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Some curves on three-dimensional trans-Sasakian manifolds with semisymmetric metric connection

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Abstract. The object of the present paper is to study biharmonic Legendre curves and to study locally ϕ-symmetric Legendre curves on three-dimensional trans-Sasakian manifolds with respect to semisymmetric metric connection. As an illustrative example of three-dimensional trans-Sasakian manifold, we give the hyperbolic space $\mathbb{H}^3(-1)$ and construct Legendre curve on $\mathbb{H}^3(-1)$ with respect to semisymmetric metric connection.

1 Introduction

In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [4]. The concept of Legendre curves can also be extended to almost contact metric manifolds [17], [32]. There exists no non-geodesic biharmonic Legendre curves in S^3 with respect to Levi-Civita connection [7], [16]. The study of Legendre curves on S^3 with pseudo-Hermitian connection has been initiated by J. T. Cho [8]. Recently, besides contact manifolds, Legendre curves on almost contact manifolds have been studied by several authors [17], [32]. Legendre curves on almost contact metric manifolds are also known as almost contact curves [17]. We are interested to find conditions under which a non-geodesic Legendre curve is biharmonic with respect to semisymmetric metric connection on a three-dimensional trans-Sasakian manifold. We also consider locally ϕ-symmetric Legendre curves with respect to semisymmetric metric connection on a three-dimensional trans-Sasakian manifold. Legendre curves with respect to pseudo-Hermitian connection have been studied in the papers [8], [17], [19], [23]. We may find works on biharmonic curves and maps in the papers [12], [22]. The first author of the present paper has also studied Legendre curves in the papers [26], [27], [28].

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzalez [6], and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [14], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [24] if the product manifold $M \times R$ belongs to the class W_4. The class $C_6 \oplus C_5$ [20], [21] coincides with the class of trans-Sasakian structures of type (α, β). In [21], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [5], some curvature identities and sectional curvatures for C_5, C_6 and trans-Sasakian manifolds are obtained. It is known that [18] trans-Sasakian structures of type (0,0), $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu and α-Sasakian respectively.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero [20]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. So proper trans-Sasakian manifolds exist only for dimension three. The first author of the present paper have studied trans-Sassakian manifold of dimension three [9]. In this paper we are involved with three-dimensional trans-Sasakian manifolds.

In 1924, Friedmann and Schouten [13], introduced the notion of semisymmetric linear connection in a differentiable manifold. Hayden [15] introduced a metric connection with a non-zero torsion on a Riemannian manifold in 1932. In 1970, K. Yano [33], studied semisymmetric metric connection systematically on a Riemannian manifold. The study of semisymmetric metric connection in the area of almost contact manifolds was introduced by A. Sharfuddin [30]. Semisymmetric metric connection on almost contact manifolds has been studied by several authors [11], [25]. Semisymmetric metric connection on trans-Sasakian manifolds has been studied by C. S. Bagewadi in the paper [3]. The geometrical interpretation of semisymmetric metric connection can be found in the book [29], page 143.

The present paper is organized as follows: After the introduction, we give required preliminaries in Section 2. In Section 3, we study biharmonic Legendre curves with respect to semisymmetric metric connection. In Section 4, we give the notion of locally ϕ-symmetric Legendre curves on three-dimensional trans-Sasakian manifolds and study them with respect to semisymmetric metric connection. The last section contains, as illustrative example, the study of Legendre curves on the hyperbolic 3-space $\mathbb{H}^3(-1)$.

2 Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X,\xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [1]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X,Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi,\phi](X,Y) + 2d\eta(X,Y)\xi = 0$.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called trans-Sasakian structure [24] if $(M \times R, J, G)$ belongs to the class W_4 [14], where J is the almost complex structure on $M \times R$ defined by

$$J(X, fd/dt) = (\phi X - f\xi, \eta(X)d/dt),$$

for all vector fields X on M, a smooth function f on $M \times R$ and the product metric G on $M \times R$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for smooth functions α and β on M. α, β are called the structure functions of the manifold. Here ∇ is Levi-Civita connection on M. We say M as the trans-Sasakian manifold of type (α, β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.6}$$

From [10], we get the Riemannian curvature tensor R with respect to Levi-Civita connection of a three-dimensional trans-Sasakian manifold as the following:

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y) \\ &\quad -g(Y,Z)[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\xi \\ &\quad -\eta(X)(\phi\text{grad}\alpha-\text{grad}\beta)+(X\beta+(\phi X)\alpha)\xi] \\ &\quad +g(X,Z)[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\xi \\ &\quad -\eta(Y)(\phi\text{grad}\alpha-\text{grad}\beta)+(Y\beta+(\phi Y)\alpha)\xi] \\ &\quad -[(Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\eta(Z)]X \\ &\quad +[(Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.7}$$

where S is the Ricci tensor of type $(0,2)$, and r is the scalar curvature of the manifold M with respect to Levi-Civita connection.

