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Homothetic Bishop Motion Of Euclidean Submanifolds in Euclidean 3-Space

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Abstract. In this study, we gave an alternative kinematic model for two smooth submanifolds M and N both on another and inside of another, along given any two curves which are tangent to each other on M and N at every moment , which the motion accepted that these curves are trajectories of the instantaneous rotation centers at the contact points of these submanifolds and we gave some remarks for the kinematic model at every moments by using Bishop frame. In addition, we established the relationships between Bishop curvatures of the moving and fixed pole curves.

1 Introduction

R.Müller generalized 1-parameter motions in an n-dimensional Euclidean space which is given by the equation $Y = AX + C$ and investigated axoid surfaces[7]. K. Nomizu defined the 1-parameter motion model along the pole curves on the tangent plane of the sphere, by using parallel vector fields and obtained some results of the motion in the cases that the motion is only sliding or only rolling [9]. H.H.Hacısalihoğlu investigated 1-parameter homothetic motion and obtained some important results in an n-dimensional Euclidean space[5]. B.Karakaş adapted K. Nomizu's motion model to the homothetic motion, again by defining parallel vector fields along the curves[3]. Y. Tuncer, Y. Yaylı and M. K. Sağel showed that a smooth manifold M can be rolling, sliding and spining on (or in side of) another smooth manifold N along not only special curves but also any regular curves (which are the pole curves of the homothetic motion) on M and N by using Frenet vectors, curvatures and torsions[13].

In this study, our aim is to show that a smooth manifold M can be rolling, sliding and spining on (or in side of) another smooth manifold N along not only special curves but also any regular curves (which are the pole curves of the homothetic motion) on M and N, by using Bishop frames, curvatures and the other special orthonormal frames along these curves and obtain the equation of this motion. Consequently, we will have obtained the equation of the homothetic motion of M on N along the pole curves.

The homothetic motion of the smooth submanifold M on (or in side of) another N in a 3-dimensional Euclidean space is generated by the transformation

$$\begin{aligned} F : M &\rightarrow N \\ X(s) &\rightarrow Y(s) = hAX(s) + C \end{aligned} \tag{1.1}$$

where A is a proper orthogonal 3×3 matrix, X and C are 3×1 vectors and $h \neq 0$ is a homothetic scale. The elements of A, C and h are continuously differentiable functions of the time-dependent parameter s and the elements of X are the coordinates of a point on M according to the Euclidean coordinate system $\{x_1, x_2, x_3\}$. We take B as hA with differentiating (1.1) and we obtain

$$\frac{dY}{ds} = B\frac{dX}{ds} + \frac{dB}{ds}X + \frac{dC}{ds} \tag{1.2}$$

where $\frac{dB}{ds}X + \frac{dC}{ds}$, $B\frac{dX}{ds}$ and $\frac{dY}{ds}$ are called sliding velocity, relative velocity and absolute velocity of the point X . We called X is a center of the instantaneous rotation if its sliding velocity is vanished. If X is a center of the instantaneous rotation then X is a pole point at the time s of the motion F given in (1.1) [5, 13, 14, 15]. Since $det(\frac{dB}{ds}) \neq 0$ then every homothetic motion in E^3 is a regular motion[5]. Let $X(s)$ be a regular curve on M which is defined on closed interval $I \subset IR$ so that all of its points are the pole points. In this case, we called

$$X(s) = -[\frac{dB}{ds}]^{-1}[\frac{dC}{ds}]$$

and

$$Y(s) = -B[\frac{dB}{ds}]^{-1}[\frac{dC}{ds}] + C$$

are the moving and fixed pole curves, respectively, where the matrix $B[\frac{dB}{ds}]^{-1}$ is as follows.

$$-B\,[\frac{dB}{ds}]^{-1} = \left((\frac{dh}{ds}A + h\frac{dA}{ds}\right) h^{-1}A^{-1} = \underbrace{\frac{dh}{ds}h^{-1}I_3}_{\varphi} + \underbrace{\frac{dA}{ds}A^{-1}}_{S}$$

We called φ and S are sliding part and rolling part of the motion F, respectively. For $S \neq 0$, there is a uniquely determined vector $W(s)$ such that $S(U)$ is equal to the cross product $W(s) \wedge U$ for every vector $U \in IR^3$. The vector $W(s)$ is called the angular velocity vector of the point $X(s)$ at instant s. If $W(s)$ is normal to N at $Y(s)$ then we have a spinning at instant s. If $W(s)$ is tangent to N at $Y(s)$ then we say that motion is a rolling with sliding , if $\varphi = 0$ and $S \neq 0$ then F is a pure rolling motion, if $\varphi \neq 0$ and $S = 0$ then F is a pure sliding motion[3, 5, 9, 15]. Since the motion F is a homothetic motion then it contains sliding part absolutely.

The ability to "ride" along a three-dimensional space curve and illustrate the properties of the curve, such as curvature and torsion, would be a great asset for mathematicians. The classic Serret-Frenet frame provides such ability,however the Serret-Frenet frame is not defined for all points along every curve. A new frame is needed for the kind of mathematical analysis that is typically done with the computer graphics.

Denote by $\{T(s), N(s), B(s)\}$ the moving Frenet-Serret frame along the curve $\alpha(s)$ in the space E^3. For an arbitrary curve $\alpha(s)$ with first and second curvature, $\kappa(s)$ and $\tau(s)$ in the space E^3, the following Frenet-Serret formulae are given in [4] written under matrix form

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

where

$$\begin{aligned} \langle T, N \rangle &= \langle T, B \rangle = \langle N, B \rangle = 0, \\ \langle T, T \rangle &= \langle N, N \rangle = \langle B, B \rangle = 1. \end{aligned}$$

Here, curvature functions are defined by

$$\kappa(s) = \|\alpha''(s)\|, \text{ and } \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\|\alpha''(s)\|^2}.$$

The Relatively Parallel Adapted Frame or Bishop Frame could provide the desired means to ride along any given space curve.The Bishop Frame has many properties that make it ideal for mathematical research. Another area about interested in the Bishop Frame is so-called Normal Development, or the graph of the twisting motion of the Bishop Frame. This information with the initial position and the orientation of the the Bishop Frame provide all of the information which is necessary to define the curve.

