2\beta-1}\psi(x)\right]\right| < \infty \tag{2.3}$$

for each $m,k = 0,1,2\dots$ where $D = d/dx$.
Now we will need following theorem which is an important result for the development of our theory.

Theorem 2.1. *Let* $(\alpha-\beta),(a-b)\neq -1/2$*;* a,b *are real numbers. Then*

(i) The operation $\phi\to x^{2\alpha}\phi$ *is an homeomorphism from* F *onto* $H_{\alpha,\beta}$

(ii) $\left(x^{-1}D\right)^n : F\to F$ *is an automorphism of* F.

(iii) (F,T_{a-b}) *and* $(F,T_{\alpha-\beta})$ *are equivalent topological spaces.*

(iv) $h_{\alpha,\beta}(\phi) = (-1)^n\left[h_{\alpha,\beta,n}\left(x^{-1}D\right)^n\right](\phi), \phi\in F, (\alpha-\beta)\geq -1/2, n = 0,1,2\dots.$

Notation: In view of Theorem 2.1((i),(ii)), we will always write $x^{2\alpha}\phi = \overline{\phi(x)} \in H_{\alpha,\beta}$ for $\phi \in F$ and drop the suffix, $(\alpha-\beta)$ from $T_{\alpha-\beta}$. So henceforth the topological linear space (F,T) will be denoted by F.

Proof of Theorem 2.1:(i)We use induction on n and noting that

$$\triangle_{\alpha,\beta,x} = x^2(x^{-1}D)^2 + 2(3\alpha+\beta)(x^{-1}D) \tag{2.4}$$

It can be proved that

$$\triangle^n_{\alpha,\beta,x} = x^{2n}(x^{-1}D)^{2n} + a_1 x^{2(n-1)}(x^{-1}D)^{2n-1} + \cdots + a_n(x^{-1}D)^n \tag{2.5}$$

where a_i are the constants depending on $\alpha-\beta$. Now $\phi \in F$ if and only if $\overline{\phi} \in H_{\alpha,\beta}$ (see remark II of Lee [3]) and taking $a_0 = 1$, it follows from (2.5) that

$$\rho^{\alpha,\beta}_{m,k}(\phi) \leq \sum_{i=k}^{2k} a_{2k-i}\overline{\rho}^{\alpha,\beta}_{m+2(i-k),i}(\overline{\phi}) \tag{2.6}$$

Proving the continuity of the inverse operation $\overline{\phi} \to x^{2\beta-1}\phi$. By using open mapping theorem [6, page.172], F being Frechet space, we complete the proof.
(ii) Letϕ_j be a sequence tending to zero in F. Then $\phi_j \to 0$ in $H_{\alpha,\beta}$ for arbitrary $(\alpha-\beta) \neq -1/2$. Hence

$$\rho^{\alpha,\beta}_{m,k}\left[(x^{-1}D)^n\phi_j(x)\right] \leq \sum_{i=k}^{2k} a_{2k-i}\overline{\rho}^{\alpha,\beta}_{m+2(i-k),i+n}\overline{\phi}_j \to 0,\ \text{as } j \to \infty (by (2.6))$$

Now it remains to be shown that $(x^{-1}D)^n$ is bijective. It is enough to prove this for $n = 1$. So, let $x^{-1}D\phi_1(x) = x^{-1}D\phi_2(x)$ for $\phi_1, \phi_2 \in F$. Hence $\phi_1(x) - \phi_2(x) = constant$. But $\phi_1(x)$and $\phi_2(x)$ are of rapid descent as $x \to \infty \Rightarrow \phi_1 = \phi_2(x)$. Now let $\psi \in F$, then $\phi(x) = -\int_x^\infty t\psi(t)dt$, defined uniquely (since ψ is of rapid descent as $x \to \infty$) in F, is such that $x^{-1}D\phi(x) = \psi(x)$. Thus we see that $(x^{-1}D)^n$ is a continuous bijection on F. The space F being a Frechet space, the Open Mapping Theorem shows that$(x^{-1}D)^n$ is a bicontinuous bijection on $(F, T_{\alpha-\beta})$ for each $(\alpha-\beta) \in R - \{1/2\}$.
(iii)Let $\alpha - \beta = a - b + d, d \in R$ and ϕ_n be a sequence tending to zero in (F, T_{a-b}). Then