A linear connection on a Riemannian manifold is called symmetric if the torsion tensor of the connection is zero on the manifold. The connection is called semisymmetric [13], [15], [33] if the torsion tensor $\tilde{T}$ is given by

$$\tilde{T}(X,Y)=\pi(Y)X-\pi(X)Y,$$

where π is a 1-form given by $\pi(X)=g(X,\rho)$, ρ is a unit vector field. A semisymmetirc connection $\tilde{\nabla}$ is called semisymmetric metric connection if $\tilde{\nabla}g=0$. In an almost contact manifold the notion of semisymmetric metric connection was given in the paper [30]. The torsion tensor of a semisymmetric metric connection on an almost contact metric manifold is given by

$$\tilde{T}(X,Y)=\eta(Y)X-\eta(X)Y. \tag{2.8}$$

The semisymmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on an almost contact metric manifold are related by

$$\tilde{\nabla}_XY=\nabla_XY+\eta(Y)X-g(X,Y)\xi, \tag{2.9}$$

for all vector fields X,Y on M.

Let M be a $3-$dimensional Riemannian manifold. Let $\gamma:I\rightarrow M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said that γ is a Frenet curve if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\nabla_tt=0$ (geodesic), $t=\dot{\gamma}$. Here, . denotes differentiation with respect to arc parameter.

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(=\dot{\gamma})$, n and a non-negative function k (curvature) along γ such that $\nabla_tt=kn$, $\nabla_tn=-kt$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(=\dot{\gamma})$, n, b and two non-negative functions k(curvature) and τ(torsion) along γ such that

$$\nabla_tt=kn, \tag{2.10}$$

$$\nabla_tn=-kt+\tau b, \tag{2.11}$$

$$\nabla_tb=-\tau n. \tag{2.12}$$

With respect to Levi-Civita connection, a Frenet curve of osculating order 3 for which k is a positive constant and $\tau=0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if k and τ both are positive constants and the curve is called a generalized helix if $\frac{k}{\tau}$ is a constant.

Let $\tilde{\nabla}_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to semisymmetric metric connection on M. We shall say that γ is a Frenet curve with respect to semisymmetric metric connection if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\tilde{\nabla}_t t = 0$ (geodesic).

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(=\dot{\gamma})$, N and a non-negative function $\tilde{k}$ (curvature) along γ such that $\tilde{\nabla}_t t = \tilde{k}n$, $\tilde{\nabla}_t n = -\tilde{k}t$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(=\dot{\gamma})$, n, b and two non-negative functions $\tilde{k}$(curvature) and $\tilde{\tau}$(torsion) along γ such that

$$\tilde{\nabla}_t t = \tilde{k}n, \tag{2.13}$$

$$\tilde{\nabla}_t n = -\tilde{k}t + \tilde{\tau}b, \tag{2.14}$$

$$\tilde{\nabla}_t b = -\tilde{\tau}n. \tag{2.15}$$

With respect to semisymmetric metric connection, a Frenet curve of osculating order 3 for which $\tilde{k}$ is a positive constant and $\tilde{\tau} = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if $\tilde{k}$ and $\tilde{\tau}$ both are positive constants and the curve is called a generalized helix if $\frac{\tilde{k}}{\tilde{\tau}}$ is a constant.

A Frenet curve γ in an almost contact metric manifold is said to be a Legendre curve or almost contact curve if it is an integral curve of the contact distribution $\mathcal{D} = \ker\eta$. Formally, it is also said that a Frenet curve γ in an almost contact metric manifold is a Legendre curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. For more details we refer [4], [8], [19], [32].

3 Biharmonic Legendre curves with respect to semisymmetric metric connection

In this section we study biharmonic Legendre curves on a three-dimensional trans-Sasakian manifold with respect to semisymmetric metric connection.

Definition 3.1. *A Legendre curve γ on a three-dimensional trans-Sasakian manifold will be called biharmonic with respect to semisymmetric metric connection if it satisfies the biharmonic equation*

$$\tilde{\nabla}_t^3 t + \tilde{\nabla}_t \tilde{T}(\tilde{\nabla}_t t, t)t + \tilde{R}(\tilde{\nabla}_t t, t)t = 0, \tag{3.1}$$

where $\dot{\gamma} = t$. Since for a Legendre curve $\eta(\dot{\gamma}) = 0$, in view of (2.8) the biharmonic equation reduces to

$$\tilde{\nabla}_t^3 t + \tilde{R}(\tilde{\nabla}_t t, t)t = 0. \tag{3.2}$$

For the definition of biharmonic equation we have followed the papers [8] and [17].