The Bishop frame may have the applications in the area of Biology and Computer Graphics. For example, it may be possible to compute the information about the shape of the sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animations[2, 10, 11].

The Bishop frame or parallel transport frame is an alternative approach to define a moving frame that is well defined even when the curve is vanished the second derivative. We can transport by parallel an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $\{N_1(s), N_2(s)\}$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point. If the derivatives of $\{N_1(s), N_2(s)\}$ depend only on $T(s)$ and not each other, we can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature.

In addition, suppose that the curve α is an arclength-parametrized C^2 curve and we have C^1 unit vector fields N_1 and $N_2 = T\wedge N_1$ along the curve α so that

$$\langle T, N_1\rangle = \langle T, N_2\rangle = \langle N_1, N_2\rangle = 0,$$

i.e., T , N_1, N_2 will be a smoothly varying right-handed orthonormal frame as we move along the curve (to this point, the Frenet frame would work just fine if the curve were C^3 with $\kappa \neq 0$). But now we want to impose the extra condition that $\langle N_1', N_2\rangle = 0$.We say that the unit first normal vector field N_1 is parallel along the curve α. This means that the change of N_1 is only in the direction of T. A Bishop frame can be defined even when a Frenet frame can not (e.g., when there are points with $\kappa = 0$). Therefore, we have the alternative frame equations

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} \tag{1.3}$$

where $\kappa(s) = \sqrt{k_1^2 + k_2^2}$, $\delta(s) = \arctan\left(\frac{k_2}{k_1}\right)$, $\tau(s) = -\frac{d\delta(s)}{ds}$ so that $k_1(s)$ and $k_2(s)$ effectively correspond to a cartesian coordinate system for the polar coordinates $(\kappa(s), \delta(s))$, with $\delta(s) = -\int \tau(s)\, ds$. The orientation of the parallel transport frame includes an arbitrary choice of the integration constant δ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation [2, 10].

Let us consider the smooth manifolds M and N which are tangent (inside or outside) to each other, $X(s)$ on M and $Y(s)$ on N be the moving and fixed regular pole curves and the tangent planes of M and N (along $X(s)$ and $Y(s)$) coincide at the contact points. We shall take a rectangular coordinate system in E^3. Let e_1, e_2 and e_3 be the unit vectors$(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. We denote $\xi = \xi(s)$ and $\eta = \eta(s)$ as the normal vector fields of M and N along the curves $X(s)$ and $Y(s)$, respectively. In addition, we denote the systems $\{T, N_1, N_2\}$ and $\{\overline{T}, \overline{N}_1, \overline{N}_2\}$ as the Bishop vector fields of the curves $X(s)$ and $Y(s)$, respectively. Since the homothetic motion $F : M \to N$ consists of rolling then $W(s)$ is tangent to both $X(s)$ on M and $Y(s)$ on N at every moments[11]. Since ξ and η have same or opposite directions depending on the orientation of M and N, we have $B\xi = \epsilon h\eta$ at the contact points, where ϵ is the sign such that; if $\epsilon = +1$ then M moves inside of N along the pole curves, if $\epsilon = -1$ then M moves out side of N along the pole curves.

Suppose that $\{b_1 = b_1(s), b_2 = b_2(s)\}$ and $\{a_1 = a_1(s), a_2 = a_2(s)\}$ be orthonormal systems along the regular pole curves $X(s)$ and $Y(s)$ respectively, and let b_1, b_2 and a_1, a_2 transform to each another as $b_1 = hB^{-1}a_1$ and $b_2 = hB^{-1}a_2$, respectively. Hence $\{b_1, b_2, \xi\}$ and $\{a_1, a_2, \eta\}$ will be the moving and fixed orthonormal systems for $(X) = X(s)$ and $(Y) = Y(s)$, respectively. Since (X) is the pole curve, we can write the equation $\frac{dY}{ds} = B\frac{dX}{ds}$ by using (1.2). Let the parameter s be arc-length parameter for the curve (X). Thus we can write $\frac{dY}{ds} = hAT$ and then we obtain

$$h = \left\| \frac{dY}{ds} \right\|$$

furthermore the tangent vector of (Y) will be $\overline{T} = \frac{1}{h}\frac{dY}{ds}$.

On the other hand, since $\xi \in Sp\{N_1, N_2\}$ then we can write

$$\xi(s) = \cos\psi(s) N_1(s) + \sin\psi(s) N_2(s) \tag{1.4}$$

We must construct the frames $\{b_1, b_2, \xi\}$ and $\{a_1, a_2, \eta\}$ for determining the orthogonal matrix A in (1.1). During this operations, we used the frames $\{T, \xi\Lambda T, \xi\}$ and $\{\overline{T}, \eta\Lambda\overline{T}, \eta\}$ which are called Darboux frames along (X) and (Y) at contact points on M and N, respectively. We can easily find the orthogonal matrices Q, P and R which transform $\{T, N_1, N_2\}$ to $\{T, \xi\Lambda T, \xi\}$, $\{e_1, e_2, e_3\}$ to $\{T, N_1, N_2\}$ and $\{T, \xi\Lambda T, \xi\}$ to $\{b_1, b_2, \xi\}$ by using (1.4), respectively. The matrix $A_1 = P^T Q^T R^T$ transforms b_1 to e_1, b_2 to e_2 and ξ to e_3. We obtain that the skew symmetric matrix $w_1 = \frac{dA_1^T}{ds} A_1$ is