$$\begin{aligned}
\rho^{\alpha,\beta}_{m,k}(\phi_n) &= Sup_{x\in I}\left|x^m\left[\triangle_{a,b,x} + 2d(x^{-1}D)\right]^k\phi_n(x)\right| < \infty \\
&\leq Sup_{x\in I}x^m\left[\left|\sum_{i=0}^{k}\triangle^{k-i}_{a,b,x}(2dx^{-1}D)^i\phi_n(x)\right|\right. \\
&+ \sum_{i=0}^{k}\left|(2dx^{-1}D)^{k-i}\triangle^i_{a,b,x}\phi_n(x)\right| \\
&+ \text{terms of type}\left|\triangle^{i_1}_{a,b,x}(2dx^{-1}D)^{i_2}\triangle^{i_3}_{a,b,x}\cdots\phi_n(x)\right| \\
&\left.\text{and}\left|(2dx^{-1}D)^{j_1}\triangle^{j_2}_{a,b,x}(2dx^{-1}D)^{j_3}\cdots\phi_n(x)\right|\right]
\end{aligned}$$

(where $i_1 + i_2 + i_3 + \cdots = j_1 + j_2 + j_3 + \cdots = k) \to 0$ as $n \to \infty$, for each $m, k = 0, 1, 2, \ldots$
Since $\triangle^i_{a,b,x}$ and $(x^{-1}D)^i$ are continuous on (F, T_{a-b}).
This follows from integration by parts and induction on n.

Remark 2.2. It can be shown that on F

$$\triangle^k_{\alpha,\beta,x} \circ (x^{-1}D)^n = (x^{-1}D)^n \circ \triangle^k_{\alpha,\beta-n,x}$$

This proof follows by induction on k.

Definition 2.3. In view of Theorem 2.1(iv), we define the Hankel-type transform $h_{\alpha,\beta}$ formally for any $(\alpha-\beta)\in R$ as

$$h_{\alpha,\beta}(\phi)=h_{\alpha,\beta,n}\circ(x^{-1}D)^n\phi,\quad \phi\in F \tag{2.7}$$

where n is chosen such that $\alpha-\beta+n>-1/2$.
This is well defined definition as $(x^{-1}D)^n$ is an automorphism.

Definition 2.4. Let F' be the dual space of F.Then for $f\in F'$ define the generalized Hankel-type transform $h_{\alpha,\beta}f=\hat{f}$ of f by

$$\langle h_{\alpha,\beta}f,h_{\alpha,\beta}\phi\rangle=\langle f,\phi\rangle\,,\phi\in F,(\alpha-\beta)\in R$$

Theorem 2.5. *For $(\alpha-\beta)\in R, h_{\alpha,\beta}$ is an automorphism on F and hence on F'.*

Proof: Let $\phi(x)\in F$. Then

$$h_{\alpha,\beta}(\phi)=\Phi(y)=\int_0^\infty(x^{-1}D)^{2n}\phi(x)\mathcal{J}_{\alpha,\beta,2n}(xy)dm(x)=y^{2\beta-1}\overline{h}_{a,b}\left(\overline{\psi}(x)\right)(y), \tag{2.8}$$

where $a-b=\alpha-\beta+2n>-1/2$ and $\overline{\psi}(x)=x^{2a}\psi(x)=x^{2a}(x^{-1}D)^{2n}\phi(x)$.
Let $\phi_n(x)\to 0$ in $F\Rightarrow\overline{\psi}_m(x)\to 0$ in $H_{a,b}$
$\Rightarrow\overline{h}_{a,b}(\overline{\psi}_m)\to 0$ in $H_{a,b}$
$\Rightarrow h_{a,b}(\phi_m)\to 0$ in F.
Now $\overline{h}_{a,b}$,the Hankel-type transform being bijective , (2.7) shows that $h_{\alpha,\beta}$is a bijection. Finally by making use of the Open Mapping Theorem we can complete the proof.
Writing $\alpha=\beta=0$, in (2.7) and $h_{0,0}=h_0$ in definition(2.1) we get

$$h_0(\phi)=h_n\circ(-x^{-1}D)^n\phi(x),\qquad\phi\in F.$$

The above equation motivates us to propose the following definition.