Theorem 3.1. *A necessary condition for a non-geodesic Legendre curve on a three-dimensional trans-Sasakian manifold to be biharmonic with respect to semisymmetric metric connection is $\tilde{k}$ = a non zero constant and $\tilde{k}^2 + \tilde{\tau}^2 = 2(\alpha^2 - \beta^2) + 2(\beta + \frac{1}{2}) - \frac{r}{2} - 2\xi\beta$. The condition is sufficient when $\tilde{\tau}$ and the structure functions α and β of the manifold are constants. Here r is the scalar curvature of the manifold with respect to Levi-Civita connection.*

Proof. Let us consider a Legendre curve γ. Let T be the unit tangent vector of the Legendre curve. To maintain orientation let $T, \phi T, \xi$ be a orthonormal right handed system where $\phi T = -N, \phi N = T$. It is to be mentioned that such assumption is compatible with almost contact structure. Then $t = T, n = \phi T = -N, b = B = \xi$ and by Serret-Frenet formula

$$\begin{aligned} \tilde{\nabla}_t t &= \tilde{\nabla}_T T \\ &= \tilde{k}(\phi T) \\ &= -\tilde{k}N. \end{aligned}$$

Then the equation (3.2) reduces to the following:

$$\tilde{\nabla}_T^3 T - \tilde{k}\tilde{R}(N, T)T = 0. \tag{3.3}$$

Let $\tilde{R}$ and R be the Riemannian curvature tensor with respect to semisymmetric metric connection and Levi-Civita connection respectively. Then the relation between $\tilde{R}$ and R is given by [25], [33]

$$\tilde{R}(X,Y)Z = R(X,Y)Z - L(Y,Z)X + L(X,Z)Y - g(Y,Z)FX + g(X,Z)FY, \tag{3.4}$$

where L is a tensor field of type $(0,2)$ given by

$$L(Y,Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y,Z) \tag{3.5}$$

and F is a tensor field of type $(1,1)$ given by $g(FY,Z) = L(Y,Z)$ for any vector field Y,Z on M. Using (2.6) in (3.5) we have, after some calculation

$$L(Y,Z) = -\alpha g(\phi Y,Z) + (\beta + \frac{1}{2})g(Y,Z) - (\beta+1)\eta(Y)\eta(Z). \tag{3.6}$$

Using $g(FY,Z) = L(Y,Z)$ in the above equation we obtain

$$FY = -\alpha\phi Y + (\beta + \frac{1}{2})Y - (\beta+1)\eta(Y)\xi. \tag{3.7}$$

Using (3.6) and (3.7) in (3.4) we get for a three-dimensional trans Sasakian manifold

$$\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha(g(\phi Y,Z)X - g(\phi X,Z)Y) \\
&+ \alpha(g(Y,Z)\phi X - g(X,Z)\phi Y) - 2(\beta + \frac{1}{2})(g(Y,Z)X - g(X,Z)Y) \\
&+ (\beta+1)(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi) \\
&+ (\beta+1)\eta(Z)(\eta(Y)X - \eta(X)Y).
\end{aligned} \tag{3.8}$$

For a Legendre curve $\eta(T) = 0, \eta(N) = 0$ because we have considered the Frenet frame as $T, \phi T, \xi$, where $\phi T = -N$. Using these facts in (3.8) and considering (2.7) we get, after simplification

$$\tilde{R}(N,T)T = (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))N - (N\beta + T\alpha)B. \tag{3.9}$$

By serret-Frenet formula we get

$$\tilde{\nabla}_T^3 T = -3\tilde{k}\tilde{k}'T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2)N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}')B. \tag{3.10}$$

In view of (3.9) and (3.10), it follows that

$$\begin{aligned}
\tilde{\nabla}_T^3 T - \tilde{k}\tilde{R}(N,T)T &= -3\tilde{k}\tilde{k}'T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 - \tilde{k}(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2})))N \\
&+ (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' + \tilde{k}(N\beta + T\alpha))B.
\end{aligned} \tag{3.11}$$

For biharmonicity by virtue of (3.3) and observing the components of the right hand side of (3.11) we get $\tilde{k}$ is a constant and consequently the theorem is proved.