$$w_1 = \begin{bmatrix} 0 & \begin{matrix} \rho' + \epsilon k_1 \sin\psi \\ -\epsilon k_2 \cos\psi \end{matrix} & \left\{ \begin{matrix} \epsilon k_1 \cos\rho \cos\psi \\ +\epsilon k_2 \sin\psi \cos\rho \\ +\psi' \sin\rho \end{matrix} \right\} \\ \begin{matrix} -\rho' - \epsilon k_1 \sin\psi \\ +\epsilon k_2 \cos\psi \end{matrix} & 0 & \left\{ \begin{matrix} -\epsilon k_1 \sin\rho \cos\psi \\ -\epsilon k_2 \sin\psi \sin\rho \\ +\psi' \cos\rho \end{matrix} \right\} \\ -\left\{ \begin{matrix} \epsilon k_1 \cos\rho \cos\psi \\ +\epsilon k_2 \sin\psi \cos\rho \\ +\psi' \sin\rho \end{matrix} \right\} & -\left\{ \begin{matrix} -\epsilon k_1 \sin\rho \cos\psi \\ -\epsilon k_2 \sin\psi \sin\rho \\ +\psi' \cos\rho \end{matrix} \right\} & 0 \end{bmatrix} \quad (1.5)$$

where $k_1 = k_1(s)$ and $k_2 = k_2(s)$ are the Bishop curvatures of the pole curve (X) and $\rho = \rho(s)$ is the rotation angle of $\{b_1, b_2\}$ according to $\{T, \xi\Lambda T\}$..

Corollary 1.1. *The vector fields b_1 and b_2 are the parallel vector fields along curve (X) according to the connection of M if and only if*

$$\rho' + \epsilon(k_1 \sin\psi - k_2 \cos\psi) = 0$$

is satisfied.

Proof. Let $\overline{\nabla}$ be Levi Civita connection and S_M be the shape operator of M. We can write b_1 as follows by using the matrices R and P.

$$b_1 = \cos\rho T + \sin\rho \sin\psi N_1 - \sin\rho \cos\psi N_2$$

Using the Gauss equation

$$\overline{\nabla}_T b_1 = \nabla_T b_1 + \langle S_M(T), b_1 \rangle \xi$$

and after routine calculations, we obtain

$$\overline{\nabla}_T b_1 = -\{\rho' + k_1 \sin\psi - k_2 \cos\psi\}\{\sin\rho T - \sin\psi \cos\rho N_1 + \cos\psi \cos\rho N_2\}$$

It is easily to see that $\overline{\nabla}_T b_1 = 0$ if and only if $\rho' + k_1 \sin\psi - k_2 \cos\psi = 0$. Hence, b_1 is a parallel vector field along curve (X) according to the connection of M if and only if $\rho' + k_1 \sin\psi - k_2 \cos\psi = 0$ is satisfied. Similarly, we can easily proof that b_2 is a parallel vector field along curve (X) according to the connection of M if and only if $\rho' + k_1 \sin\psi - k_2 \cos\psi = 0$ is satisfied, too.

On the other hand, since

$$\eta(s) = \cos\overline{\psi}(s)\overline{N}_1(s) + \sin\overline{\psi}(t)\overline{N}_2(s) \quad (1.6)$$

then we can easily find the orthogonal matrices $\overline{Q}$, $\overline{P}$ and $\overline{R}$ by using (1.6) which transform $\{\overline{T}, \overline{N}_1, \overline{N}_2\}$ to $\{\overline{T}, \eta\Lambda\overline{T}, \eta\}$, $\{e_1, e_2, e_3\}$ to $\{\overline{T}, \overline{N}_1, \overline{N}_2\}$ and $\{\overline{T}, \eta\Lambda\overline{T}, \eta\}$ to $\{a_1, a_2, \eta\}$, respectively. The matrix $A_2 = \overline{P}^T \overline{Q}^T \overline{R}^T$ transforms a_1 to e_1, a_2 to e_2 and η to e_3. We obtain that the skew symmetric matrix $w_2 = \frac{dA_2^T}{ds} A_2$ is

$$w_2 = \begin{bmatrix} 0 & \begin{matrix} \overline{\rho}' + \overline{k}_1 \sin \overline{\psi} \\ -\overline{k}_2 \cos \overline{\psi} \end{matrix} & \left\{ \begin{matrix} \overline{k}_1 \cos \overline{\rho} \cos \overline{\psi} \\ +\overline{k}_2 \sin \overline{\psi} \cos \overline{\rho} \\ +\overline{\psi}' \sin \overline{\rho} \end{matrix} \right\} \\ \begin{matrix} -\overline{\rho}' - \overline{k}_1 \sin \overline{\psi} \\ +\overline{k}_2 \cos \overline{\psi} \end{matrix} & 0 & \left\{ \begin{matrix} -\overline{k}_1 \sin \overline{\rho} \cos \overline{\psi} \\ -\overline{k}_2 \sin \overline{\psi} \sin \overline{\rho} \\ +\overline{\psi}' \cos \overline{\rho} \end{matrix} \right\} \\ -\left\{ \begin{matrix} \overline{k}_1 \cos \overline{\rho} \cos \overline{\psi} \\ +\overline{k}_2 \sin \overline{\psi} \cos \overline{\rho} \\ +\overline{\psi}' \sin \overline{\rho} \end{matrix} \right\} & -\left\{ \begin{matrix} -\overline{k}_1 \sin \overline{\rho} \cos \overline{\psi} \\ -\overline{k}_2 \sin \overline{\psi} \sin \overline{\rho} \\ +\overline{\psi}' \cos \overline{\rho} \end{matrix} \right\} & 0 \end{bmatrix} \quad (1.7)$$

where $\overline{k}_1 = \overline{k}_1(s)$ and $\overline{k}_2 = \overline{k}_2(s)$ are the Bishop curvatures of the pole curve (Y) and $\overline{\rho} = \overline{\rho}(s)$ is the rotation angle of $\{a_1, a_2\}$ according to $\{\overline{T}, \eta \Lambda \overline{T}\}$.

Corollary 1.2. *The vector fields a_1 and a_2 are the parallel vector fields along curve (Y) according to the connection of N if and only if*

$$\overline{\rho'} + \overline{k}_1 \sin \overline{\psi} - \overline{k}_2 \cos \overline{\psi} = 0$$

is satisfied.