Definition 2.6. For $(\alpha-\beta)\in R$, define $(-x^{-1}D)^{\alpha-\beta}$ by

$$(-x^{-1}D)^{\alpha-\beta}(\phi)=h^{-1}_{\alpha,\beta}\circ h_0(\phi), \tag{2.9}$$

Then $(-x^{-1}D)^{\alpha-\beta}$ is clearly an automorphism on F for each real $(\alpha-\beta)$. From equation (2.9) we get

$$(-x^{-1}D)^{\alpha-\beta}\phi(x)=\int_0^\infty dm(y)\mathcal{J}_{\alpha,\beta}(xy)\int_0^\infty dm\phi(x)\mathcal{J}_0(xy) \tag{2.10}$$

For distribution $f\in F'$, define $(-x^{-1}D)^{\alpha-\beta}$ by

$$\left\langle(-x^{-1}D)^{\alpha-\beta}f,\phi\right\rangle=\left\langle f,(-x^{-1}D)^{\alpha-\beta}\phi\right\rangle,\phi\in F$$

Now we modify theorem 2.1 (ii) to give our main result.

Theorem 2.7. *The pseudo differential operator $(-x^{-1}D)^{\alpha-\beta}$ is an automorphism on F and hence on F' for each $(\alpha-\beta)\in R$.*

3 The Fourier-Bessel type Series expansion of $(-x^{-1}D)^{\alpha-\beta}$

Equation (2.10) gives the integral representation of the operator $(-x^{-1}D)^{\alpha-\beta}$. To get the Fourier-Bessel type series expansion, we modify our leading function space F suitably as follows (similar to the ones as in Zemanian [7, 9]).
For $b>0$, define

$$F_b=\{\phi\in F|\phi=0\text{ for }\quad x>b\} \tag{3.1}$$

The topology of F_b is generated by a countable family of seminorms

$$\rho_k^{\alpha,\beta}(\phi)=Sup_{0<x<b}\left|\Delta^k_{\alpha,\beta,x}\,\phi(x)\right|<\infty\qquad k=0,1,2\ldots \tag{3.2}$$

Clearly all the topologies obtained by choosing different $(\alpha-\beta)$'s are equivalent.

Remark 3.1. Without loss of generality, we may take $(\alpha-\beta) > -1/2$.

Definition 3.2. We define finite Hankel type transform $h_{\alpha,\beta}$ by

$$\Phi(z) = [h_{\alpha,\beta}\phi](z) = \int_0^b \phi(x)\mathcal{J}_{\alpha,\beta}(xz)dm(x) \tag{3.3}$$

Then $\Phi(z)$ is an even entire function by Griffith Theorem [2, 9]. Let $z = y + iw$ and $G_b = \{\Phi(z)|\Phi(z)$is an even entire function satisfying (3.4)$\}$.

$$\eta_b^k(\Phi) = Sup_{z\in\mathbb{C}}\left|e^{-b|w|}z^{2k}\Phi(z)\right| < \infty \tag{3.4}$$

for $k = 0, 1, 2 \ldots$. Then G_b is a linear topological space with η_b^k as seminorms.
Both spaces F_b and G_b are Hausdorff locally convex topological linear spaces satisfying the axiom of first countability. They are sequentially complete spaces.

Theorem 3.3. *$h_{\alpha,\beta}$ is an homeomorphism from F_b onto G_b.*

Proof: Let $\phi \in F_b$. Then

$$\Phi(z) = h_{\alpha,\beta,2m}\left[(-x^{-1}D)^{2m}\phi(x)\right] \quad m \in \mathbb{N}$$

Hence

$$z^{2m}\Phi(z) = \int_0^b x^{4\alpha+2m}\left[(-x^{-1}D)^{2m}\phi(x)\right](xz)^{-(\alpha-\beta)}J_{\alpha-\beta+2m}(xz)dz$$

