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A generalization of the Laplacian differential operator

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Dedicated to my wife Romina and
my sons Maximiliano and Benjamin.

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Abstract. In this paper, a generalization of the Laplacian differential operator in fractional derivative is presented. Using the Fourier and Laplace Transforms, we study the fundamental solution of an initial value problem. The solution presented in this paper is given by a function in term of the Wright function.

1 Preliminaries

In recent years several authors have studied Differential operators in the context of the Fractional Calculus.

The Laplacian operator $\triangle$ in $\mathbb{R}^n$ is defined by the expression

$$\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1.1}$$

where if $n = 2$, then we have

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} \tag{1.2}$$

In this note, we will be using the following important definitions.

Definition 1.1. Let be a function f, an exponential order function and piecewise continuous. The Laplace transform (cf. [2]) of the function f is given by

$$\mathfrak{L}\left[f\right](z) = \int_0^\infty e^{-zt} f(t)dt, \qquad z \in \mathbb{C} \tag{1.3}$$

Definition 1.2. Let f be a function of the space $S(\mathbb{R})$ the Schwartzian space of functions that decay rapidly at infinity together with all derivatives.

The Fourier transform (cf. [2]) $\hat{f}(\omega)$ is given by the integral

$$\hat{f}(\omega) = \mathfrak{F}[f](\omega) = \int_{\mathbb{R}} f(t)e^{i\omega t}dt \tag{1.4}$$

With respect to the fractional calculus, we study the Caputto fractional derivative given by the following expression

$$D_+^\nu f(t) = \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\nu)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\nu+1-m}}\right], & m-1<\nu<m \\ \frac{d^m}{dt^m} f(t) \qquad\qquad , & \nu = m \end{cases} \tag{1.5}$$

and we present a fundamental function of the fractional calculus that is solution of the different fractional differential equation using Wright function.

It is known that the Wright function $W_{\alpha,\beta}$ is defined (cf. [6]) by the series

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n + \beta)};\ \alpha > -1;\ \beta \in \mathbb{C} \tag{1.6}$$

where $\Gamma(z)$ is the Euler Gamma function given by the integral

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \quad \text{(cf. [6])}$$

.

2 Main Result

In this section, we present the function $\mathcal{E}(\alpha, t)$ and its Laplace transform is obtained. Then we will define the AFA Fractional Differential Operator as a generalization of the Laplacian and then we will obtain fundamental solution.

2.1 The function $\mathcal{E}(\alpha, t)$

Definition 2.1. The function $\mathcal{E}(\alpha, t)$ is defined by the following expression

$$\mathcal{E}(\alpha, t) = t^{-\alpha}\, W_{-\alpha,1-\alpha}(a\, t^{-\alpha}) \tag{2.1}$$

where $W_{\alpha,\beta}(t)$ is the Wright function, $t \in \mathbb{R}$ and a is be a real number.

Theorem 2.2. *The Laplace Transform of the $\mathcal{E}(\alpha, t)$ function is given by*

$$\mathfrak{L}\left[\mathcal{E}(\alpha, t)\right](s) = s^{\alpha-1} e^{a s^{\alpha}} \tag{2.2}$$

Proof. From definition (1.3) and (2.1) we have

$$\mathfrak{L}\left[\mathcal{E}(\alpha, t)\right](s) = \int_0^{+\infty} e^{-st} t^{-\alpha}\, W_{-\alpha,1-\alpha}(a\, t^{-\alpha}) dt =$$

$$\int_0^{+\infty} e^{-st} t^{-\alpha} \sum_{k=0}^{\infty} \frac{a^k\, t^{-k\alpha}}{n!\Gamma(-k\alpha + 1 - \alpha)} dt$$

$$\int_0^{+\infty} e^{-st} \sum_{k=0}^{\infty} \frac{a^k\, t^{-(k+1)\alpha}}{n!\Gamma(-k\alpha + 1 - \alpha)} dt \tag{2.3}$$

As the Wright function converges uniformly, (2.3) implies

$$\mathfrak{L}\left[\mathcal{E}(\alpha, t)\right](s) = \sum_{k=0}^{\infty} \frac{a^k}{n!\Gamma(-k\alpha + 1 - \alpha)} \int_0^{+\infty} e^{-st} t^{-(k+1)\alpha} dt \tag{2.4}$$

Taking into account that the integral in (2.4) is

$$\int_0^{+\infty} e^{-st} t^{-(k+1)\alpha} dt = \frac{\Gamma(-k\alpha - \alpha + 1)}{s^{-k\alpha-\alpha+1}} \tag{2.5}$$

From (2.5) and (2.4) we have

$$\mathfrak{L}\left[\mathcal{E}(\alpha, t)\right](s) = \sum_{k=0}^{\infty} \frac{a^k}{n!\Gamma(-k\alpha + 1 - \alpha)} \frac{\Gamma(-k\alpha - \alpha + 1)}{s^{-k\alpha-\alpha+1}} =$$

$$s^{\alpha-1} \sum_{k=0}^{\infty} \frac{(a\, s^{\alpha})^k}{k!} = s^{\alpha-1} e^{a s^{\alpha}} \tag{2.6}$$

□

2.2 AFA Fractional Differential Operator

Definition 2.3. The AFA Fractional Differential Operator is defined as

$$\triangle_{AFA} = \frac{\partial^2}{\partial x^2} + \frac{\partial^\alpha}{\partial t^\alpha} \tag{2.7}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputto derivative for $1 < \alpha \leq 2$.
If $\alpha = 2$, the AFA Fractional Differential Operator is a generalization of the Laplacian Differential Operator.