Proof. We can proof similarly to corollary 1.1.

Therefore, we obtain the matrix A using A_1 and A_2 as $A = A_2 A_1^T$ so that A transforms b_1 to a_1, b_2 to a_2 and ξ to $\epsilon\eta$, respectively. The skew-symmetric matrix $S = \frac{dA}{ds} A^T$ is an instantaneous rotation matrix and S represents a linear ishomorphism as $T_{Y(t)}N \longrightarrow Sp\{\eta\}$. We can find the matrix S by using (1.5) and (1.7) as $S = A_2(-w_2 + w_1) A_2^T$. Consequently the matrix S determines an unique vector $w \in Sp\{a_1, a_2, \eta\}$ as follows.

$$w = u_1 a_1 + u_2 a_2 + u_3 \eta \quad (1.8)$$

where

$$\begin{aligned} u_1 &= -\left(\overline{k}_1 \cos \overline{\psi} + \overline{k}_2 \sin \overline{\psi}\right) \sin \overline{\rho} + \overline{\psi}' \cos \overline{\rho} + \{\epsilon (k_1 \cos \psi + k_2 \sin \psi) \sin \rho - \psi' \cos \rho\} \\ u_2 &= -\left(\overline{k}_1 \cos \overline{\psi} + \overline{k}_2 \sin \overline{\psi}\right) \cos \overline{\rho} - \overline{\psi}' \sin \overline{\rho} + \{\epsilon (k_1 \cos \psi + k_2 \sin \psi) \cos \rho + \psi' \sin \rho\} \\ u_3 &= \overline{\rho'} + \overline{k}_1 \sin \overline{\psi} - \overline{k}_2 \cos \overline{\psi} - \rho' - \epsilon k_1 \sin \psi + \epsilon k_2 \cos \psi \end{aligned}$$

Thus, we obtained the main condition for two moving smooth submanifolds on (or inside of) another, along the regular pole curves. So, we prove the following theorem.

Theorem 1.3. *F is rolling with sliding motion defined as $hAb_1 = a_1$, $hAb_2 = a_2$ and $hA\xi = \epsilon\eta$ along the regular pole curves if and only if*

$$\overline{\rho'} + \overline{k}_1 \sin \overline{\psi} - \overline{k}_2 \cos \overline{\psi} - \rho' - \epsilon k_1 \sin \psi + \epsilon k_2 \cos \psi = 0$$

This condition shows that any smooth submanifolds can be rolling with sliding, pure sliding or sliding with spining on (or inside of) another along the pole curves which are tangent to each other at every moment. This is possible by choosing one of ρ and $\overline{\rho}$ as a constant even if we face hard integrals. In addition, ρ and $\overline{\rho}$ show that how we must define the vector fields a_1, a_2 and b_1, b_2 along the pole curves according to what we desire a homothetic motion. We can also find the geodesic and normal curvatures and geodesic torsions of M and N in the Bishop means, along the curves (X) and (Y) as follows. The curvatures of M along (X) are

$$\kappa_g = k_1 \sin \psi - k_2 \cos \psi \ , \ \kappa_\xi = k_1 \cos \psi + k_2 \sin \psi \ , \ \tau_g = \psi' \quad (1.9)$$

and the curvatures of N along (Y) are

$$\overline{\kappa}_g = \overline{k}_1 \sin \overline{\psi} - \overline{k}_2 \cos \overline{\psi} \ , \ \overline{\kappa}_\eta = \overline{k}_1 \cos \overline{\psi} + \overline{k}_2 \sin \overline{\psi} \ , \ \overline{\tau}_g = \overline{\psi}' \quad (1.10)$$

Hence we restore (1.8) as follows.

$$\begin{aligned} u_1 &= -\overline{\kappa}_\eta \sin\overline{\rho} + \overline{\tau}_g \cos\overline{\rho} + \epsilon\kappa_\xi \sin\rho - \tau_g \cos\rho \\ u_2 &= -\overline{\kappa}_\eta \cos\overline{\rho} - \overline{\tau}_g \sin\overline{\rho} + \epsilon\kappa_\xi \cos\rho + \tau_g \sin\rho \\ u_3 &= \overline{\rho'} - \rho' + \overline{\kappa}_g - \epsilon\kappa_g \end{aligned} \tag{1.11}$$

If M is rolling on (or inside of) N along the curves (X) and (Y) then $\overline{\rho'} - \rho' + \overline{\kappa}_g - \epsilon\kappa_g = 0$. If b_1, b_2, a_1 and a_2 are the parallel vector fields then the motion is rolling with sliding automatically. In the same conditions, the following equalities are satisfied at the points that the motion is pure sliding.

$$\begin{aligned} \kappa_\xi &= \epsilon\overline{\kappa}_\eta \cos\left(\int (\epsilon\kappa_g - \overline{\kappa}_g)\, ds + c\right) + \epsilon\overline{\tau}_g \sin\left(\int (\epsilon\kappa_g - \overline{\kappa}_g)\, ds + c\right) \\ \tau_g &= -\overline{\kappa}_\eta \sin\left(\int (\epsilon\kappa_g - \overline{\kappa}_g)\, ds + c\right) + \overline{\tau}_g \cos\left(\int (\epsilon\kappa_g - \overline{\kappa}_g)\, ds + c\right) \end{aligned}$$

where c is a constant. In this case, $u_1 = u_2 = u_3 = 0$. In the case, b_1, b_2, a_1, a_2 are not the parallel vector fields and $\kappa_\xi^2 + \tau_g^2 \neq 0$ and $\overline{\kappa}_\eta^2 + \overline{\tau}_g^2 \neq 0$ then

$$\overline{\rho} - \rho = \arccos\left(\frac{\epsilon\kappa_\xi\overline{\kappa}_\eta + \tau_g\overline{\tau}_g}{\overline{\kappa}_\eta^2 + \overline{\tau}_g^2}\right)$$

or

$$\overline{\rho} - \rho = \arcsin\left(\frac{\epsilon\kappa_\xi\overline{\tau}_g - \overline{\kappa}_\eta\tau_g}{\overline{\kappa}_\eta^2 + \overline{\tau}_g^2}\right)$$

$\overline{\rho'} - \rho' + \overline{\kappa}_g - \epsilon\kappa_g \neq 0$ and $\kappa_\xi^2 + \tau_g^2 = \overline{\kappa}_\eta^2 + \overline{\tau}_g^2$ are satisfied at the points that the motion is sliding with spining. If the curves (X) and (Y) are both the principal curves and geodesics of M and N then $\tau_g = \overline{\tau}_g = \overline{\kappa}_g = \kappa_g = 0$ and also ψ and $\overline{\psi}$ are constants.