From the asymptotic formula

$$J_{\alpha-\beta)}(z) \sim \sqrt{2/\pi z}\,\cos\left(z - \frac{(\alpha-\beta)\pi}{2} - \frac{\pi}{4}\right) \quad |z| \to \infty, |argz| < \pi$$

and from the fact that $z^{-(\alpha-\beta)}J_{\alpha-\beta+m}(z)$ is an entire function, it follows that for all x, z,

$$\left|e^{-b|w|}(xz)^{-(\alpha-\beta)}J_{\alpha-\beta+2m}(xz)\right| < C_{m(\alpha-\beta)} \qquad \text{(a constant)}$$

Thus

$$\eta_b^m(\Phi) \le C_{m(\alpha-\beta)}b^{2(m+3\alpha+\beta)}\rho_0^{\alpha,\beta}\left[(x^{-1}D)^{2m}\phi(x)\right] < \infty \tag{3.5}$$

$(x^{-1}D)^{2m}$ being an automorphism (also on F_b), (3.5) implies the continuity of $h_{\alpha,\beta}$. $h_{\alpha,\beta}$ is clearly injective. For any $\Phi(z) \in G_b$, take

$$\phi(x) = \int_0^\infty \Phi(y)\mathcal{J}_{\alpha,\beta}(xz)dm(y)$$

Then it follows from Griffith Theorem [2] that ϕ is zero almost everywhere for $x > b$. Also

$$\begin{aligned}\rho_k^{\alpha,\beta}(\phi) &= Sup_{0<x<b}\left|\triangle_{\alpha,\beta,x}^k\int_0^\infty \Phi(y)\mathcal{J}_{\alpha,\beta}(xz)dm(y)\right| \\ &= Sup_{0<x<b}\left|\int_0^\infty \Phi(y)(-1)^k y^{4\alpha+2k}(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)dy\right| < \infty,\end{aligned}$$

Since $\triangle_{\alpha,\beta,x}^k\left[(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)\right] = (-1)^k y^{2k}(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)$, $\Phi(y)$ is of rapid descent as $y \to \infty$, and $\left[(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)\right]$ is bounded for $0 < y < \infty$. Therefore, $\phi \in F_b$. Hence $h_{\alpha,\beta}$ is surjective. Now the Open Mapping Theorem completes the proof.

Theorem 3.4. *Let $\phi \in F_b$. Then*

$$\phi(x) = \lim_{\epsilon\to 0+}\frac{2}{b^2}\sum_{n=1}^\infty \lambda_\epsilon(x)\left(\frac{\lambda_n}{x}\right)^{\alpha-\beta}\frac{J_{\alpha-\beta}(x\lambda_n)}{J_{3\alpha+\beta}^2(b\lambda_n)}\Phi(\lambda_n) \tag{3.6}$$

where the λ_n's are the positive roots of $J_{\alpha-\beta}(bz)=0$ arranged in the ascending order and for $0<\epsilon<b/4$,

$$\lambda_\epsilon(x)=\begin{cases} E\left(\frac{x}{2\epsilon}\right) & 0<x<2\epsilon \\ 1 & 2\epsilon\le x\le b-2\epsilon \\ 1-E\left(\frac{x-b+2\epsilon}{2\epsilon}\right) & b-2\epsilon<x<b \\ 0 & x\ge b \end{cases}$$

and

$$E(u)=\frac{\int_0^u e^{\frac{1}{x(x-1)}}\,dx}{\int_0^1 e^{\frac{1}{x(x-1)}}\,dx}$$

Proof: By following [5] proof can be completed. Theorem 3.4 gives the required Fourier-Bessel type series expansion for the pseudo differential type operator $(-x^{-1}D)^{\alpha-\beta}$, which we obtain in the following.