4 Locally ϕ-symmetric Legendre curves with respect to semisymmetric metric connection.

The notion of locally ϕ-symmetric manifolds was introduced by T. Takahashi [31]. Since every smooth curve is one-dimensional differentiable manifold we may apply the concept of local ϕ-symmetry on a smooth curve. In [26], locally ϕ-symmetric Legendre curves have been studied in Heisenberg group. In the following we give the definition of locally ϕ-symmetric Legendre curves with respect to semisymmetric metric connection.

Definition 4.1. *With respect to semisymmetric metric connection a Legendre curve γ on a three-dimensional trans-Sasakian manifold will be called locally ϕ-symmetric if it satisfies*

$$\phi^2(\tilde{\nabla}_t \tilde{R})(\tilde{\nabla}_t t, t)t = 0, \tag{4.1}$$

where $\dot{\gamma} = t$.

Theorem 4.1. *A necessary and sufficient condition for a non-geodesic Legendre curve on a three-dimensional trans-Sasakian manifold with constant structure function to be locally ϕ-symmetric with respect to semisymmetric metric connection is $r = 4(\alpha^2 - \beta^2) + 4(\beta + \frac{1}{2})$, where r is the scalar curvature of the manifold with respect to Levi-Civita connection.*

Proof. Let us consider a Legendre curve γ. Now, proceeding in the same way as in the previous section we get

$$\begin{aligned} \tilde{R}(\tilde{\nabla}_T T, T)T &= \tilde{R}(\tilde{k}\phi T, T)T \\ &= -\tilde{k}\tilde{R}(N, T)T \\ &= -\tilde{k}(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))N + \tilde{k}(N\beta + \alpha T). \end{aligned} \tag{4.2}$$

By virtue of (3.8) and (2.7) we have

$$\begin{aligned} \tilde{R}(B, T)T &= \tilde{R}(\xi, T)T \\ &= (2\xi\beta + \alpha^2 - \beta^2 - \beta)\xi - (\phi\text{grad}\alpha - \text{grad}\beta) + (T\beta - N\alpha)T. \end{aligned} \tag{4.3}$$

By definition of covariant derivative of $\tilde{R}$ we get

$$(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = \tilde{\nabla}_T\tilde{R}(\tilde{\nabla}_T T, T)T - \tilde{R}(\tilde{\nabla}_T^2 T, T)T - \tilde{R}(\tilde{\nabla}_T T, \tilde{\nabla}_T T)T - \tilde{R}(\tilde{\nabla}_T T, T)\tilde{\nabla}_T T.$$

Using Serret-Frenet formula we get, after simplification, from above equation

$$(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = \tilde{\nabla}_T\tilde{R}(\tilde{\nabla}_T T, T)T + \tilde{k}'\tilde{R}(N, T)T + \tilde{k}\tilde{\tau}\tilde{R}(B, T)T.$$

After some straight forward calculations, the above equation together with (3.9), (4.2) and (4.3) yields

$$\begin{aligned} (\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = & -\tilde{k}(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))'N \\ & -\tilde{k}(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))(-\tilde{k}T + \tilde{\tau}B) \\ & +\tilde{k}(N\beta + T\alpha)'B \\ & +\tilde{k}\tilde{\tau}(N\beta + T\alpha)N \\ & +\tilde{k}\tilde{\tau}(2\xi\beta + \alpha^2 - \beta^2 - \beta)B \\ & -\tilde{k}\tilde{\tau}(\phi\text{grad}\alpha - \text{grad}\beta) \\ & +\tilde{k}\tilde{\tau}(T\beta - N\alpha)T. \end{aligned} \tag{4.4}$$

Applying ϕ^2 in both sides of the above equation and using (2.1) we obtain

$$\begin{aligned} \phi^2(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = & \tilde{k}(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))'N \\ & -\tilde{k}^2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))T \\ & -\tilde{k}\tilde{\tau}(N\beta + T\alpha)N \\ & -\tilde{k}\tilde{\tau}\phi^2(\phi\text{grad}\alpha - \text{grad}\beta) \\ & -\tilde{k}\tilde{\tau}(T\beta - N\alpha)T. \end{aligned} \tag{4.5}$$

If the structure functions α, β are constants, then the above equation reduces to

$$\begin{aligned} \phi^2(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = & \tilde{k}(\frac{r}{2} - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))'N \\ & -\tilde{k}^2(\frac{r}{2} - 2(\alpha^2 - \beta^2) - 2(\beta + \frac{1}{2}))T. \end{aligned} \tag{4.6}$$

By virtue of Definition 4.1 and the above equation the theorem follows.