If M is any manifold in E^3 and N is a plane then angular velocity vector at the contact points will be as follows

$$\begin{aligned} w = &\left\{-\left(\overline{k}_1 \cos\overline{\psi} + \overline{k}_2 \sin\overline{\psi}\right) \sin\overline{\rho} + \left\{\epsilon\left(k_1 \cos\psi + k_2 \sin\psi\right) \sin\rho - \psi' \cos\rho\right\}\right\} a_1 \\ &- \left\{\left(\overline{k}_1 \cos\overline{\psi} + \overline{k}_2 \sin\overline{\psi}\right) \cos\overline{\rho} - \left\{\epsilon\left(k_1 \cos\psi + k_2 \sin\psi\right) \cos\rho + \psi' \sin\rho\right\}\right\} a_2 \\ &+ \left\{\overline{\rho'} + \overline{k}_1 \sin\overline{\psi} - \overline{k}_2 \cos\overline{\psi} - \rho' - \epsilon k_1 \sin\psi + \epsilon k_2 \cos\psi\right\} \eta \end{aligned}$$

In this case, F is a rolling with sliding if and only if

$$\overline{\rho} - \rho = \left(\overline{k}_2 \cos\overline{\psi} - \overline{k}_1 \sin\overline{\psi}\right) s + \epsilon \int (k_1 \sin\psi - k_2 \cos\psi)\, ds + c$$

is satisfied, where c, $\overline{\psi}$, $\overline{k}_1$ and $\overline{k}_2$ are constants. We can restate (**??**) as follows by using (1.9) and (1.10).

$$\begin{aligned} w = &\left\{-\overline{\kappa}_\eta \sin\overline{\rho} + \epsilon\kappa_\xi \sin\rho - \tau_g \cos\rho\right\} a_1 + \left\{-\overline{\kappa}_\eta \cos\overline{\rho} + \epsilon\kappa_\xi \cos\rho + \tau_g \sin\rho\right\} a_2 \\ &+ \left\{\overline{\rho'} - \rho' + \overline{\kappa}_g - \epsilon\kappa_g\right\} \eta \end{aligned}$$

Thus, F is a rolling with sliding if and only if

$$\overline{\rho} - \rho = s\overline{\kappa}_g - \epsilon \int \kappa_g ds + c$$

is satisfied, where c, and $\overline{\kappa}_g$ are constants, too.

Corollary 1.4. *If (X) and (Y) are geodesics of M and N, respectively, then F is a rolling with sliding motion if and only if $\overline{\rho'} - \rho'$ =constant.*

Theorem 1.5. *Let M and N be two submanifolds and (X) and (Y) be the smooth curves on M and N, respectively, which are satisfied given condition in theorem 1.3 and be tangent to each other at the contact points. Then we can find a unique homothetic motion F of M on (or inside of) N along the pole curves (X) and (Y).*

Theorem 1.6. *Let S_M and S_N be the shape operators of M and N along the curves (X) and (Y) respectively. If*

$$h^{-1}S_M\left(\frac{dX}{ds}\right) = S_N\left(\frac{dY}{ds}\right)$$

then F is sliding motion without rolling.

Proof. We can write the following equations along the curves (X) and (Y), respectively.

$$S_M\left(\frac{dX}{ds}\right) = \frac{d\xi}{ds} \quad \text{and} \quad S_N\left(\frac{dY}{ds}\right) = \frac{d\eta}{ds}$$

By differentiating (1.4) and by using (1.3), we obtain

$$\frac{d\xi}{ds} = -\left\{\epsilon\kappa_\xi \cos\rho + \tau_g \sin\rho\right\} b_1 - \left\{-\epsilon\kappa_\xi \sin\rho + \tau_g \cos\rho\right\} b_2$$

since $b_1 = hB^{-1}a_1$, $b_2 = hB^{-1}a_2$ and $\xi = \epsilon hB^{-1}\eta$,

$$h^{-1}B\left(\frac{d\xi}{ds}\right) = -\left\{\epsilon\kappa_\xi \cos\rho + \tau_g \sin\rho\right\} a_1 - \left\{-\epsilon\kappa_\xi \sin\rho + \tau_g \cos\rho\right\} a_2$$

by differentiating (1.6) and by using (1.3), we obtain

$$\frac{d\eta}{ds} = -\left\{\overline{\kappa}_\eta \cos\overline{\rho} + \overline{\tau}_g \sin\overline{\rho}\right\} a_1 - \left\{-\overline{\kappa}_\eta \sin\overline{\rho} + \overline{\tau}_g \cos\overline{\rho}\right\} a_2$$

Since $h^{-1}B\left(\frac{d\xi}{ds}\right) = \frac{d\eta}{ds}$, we can write

$$\epsilon\kappa_\xi \cos\rho + \tau_g \sin\rho = \overline{\kappa}_\eta \cos\overline{\rho} + \overline{\tau}_g \sin\overline{\rho} \tag{1.12}$$

and

$$\epsilon\kappa_\xi \sin\rho + \tau_g \cos\rho = \overline{\kappa}_\eta \sin\overline{\rho} + \overline{\tau}_g \cos\overline{\rho}. \tag{1.13}$$

we substitute (1.12) and (1.13) in (1.11) and from (1.8), we obtain that F is sliding motion without rolling. □

Corollary 1.7. *If F is rolling with sliding motion then the shape operators of M and N satisfy the following inequality.*

$$h^{-1}S_M\left(\frac{dX}{ds}\right) \neq S_N\left(\frac{dY}{ds}\right)$$

Corollary 1.8. *Let (X) and (Y) be the smooth curves on M and N such the curves not passing through the flat points of M and N. In this case M is sliding and rolling on (or inside of) N along these curves. M is sliding without rolling (or inside of) N at the flat-contact points.*

All of the corollaries, theorems and the things we said in this study are consistent with[3] and [13]. If $h = 1$ then this study gives us a one parameter kinematic model for the smooth submanifolds in Euclidean 3-space. In this case, the notions rolling with sliding and sliding with spining transform to pure rolling and pure spining, respectively.