Theorem 3.5 (The Fourier-Bessel type series). *For $\phi\in F_b$, we have*

$$\left[(-x^{-1}D)^{\alpha-\beta}\right]\phi(x)=\lim_{\epsilon\to 0+}\frac{2}{b^2}\sum_{n=1}^{\infty}\lambda_\epsilon(x)\left(\frac{\lambda_n}{x}\right)^{\alpha-\beta}\frac{J_{\alpha-\beta}(x\lambda_n)}{J^2_{3\alpha+\beta}(b\lambda_n)}\Phi_0(\lambda_n) \tag{3.7}$$

where

$$\Phi_0(y)=h_0[\phi(x)](y)$$

Proof: Equation (2.9) along with Theorem 3.4 gives the required proof.
Note that,

$$\left|\lambda_n^{2\alpha}\Phi_0(\lambda_n)\right|\le A_{k(\alpha-\beta)}\lambda_n^{2\alpha-2k}$$

$A_{k(\alpha-\beta)}$ constant and $\dfrac{J_{\alpha-\beta}(x\lambda_n)}{x^{\alpha-\beta}\lambda_n^{1/2}J^2_{3\alpha+\beta}(b\lambda_n)}$ is smooth and bounded on $0<x<b, 0<\lambda_n<\infty$.
Hence the truncation error

$$E_N=\lim_{\epsilon\to 0+}\frac{2}{b^2}\sum_{n=N+1}^{\infty}\lambda_\epsilon(x)\left(\frac{\lambda_n}{x}\right)^{\alpha-\beta}\frac{J_{\alpha-\beta}(x\lambda_n)}{J^2_{3\alpha+\beta}(b\lambda_n)}\Phi_0(\lambda_n)$$

has exponential decay for large N. This completes the proof.
Theorem 3.5 gives the Fourier-Bessel type series representation of the operator $(-x^{-1}D)^{\alpha-\beta}$ on the testing function space F_b. We wish to investigate the nature of the Fourier-Bessel series for the pseudo-differential type operator $(-x^{-1}D)^{\alpha-\beta}$ on the distribution space F'_b.
The spaces F'_b and G'_b are dual spaces of F_b and G_b respectively. They are assigned the weak topologies generated by the seminorms

$$P_\phi(f)=|\langle f,\phi\rangle|,\quad \phi\in F_b, f\in F'_b$$

and

$$P_\phi\left(h_{\alpha,\beta}f\right)=|\langle h_{\alpha,\beta}f,h_{\alpha,\beta}\phi\rangle|,\quad h_{\alpha,\beta}\phi\in G_b, h_{\alpha,\beta}f\in G'_b$$

respectively.Both the spaces are sequentially complete.

Definition 3.6. For $f\in F'_b, \phi\in F_b$, we define the generalized finite hankel-type transform $h_{\alpha,\beta}f$ by

$$\langle h_{\alpha,\beta}f,h_{\alpha,\beta}\phi\rangle=\langle f,\phi\rangle \tag{3.8}$$

Theorem 3.7. *For $(\alpha-\beta)\in R, h_{\alpha,\beta}$ is an homeomorphism from F'_b onto G'_b.*

Theorem 3.8. *For every $\epsilon\in(0,b/4)$ and each $f\in F'_b$, the function*

$$\hat{f}_\epsilon(y)=\left\langle f(x),y^{2\beta-1}\lambda_\epsilon(x)m'(y)\mathcal{J}_{\alpha,\beta}(xy)\right\rangle \tag{3.9}$$

where $\lambda_\epsilon(x)$ is defined as in Theorem 3.4 is a smooth function of slow growth and defined a regular generalized function in G'_b

Proof: We note that $(x^{-1}D)^k\lambda_\epsilon(x)$ is bounded on $0<x<b$ for each k.Using (2.6), it is simple to see that $y^{2\beta-1}\lambda_\epsilon(x)m'(y)\mathcal{J}_{\alpha,\beta}(xy)\in F_b$. Hence (3.9) is well defined . The result of the proof is similar to that of Zemanian [Lemma 12]H.

Theorem 3.9. *The finite Hankel-type transform $h_{\alpha,\beta}f$ of a generalized function $f\in F'_b$ is the distributional limit, as $\epsilon\to 0+$ of the family $\hat{f}_\epsilon(z)$ defined by (3.9).*

Proof: Proof is simple and hence omitted.