5 Legendre curves on a hyperbolic 3-space $\mathbb{H}^3(-1)$ with semisymmetric metric connection

In this section we give a hyperbolic space $\mathbb{H}^3(-1)$ as an example of three-dimensional trans-Sasakian manifold and then study Legendre curves on it with semisymmetric metric connection.

We consider the well-known three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -e_2, \qquad [e_1, e_3] = -e_1.$$

By Koszul formula

$$\begin{aligned}
&\nabla_{e_1} e_3 = -e_1, \quad &&\nabla_{e_1} e_2 = 0, \quad &&\nabla_{e_1} e_1 = e_3, \\
&\nabla_{e_2} e_3 = -e_2, \quad &&\nabla_{e_2} e_2 = e_3, \quad &&\nabla_{e_2} e_1 = 0, \\
&\nabla_{e_3} e_3 = 0, \quad &&\nabla_{e_3} e_2 = 0, \quad &&\nabla_{e_3} e_1 = 0.
\end{aligned}$$

From above we see that the manifold satisfies (2.5) for $\alpha = 0$, $\beta = -1$, and $e_3 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0, -1)$. With the help of the above results it can be verified that

$$\begin{aligned}
&R(e_1, e_2)e_3 = 0, \quad &&R(e_2, e_3)e_3 = -e_2, \quad &&R(e_1, e_3)e_3 = -e_1, \\
&R(e_1, e_2)e_2 = -e_1, \quad &&R(e_2, e_3)e_2 = e_3, \quad &&R(e_1, e_3)e_2 = 0, \\
&R(e_1, e_2)e_1 = e_2, \quad &&R(e_2, e_3)e_1 = 0, \quad &&R(e_1, e_3)e_1 = e_3.
\end{aligned}$$

From above we see that the manifold satisfies $R(X, Y)Z = (-1)(g(Y, Z)X - g(X, Z)Y)$, i. e., the manifold is of constant curvature -1. Hence the manifold is a hyperbolic space. We denote it by $\mathbb{H}^3(-1)$.

Using (2.9), it can be easily calculated that

$$\begin{aligned}
&\tilde{\nabla}_{e_1} e_1 = 0, \qquad &&\tilde{\nabla}_{e_2} e_1 = 0, \\
&\tilde{\nabla}_{e_1} e_2 = 0, \qquad &&\tilde{\nabla}_{e_2} e_2 = 0.
\end{aligned} \tag{5.1}$$

Using these results we construct a Legendre curves on this trans-Sasakian manifold of type $(0, -1)$ in the following.

Example 5.1. Consider a curve $\gamma : I \to M$ defined by $\gamma(s) = (\sqrt{\frac{2}{3}}s, \sqrt{\frac{1}{3}}s, 1)$. Hence $\dot{\gamma}_1 = \sqrt{\frac{2}{3}}$, $\dot{\gamma}_2 = \sqrt{\frac{1}{3}}$ and $\dot{\gamma}_3 = 0$, $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$. Now

$$\eta(\dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, e_3) = 0.$$

$$\begin{aligned} g(\dot{\gamma},\dot{\gamma}) &= g(\dot{\gamma}_1 e_1+\dot{\gamma}_2 e_2+\dot{\gamma}_3 e_3, \dot{\gamma}_1 e_1+\dot{\gamma}_2 e_2+\dot{\gamma}_3 e_3) \\ &= \dot{\gamma}_1^2+\dot{\gamma}_2^2+\dot{\gamma}_3^2 \\ &= \dot{\gamma}_1^2+\dot{\gamma}_2^2 \\ &= \frac{2}{3}+\frac{1}{3} \\ &= 1. \end{aligned}$$

Hence the curve is a Legendre curve. For this curve $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$. Hence the curve is a geodesic with respect to semisymmetric metric connection. The curvature and torsion with respect to semisymmetric metric of this curve are zero. The curve is trivially biharmonic.

Since the manifold under consideration satisfies $R(X,Y)Z = (-1)(g(Y,Z)X - g(X,Z)Y)$, we find that the scalar curvature r of the manifold with respect to Levi-Civita connection is -6. So the manifold satisfies $r = 4(\alpha^2-\beta^2)+4(\beta+\frac{1}{2})$ for $\alpha = 0$, $\beta = -1$. Hence by Theorem 4.1, we obtain *any Legendre curves on the space* $\mathbb{H}^3(-1)$ *is locally* ϕ*-symmetric with respect to semisymmetric metric connection.*

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