Example 1.9. (For $\epsilon = -1$): Let $X(s) = (\sin(s), 0, \cos(s))$, $s \in [0, 1]$ be a unit speed curve on $\phi(u, v) = (\sin v \sin u, \sin v \cos u, \cos v)$ and $Y(s) = (\sin(s), -s, \cos(s) - 2)$ is any curve on $x^2 + (z+2)^2 = 1$. The Bishop trihedron of the curve (X) is

$$T = (\cos(s), 0, -\sin(s)), \quad N_1 = (-\sin(s), 0, -\cos(s)), \quad N_2 = (0, 1, 0)$$

and since $\delta = 0$ then the curvatures of the curve (X) are

$$k_1 = 1,\ k_2 = 0,$$

the unit normal vector field of sphere is

$$\xi(s) = (\sin(s), 0, \cos(s))$$

with the angle $\psi = \pi$. The Bishop trihedron of the curve (Y) is

$$\overline{T} = \left(\frac{1}{\sqrt{2}} \cos(s), -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \sin(s) \right)$$

$$\overline{N}_1 = \left(\left\{ \begin{array}{c} \frac{1}{\sqrt{2}} \sin\left(\frac{1}{2}s\right)\cos(s) \\ -\cos\left(\frac{1}{2}s\right)\sin(s) \end{array} \right\}, \frac{1}{\sqrt{2}} \sin\left(\frac{1}{2}s\right), \left\{ \begin{array}{c} -\frac{1}{\sqrt{2}} \sin\left(\frac{1}{2}s\right)\sin(s) \\ -\cos\left(\frac{1}{2}s\right)\cos(s) \end{array} \right\} \right)$$

$$\overline{N}_2 = \left(\left\{ \begin{array}{c} \frac{1}{\sqrt{2}} \cos\left(\frac{1}{2}s\right)\cos(s) \\ +\sin\left(\frac{1}{2}s\right)\sin(s) \end{array} \right\}, \frac{1}{\sqrt{2}} \cos\left(\frac{1}{2}s\right), \left\{ \begin{array}{c} \sin\left(\frac{1}{2}s\right)\cos(s) \\ -\frac{1}{\sqrt{2}} \cos\left(\frac{1}{2}s\right)\sin(s) \end{array} \right\} \right)$$

and the curvatures of the curve (Y) are

$$\overline{k}_1 = \frac{1}{2}\cos(\frac{1}{2}s),\ \ \overline{k}_2 = \frac{1}{2}\sin(\frac{1}{2}s)$$

with $\overline{\delta} = \frac{1}{2}s$. The unit normal vector field of cyclinder is

$$\eta(s) = -(\sin(s), 0, \cos(s))$$

with the angle $\overline{\psi} = \frac{1}{2}s$. Since $\left\| \frac{dY}{ds} \right\| = \sqrt{2}$ then the homothetic scale is $h = \sqrt{2}$ and we calculate the orthogonal matrix $A = [a_{ij}]$ and so the matrix B in (1.1) is $B = \sqrt{2}A$ where a_{ij} are

$$a_{11} = \frac{\sqrt{2}-2}{8}\cos(3s) + \frac{\sqrt{2}}{4}\cos(2s) + \frac{2-\sqrt{2}}{8}\cos(s) + \frac{\sqrt{2}}{4}$$

$$a_{12} = \frac{\sqrt{2}-2}{4}\cos(2s) + \frac{\sqrt{2}+2}{4}$$

$$a_{13} = \frac{2-\sqrt{2}}{8}\sin(3s) - \frac{\sqrt{2}}{4}\sin(2s) + \frac{2-\sqrt{2}}{8}\sin(s)$$

$$a_{21} = \frac{\sqrt{2}}{4}\cos(2s) - \frac{\sqrt{2}}{2}\cos(s) - \frac{\sqrt{2}}{4}$$

$$a_{22} = \frac{\sqrt{2}}{2}\cos(s)$$

$$a_{23} = \frac{\sqrt{2}}{2}\sin(s) - \frac{\sqrt{2}}{4}\sin(2s)$$

$$a_{31} = \frac{2-\sqrt{2}}{8}\sin(3s) - \frac{\sqrt{2}}{4}\sin(2s) + \frac{2+3\sqrt{2}}{8}\sin(s)$$

$$a_{32} = \frac{2-\sqrt{2}}{4}\sin(2s)$$

$$a_{33} = \frac{2-\sqrt{2}}{8}\cos(3s) - \frac{\sqrt{2}}{4}\cos(2s) + \frac{6+\sqrt{2}}{8}\cos(s) + \frac{\sqrt{2}}{4}$$

and the matrix C is

$$C = \left[\begin{array}{c} \frac{1-\sqrt{2}}{2}\sin(2s) + \sin(s) \\ \sin(s) - s \\ \left(1-\sqrt{2}\right)\cos^2(s) + \cos(s) - 3 \end{array} \right]$$

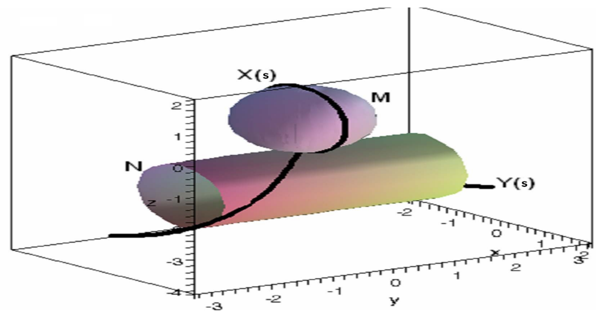

Figure 1. Sphere is rolling without sliding on the cylinder along the curves (X) and (Y).