Then, we study the following equivalent initial value problem

$$\frac{\partial^\alpha}{\partial t^\alpha}\varphi(x,t) = -\frac{\partial^2}{\partial x^2}\varphi(x,t); \qquad 1 < \alpha \leq 2 \tag{2.8}$$

$$\varphi(x,t)|_{t=0} = \delta(x) \tag{2.9}$$

where $\delta(x)$ is the distribution delta of Dirac.

$$\frac{\partial}{\partial t}\varphi(x,t)|_{t=0} = 0 \tag{2.10}$$

Now, applying the Laplace Transform with respect to the variable t in the equation (II.8) and recalling the property of the Laplace transform of the Caputto Derivative,

$$\mathfrak{L}\left[\frac{\partial^\alpha}{\partial t^\alpha}\varphi(x,t)\right](s) = s^\alpha \mathfrak{L}\left[\varphi(x,t)\right](s) - s^{\alpha-1}\varphi(x,t)|_{t=0} - s^{\alpha-2}\frac{\partial}{\partial t}\varphi(x,t)|_{t=0}$$

we have

$$s^\alpha \mathfrak{L}\left[\varphi(x,t)\right](s) - s^{\alpha-1}\varphi(x,t)|_{t=0} - s^{\alpha-2}\frac{\partial}{\partial t}\varphi(x,t)|_{t=0} = -\frac{\partial^2}{\partial x^2}\mathfrak{L}\left[\varphi(x,t)\right](s)$$

$$s^\alpha \mathfrak{L}\left[\varphi(x,t)\right](s) - s^{\alpha-1}\delta(x) = -\frac{\partial^2}{\partial x^2}\mathfrak{L}\left[\varphi(x,t)\right](s) \tag{2.11}$$

Denote $\mathfrak{L}\left[\varphi(x,t)\right](s) = \widetilde{\varphi}(x,s)$ in (II.11) we have

$$s^\alpha \widetilde{\varphi}(x,s) - s^{\alpha-1}\delta(x) = -\frac{\partial^2}{\partial x^2}\widetilde{\varphi}(x,s) \tag{2.12}$$

In (2.12) applying Fourier Transform respect to variable x result

$$s^\alpha \mathfrak{F}\left[\widetilde{\varphi}(x,s)\right](\omega) - s^{\alpha-1}\mathfrak{F}\left[\delta(x)\right](\omega) = -\mathfrak{F}\left[\frac{\partial^2}{\partial x^2}\widetilde{\varphi}(x,s)\right](\omega) \tag{2.13}$$

By recalling $\mathfrak{F}[\delta(x)](\omega) = 1$ and the property of the Fourier transform of the derivative $\mathfrak{F}\left[\frac{\partial^2}{\partial x^2}\widetilde{\varphi}(x,s)\right](\omega) = -\omega^2\mathfrak{F}\left[\widetilde{\varphi}(x,s)\right](\omega)$, in (2.13), we have

$$s^\alpha \mathfrak{F}\left[\widetilde{\varphi}(x,s)\right](\omega) - s^{\alpha-1} = \omega^2\mathfrak{F}\left[\widetilde{\varphi}(x,s)\right](\omega) \tag{2.14}$$

Hence we have

$$\mathfrak{F}\left[\widetilde{\varphi}(x,s)\right](\omega) = \frac{s^{\alpha-1}}{s^\alpha - \omega^2} = \frac{(-1)s^{\alpha-1}}{(-1)s^\alpha + \omega^2} \tag{2.15}$$

Taking into account that the Fourier Transform maps $S(\mathbb{R})$ into itself, for the function $\frac{m}{n+\omega^2}$ $(m, n \in \mathbb{R})$ belonging to $S(\mathbb{R})$, it is known that (cf. [1])

$$\frac{m}{n+\omega^2} = \mathfrak{F}\left[\frac{m}{2n^{1/2}}e^{-|x|n^{1/2}}\right]$$

and calling $m = (-1)s^{\alpha-1}$ and $n = (-1)s^{\alpha}$ from (II.15), we have

$$\widetilde{\varphi}(x,s) = \frac{(-1)s^{\alpha-1}}{2(-1)^{1/2}s^{\alpha/2}}\, e^{-|x|(-1)^{1/2}s^{\alpha/2}} =$$

$$\frac{1}{2}\, i\, s^{\alpha/2-1}\, e^{-i|x|s^{\alpha/2}} \tag{2.16}$$

From (2.2), the expression (2.16) is the Laplace Transform of the function $\mathcal{E}(\frac{\alpha}{2},t)$ multiplied by the complex number $\frac{1}{2}\, i$, for $a = -i\,|x|$. Thus we have

$$\varphi(x,s) = \frac{1}{2}\, i\, t^{-\alpha/2}\, W_{-\frac{\alpha}{2},1-\frac{\alpha}{2}}\left(-i\,|x|\; t^{-\alpha/2}\right) \tag{2.17}$$

that is the fundamental solution of the AFA Fractional Differential Operator.

If $\alpha = 2$ in the expression (2.7) the operator reduce to the classical Laplacian

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} \tag{2.18}$$

and considering the initial value problem by a analogous process to arrive

$$\mathfrak{F}\left[\widetilde{\varphi}(x,s)\right](\omega) = \frac{(-1)s}{(-1)s^2+\omega^2} \tag{2.19}$$

Hence

$$\widetilde{\varphi}(x,s) = \frac{1}{2}\, i\, e^{-i|x|s} \tag{2.20}$$

It is the Laplace transform of the function $\mathcal{E}(1,t)$ multiplied by the complex number $\frac{1}{2}\, i$, for $a = -i\,|x|$. Hence we have

$$\varphi(x,s) = \frac{1}{2}\, i\, t^{-1}\, W_{-1,0}\left(-i\,|x|\, t^{-1}\right) \tag{2.21}$$

Note that if $a = 2$ in the expression (2.17), this coincides exactly with (2.21).

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