Palestine Journal of Mathematics

Vol. 5(2) (2016) , 208–217

ON JORDAN TRIPLE HIGHER DERIVATIONS ON PRIME Γ- RINGS

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Communicated by Ayman Badawi

MSC 2012 Classifications: Primary 16W25 , 16Y30.

Keywords and phrases: Prime-rings , derivations , Γ-Ring, higher derivation

Abstract Let M be a Γ-ring and $\mathbb{N}$ be the set of non-negative integers. A family $D = \{d_n\}_{n\in\mathbb{N}}$ of additive mappings $d_n : M \to M$ such that $d_0 = I_M$ is said to be a triple higher derivation (resp. Jordan triple higher derivation) on M if $d_n(a\alpha b\beta c) = \sum\limits_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c)$ (resp. $d_n(a\alpha b\beta a) = \sum\limits_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(a)$) holds for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, and for each $n \in \mathbb{N}$. In the present paper it is shown that on prime Γ-ring M of characteristic different from two every Jordan triple higher derivation on M is a higher derivation on M.

1 Introduction

An additive mapping $d : R \to R$ is said to be a derivation (resp. Jordan derivation) on a ring R if $d(ab) = d(a)b + ad(b)$ (resp. $d(a^2) = d(a)a + ad(a)$) for any $a, b \in R$. Obviously, every derivation on R is a Jordan derivation on R. But the converse is in general not true. Herstein [12] obtained condition on R under which every Jordan derivation becomes a derivation. He proved that on a prime ring of characteristic different from two, every Jordan derivation becomes a derivation. Subsequently, various authors extended this result for semiprime ring (see [5] and [7]). Later, M. Brešar [6] introduced the concept of triple derivation and Jordan triple derivation on R. An additive mapping $d : R \to R$ is said to be a triple derivation (resp. Jordan triple derivation) on R if $d(abc) = d(a)bc+ad(b)c+abd(c)$ (resp. $d(aba) = d(a)ba+ad(b)a+abd(a)$) holds for all $a, b, c \in R$. In fact, he proved that every Jordan triple derivation on a 2-torsion free semiprime ring is a derivation.

The concept of derivation was extended to higher derivation by F. Hasse and F.K. Schmidt [11]. Let $\mathbb{N}$ be the set of all nonnegative integers and let $D = \{d_n\}_{n\in\mathbb{N}}$ be a family of additive maps on a ring R such that $d_0 = I_R$. D is said to be;

(a) a *higher derivation* on R if for each $n \in \mathbb{N}$,

$$d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$$

for all $a, b \in R$;

(b) a *Jordan higher derivation* on R if for each $n \in \mathbb{N}$,

$$d_n(a^2) = \sum_{i+j=n} d_i(a)d_j(a)$$

for all $a \in R$;

(c) a *Jordan triple higher derivation* on R if for each $n \in \mathbb{N}$,

$$d_n(aba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a)$$

for all $a, b \in R$.

The classical result due to Herstein [[12], Theorem] was extended for higher derivations by Haetinger [10], who proved that every Jordan higher derivation on a prime ring of characteristic different from two is a higher derivation. Further, in the year 2002, Ferrero and Haetinger [8] established that on a 2-torsion free semiprime ring every Jordan triple higher derivation of R is a higher derivation of R. Further similar studies have also been made by various authors in the setting of prime and semiprime rings (for reference see [1, 3], where more references can be found).

On the other hand, the notion of Γ-ring was first introduced as an extensive generalization of the concept of a classical ring by Nobusawa [15] and further, it was slightly weakened by Barnes [4]. Let M and Γ be additive abelian groups. If there exists a mapping $M \times \Gamma \times M \to M$ satisfying

(i) $a\alpha b \in M$,

(ii) $(a+b)\alpha c = a\alpha c + b\alpha c,\ a(\alpha+\beta)c = a\alpha c + a\beta c,\ a\alpha(b+c) = a\alpha b + a\alpha c,$

(iii) $(a\alpha b)\beta c = a\alpha(b\beta c),$ for all $a, b, c \in M$ and $\alpha, \beta, \in \Gamma$,

then M is called a Γ-ring.
M is said to be a prime Γ- ring if $a\Gamma M\Gamma b = \{0\}$ implies that either $a = 0$ or $b = 0$ for $a, b \in M$. M is said to be a semiprime Γ- ring if $a\Gamma M\Gamma a = \{0\}$ implies $a = 0$ for $a \in M$. Throughout this paper, M will denote a Γ-ring with center $Z(M)$ i.e., $Z(M) = \{a \in M \mid a\alpha b = b\alpha a$ for all $b \in M$ for all $\alpha \in \Gamma\}$.