Since (X) is the solution of the equation $\frac{d}{ds}BX + \frac{d}{ds}C = 0$ then (X) is a pole curve as a moving curve and (Y) is a fixed pole curve on the sphere and the cylinder $x^2+(z+2)^2 = 1$, respectively. Unit normal vectors ξ and η are the opposite direction and linear dependent at the contact points, thus the signature is $\epsilon = -1$. The components of the anti-symmetric matrix $S = [s_{ij}]$ are

$$s_{11} = s_{22} = s_{33} = 0$$

$$s_{21} = -s_{12} = \frac{\sqrt{2}-2}{8}\sin(2s) - \frac{\sqrt{2}}{4}\sin(s)$$

$$s_{31} = -s_{13} = \frac{\sqrt{2}}{4}\cos(s) + \frac{\sqrt{2}}{4}$$

$$s_{32} = -s_{23} = \frac{2-\sqrt{2}}{8}\cos(2s) + \frac{\sqrt{2}}{4}\cos(s) - \frac{1}{4}$$

with respect to the standart base of IR^3 and so the angular velocity vector is

$$W = \frac{1}{2}a_1 + \frac{1}{2}a_2$$

with respect to the base $\{a_1,\ a_2,\ \eta\}$, where the vector fields a_1 and a_2 are

$$a_1 = \left(\frac{\sqrt{2}}{2}\cos(s), -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\sin(s)\right)$$

$$a_2 = \left(\frac{2-\sqrt{2}}{4}\cos(2s) - \frac{2+\sqrt{2}}{4}, -\frac{\sqrt{2}}{2}\cos(s), \frac{\sqrt{2}-2}{4}\sin(2s)\right)$$

Since h is a constant and W lies on the tangent plane at the contact points then the sphere is rolling without sliding on the cylinder along the curves (X) and (Y).

Example 1.10. (For $\epsilon = 1$): Let $X(s) = (\sin(s), 0, \cos(s) - 1)$, $s \in [0,\pi]$ is the unit speed curve on $\phi(u,v) = (\sin v \sin u, \sin v \cos u, \cos v - 1)$ and $Y(s) = (2\sin(s), -s, 2\cos(s) - 2)$ is any curve on $x^2 + (z+2)^2 = 4$. The Bishop trihedron of the curve (X) is

$$T = (\sin(s), 0, -\cos(s)), \quad N_1 = (-\sin(s), 0, -\cos(s)), \quad N_2 = (0,1,0)$$

and since $\delta = 0$ then the curvatures of the curve (X) are

$$k_1 = 1,\ k_2 = 0,$$

the unit normal vector field of sphere is

$$\xi(s) = (\sin(s), 0, \cos(s))$$

with the angle $\psi = \pi$. The Bishop trihedron of the curve (Y) is

$$\overline{T} = \left(\frac{2\sqrt{5}}{5}\cos(s), \frac{-\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}\sin(s) \right)$$

$$\overline{N}_1 = \left(\left\{ \begin{array}{c} \frac{\sqrt{5}}{5}\sin\left(\frac{s\sqrt{5}}{5}\right)\cos(s) \\ -\cos\left(\frac{s\sqrt{5}}{5}\right)\sin(s) \end{array} \right\}, \frac{2\sqrt{5}}{5}\sin\left(\frac{s\sqrt{5}}{5}\right), \left\{ \begin{array}{c} -\frac{\sqrt{5}}{5}\sin\left(\frac{s\sqrt{5}}{5}\right)\sin(s) \\ -\cos\left(\frac{s\sqrt{5}}{5}\right)\cos(s) \end{array} \right\} \right)$$

$$\overline{N}_2 = \left(\left\{ \begin{array}{c} \frac{\sqrt{5}}{5}\cos\left(\frac{s\sqrt{5}}{5}\right)\cos(s) \\ +\sin\left(\frac{s\sqrt{5}}{5}\right)\sin(s) \end{array} \right\}, \frac{2\sqrt{5}}{5}\cos\left(\frac{s\sqrt{5}}{5}\right), \left\{ \begin{array}{c} \sin\left(\frac{s\sqrt{5}}{5}\right)\cos(s) \\ -\frac{\sqrt{5}}{5}\cos\left(\frac{s\sqrt{5}}{5}\right)\sin(s) \end{array} \right\} \right)$$

and the curvatures of the curve (Y) are

$$\overline{k}_1 = \frac{2\sqrt{5}}{5}\cos\left(\frac{s\sqrt{5}}{5}\right), \ \overline{k}_2 = \frac{2\sqrt{5}}{5}\sin\left(\frac{s\sqrt{5}}{5}\right)$$

with $\overline{\delta} = \left(\frac{\sqrt{5}}{5}s\right)$. The unit normal vector field of cyclinder is

$$\eta(s) = (\sin(s), 0, \cos(s))$$

with the angle $\overline{\psi} = \frac{s\sqrt{5}}{5}$. Since $\left\|\frac{dY}{ds}\right\| = \sqrt{5}$ then the homothetic scale is $h = \sqrt{5}$ and we calculate the orthogonal matrix $A = [a_{ij}]$ and so the matrix B in (1.1) is $B = \sqrt{5}A$ where a_{ij} are

$$a_{11} = \frac{2\sqrt{5}}{5}\cos^2(s) - \cos\left(\frac{s\sqrt{5}}{5}\right)\sin^2(s) + \frac{\sqrt{5}}{10}\sin\left(\frac{2s\sqrt{5}}{5}\right)\sin(2s)$$

$$a_{12} = \frac{-\sqrt{5}}{5}\cos\left(\frac{2s\sqrt{5}}{5}\right)\cos(s) - \sin\left(\frac{2s\sqrt{5}}{5}\right)\sin(s)$$