Theorem 3.10. *Let $f\in F'_b$and$\hat{f}=h_{\alpha,\beta}f$. Then in the sense of convergence in F'_b, we have*

$$f(x)=\lim_{N\to\infty}\frac{2}{b^2}\sum_{n=1}^{N}\frac{x^{3\alpha+\beta}}{\sqrt{\lambda_n}}\frac{J_{\alpha-\beta}(x\lambda_n)}{J^2_{3\alpha+\beta}(b\lambda_n)}\hat{f}(\lambda_n) \tag{3.10}$$

Proof: The proof follows easily from Theorem 3.4 and 3.9.
Remark 3: For $f\in F'_b$, such that either f is regular or supp$f\subset[0,b]$, the limit of $\hat{f}_\epsilon(z)$ as $\epsilon\to 0+$ exists as an ordinary function and is equivalent to the finite Hankel-type transform of f [5].

A consequence of the above theorem is the following:

Theorem 3.11. *Let $f,g\in F'_b$. If $(h_{\alpha,\beta}f)(\lambda_n)=(h_{\alpha,\beta}g)(\lambda_n)$, for $n=1,2,3\ldots$ then $f=g$ and $h_{\alpha,\beta}f=h_{\alpha,\beta}g$.*

Definition 3.12. For $f\in F'_b$, define $(-x^{-1}D)^{\alpha-\beta}f$ by

$$\left\langle(-x^{-1}D)^{\alpha-\beta}f,\phi\right\rangle=\left\langle f,(-x^{-1}D)^{\alpha-\beta}\phi\right\rangle,\quad \phi\in F_b,(\alpha-\beta)\in R \tag{3.11}$$

From equations (2.9),(3.8) and (3.11), it follows that

$$\left\langle(-x^{-1}D)^{\alpha-\beta}f,\phi\right\rangle=\left\langle f,(-x^{-1}D)^{\alpha-\beta}\phi\right\rangle=\left\langle h_0^{-1}h_{\alpha,\beta}f,\phi\right\rangle, f\in F'_b,\phi\in F_b$$

Hence

$$(-x^{-1}D)^{\alpha-\beta}f=h_0^{-1}h_{\alpha,\beta}f\quad\text{ on }\quad F'_b \tag{3.12}$$

Applying Theorem 3.10 to equation (3.12) we get

Theorem 3.13 (The Fourier-Bessel type series). *Let $f\in F'_b$ and $\hat{f}=h_{\alpha,\beta}f$. Then in the sense of convergence in F'_b, we have*

$$(-x^{-1}D)^{\alpha-\beta}f(x)=\lim_{N\to\infty}\frac{2}{b^2}\sum_{n=1}^{N}\frac{x}{\sqrt{\lambda_n}}\frac{J_0(x\lambda_n)}{J_1^2(b\lambda_n)}\hat{f}(\lambda_n). \tag{3.13}$$

References

[1] L.S.Dubey and J.N.Pandey, On the Hankel transformation of distributions, Tohoku Math.J.27(1975),337-354.

[2] J.L.Griffith, Hankel transforms of functions zero outside a finite interval, J.Proc.Roy.Soc.New South Wales 89(1955)109-115.

[3] W.Y.Lee, On Schwartz Hankel transformation of certain space of distributions, SIAM J.Math.Anal. 6(1975),427-432.

[4] A.L.Schwartz, An inversion theorem for Hankel transforms, Proc.Amer.Math.Soc. 22(1969),713-717.

[5] O.P.Singh, On distributional finite Hankel transform, Appl.Anal. 21(1986),245-260.

[6] F.Traves,Topological vector Spaces,Distributions and Kernels, Academic Press, New york and London, 1967.

[7] A.H.Zemanian, Generalized integral transformations, Interscience, New York 1968.

[8] A.H. Zemanian, A Distributional Hankel transform, J.SIAM.Appl.Math. 14(1966),561-576.

[9] A.H. Zemanian, The Hankel transformations of certain distributions of rapid growth, J.SIAM.Appl.Math. 14(1966),678-690.

Author information

B.B.Waphare, MIT ACSC, Alandi, Pune, Maharashtra, India.
E-mail: balasahebwaphare@gmail.com

S.G.Gajbhiv, Research Scholar, Savitribai Phule Pune University and Assistant Professor MIT Academy of Engineering, Alandi, Pune, Maharashtra, India.
E-mail: satishgajbhiv@yahoo.co.in

Received: May 22, 2015.

Accepted: October 23, 2015.