$$a_{13} = \frac{\sqrt{5}}{5}\sin\left(\frac{2s\sqrt{5}}{5}\right)\cos^2(s) - \left(\frac{1}{2}\cos\left(\frac{2s\sqrt{5}}{5}\right) + \frac{\sqrt{5}}{5}\right)\sin(2s)$$

$$a_{21} = \frac{\sqrt{5}}{5}\left\{2\sin\left(\frac{2s\sqrt{5}}{5}\right)\sin(s) - \cos(s)\right\}$$

$$a_{22} = \frac{-2\sqrt{5}}{5}\cos\left(\frac{2s\sqrt{5}}{5}\right)$$

$$a_{23} = \frac{\sqrt{5}}{5}\left\{2\sin\left(\frac{2s\sqrt{5}}{5}\right)\cos(s) + \sin(s)\right\}$$

$$a_{31} = -\frac{\sqrt{5}}{5}\sin\left(\frac{2s\sqrt{5}}{5}\right)\sin^2(s) - \left(\frac{\sqrt{5}}{5} + \frac{1}{2}\cos\left(\frac{2s\sqrt{5}}{5}\right)\right)\sin(2s)$$

$$a_{32} = -\frac{\sqrt{5}}{5}\cos\left(\frac{2s\sqrt{5}}{5}\right)\sin(s) - \sin\left(\frac{2s\sqrt{5}}{5}\right)\cos(s)$$

$$a_{33} = \frac{2\sqrt{5}}{5}\sin^2(s) - \cos\left(\frac{2s\sqrt{5}}{5}\right)\cos^2(s) - \frac{\sqrt{5}}{10}\sin\left(\frac{2s\sqrt{5}}{5}\right)\sin(2s)$$

and the matrix C is

$$C = \left[\begin{array}{c} (1-\cos(s))\left\{\left(\sqrt{5}\cos\left(\frac{2s\sqrt{5}}{5}\right) + 2\right)\sin(s) - \sin\left(\frac{2s\sqrt{5}}{5}\right)\cos(s)\right\} \\ 2\left(\cos(s) - 1\right)\sin\left(\frac{2s\sqrt{5}}{5}\right) - \sin(s) - s \\ (1-\cos(s))\left\{\left(\frac{10-2\sqrt{5}}{5} + 2\sqrt{5}\cos^2\left(\frac{s\sqrt{5}}{5}\right)\right)\cos(s) + \sin\left(\frac{2s\sqrt{5}}{5}\right)\sin(s)\right\} \end{array} \right]$$

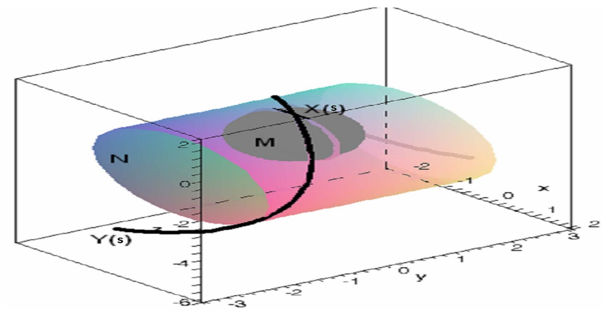

Figure 2. Sphere is rolling without sliding inside of the cylinder along the curves (X) and (Y).

Since (X) is the solution of the equation $\frac{d}{ds}BX + \frac{d}{ds}C = 0$ then (X) is a pole curve as a moving curve and (Y) is a fixed pole curve on the sphere $\phi(u,v)$ and the cylinder $x^2 + (z+2)^2 = 1$, respectively. The unit normal vectors ξ and η are the same direction and linear dependent at the contact points, thus the signature is $\epsilon = 1$. The components of the anti-symmetric matrix $S = [s_{ij}]$ are

$$s_{11} = s_{22} = s_{33} = 0$$

$$s_{21} = -s_{12} = \frac{3}{5}\sin\left(\frac{2s\sqrt{5}}{5}\right)\cos(s) + \frac{\sqrt{5}}{5}\cos\left(\frac{2s\sqrt{5}}{5}\right)\sin(s)$$

$$s_{31} = -s_{13} = \frac{2\sqrt{5}}{5}\cos(2s) - \cos\left(\frac{2s\sqrt{5}}{5}\right)\left(1 - \frac{8}{5}\cos^2(s)\right)$$

$$s_{32} = -s_{23} = \frac{-\sqrt{5}}{5}\cos(s) - \frac{4}{5}\cos\left(\frac{2s\sqrt{5}}{5}\right)\cos(s) + \frac{2\sqrt{5}}{5}\sin\left(\frac{2s\sqrt{5}}{5}\right)\sin(s)$$

with respect to the standart base of IR^3 and so the angular velocity vector is

$$W = \frac{\sqrt{5}}{5}a_1 - \frac{5+\sqrt{5}}{5}a_2$$

with respect to the base $\{a_1,\ a_2,\ \eta\}$, where the vector fields a_1 and a_2 are

$$a_1 = \left(\frac{2\sqrt{5}}{5}\cos(s), -\frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}\sin(s)\right)$$

$$a_2 = \left(\left\{\begin{array}{c} -\frac{\sqrt{5}}{5}\cos(s)\cos\left(\frac{2s\sqrt{5}}{5}\right) \\ -\sin\left(\frac{2s\sqrt{5}}{5}\right)\sin(s)\end{array}\right\}, -\frac{2\sqrt{5}}{5}\cos\left(\frac{2s\sqrt{5}}{5}\right), \left\{\begin{array}{c} \frac{\sqrt{5}}{5}\sin(s)\cos\left(\frac{2s\sqrt{5}}{5}\right) \\ -\sin\left(\frac{2s\sqrt{5}}{5}\right)\cos(s)\end{array}\right\}\right)$$

Since h is a constant and W lies on the tangent plane at the contact points then the sphere is rolling without sliding inside of the cylinder along the curves (X) and (Y).

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