Let M be a Γ-ring. An additive mapping $d : M \to M$ is said to be a derivation on M if for all $a, b \in M$ and for all $\alpha \in \Gamma$, $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, a Jordan derivation if $d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$. This notion was defined by M. Sapanci and A. Nakajima in [17]. They investigated Jordan derivation on certain types of completely prime Γ-ring. Later, many prominent mathematicians have worked out on this interesting area of research to determine many basic properties of Γ-rings and have extended numerous remarkable results in this context in the last few decades.

Very recently, in the year 2012, A.K. Joardar and A.C. Paul [14] introduced the concept of triple derivation and Jordan triple derivation on Γ-ring M. An additive mapping $d : M \to M$ is said to be a triple derivation (resp. Jordan triple derivation) if $d(a\alpha b\beta c) = d(a)\alpha b\beta c + a\alpha d(b)\beta c + a\alpha b\beta d(c)$, (resp. $d(a\alpha b\beta a) = d(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$), for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$.

In the present paper we introduce the concept of higher derivation (resp. Jordan higher derivation) and triple higher derivation (resp. Jordan triple higher derivation) on a Γ-ring M and prove that every Jordan triple higher derivation on a prime Γ-ring M of characteristic different from two is a triple higher derivation on M and finally, it is shown that every Jordan triple higher derivation is a higher derivation on M.

2 Main Results

Motivated by the existence of higher derivation in rings, the notion of higher derivation on a Γ-ring M is defined as follows. Let M be a Γ-ring and let $D = \{d_n\}_{n\in\mathbb{N}}$ be a family of additive mappings $d_n : M \to M$ such that $d_0 = I_M$. Then D is said to be

(a) a *higher derivation* on M if for each $n \in \mathbb{N}$,

$$d_n(a\alpha b) = \sum_{p+q=n} d_p(a)\alpha d_q(b)$$

for all $a, b \in M$, and $\alpha \in \Gamma$;

(b) a *Jordan higher derivation* on M if for each $n \in \mathbb{N}$,

$$d_n(a\alpha a) = \sum_{p+q=n} d_p(a)\alpha d_q(a)$$

for all $a \in M$ and $\alpha \in \Gamma$;

(c) a *triple higher derivation* on M if for each $n \in \mathbb{N}$,

$$d_n(a\alpha b\beta c) = \sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c)$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$;

(d) a *Jordan triple higher derivation* on M if for each $n \in \mathbb{N}$,

$$d_n(a\alpha b\beta a) = \sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(a)$$

for all $a, b \in M$ and $\alpha, \beta \in \Gamma$.

Example 2.1. Let R be an associative ring with unity element 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \cdot 1 \\ 0 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$ where $\mathbb{Z}$ be the set of integers.

Then M is a Γ-ring. Let $f : R \to R$ be a triple derivation on R. Now define $F : M \to M$ such that $F((x,y)) = (f(x), f(y))$. Then F is a triple derivation on M. In fact, if $a = (x_1, y_1),\ b = (x_2, y_2),\ c = (x_3, y_3) \in M$, $\alpha = \begin{pmatrix} n_1 \cdot 1 \\ 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} n_2 \cdot 1 \\ 0 \end{pmatrix} \in \Gamma$, then $a\alpha b\beta c = (x_1 n_1 x_2 n_2 x_3,\ x_1 n_1 x_2 n_2 y_3)$, and finally we get $F(a\alpha b\beta c) = F(a)\alpha b\beta c + a\alpha F(b)\beta c + a\alpha b\beta F(c)$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.
Define $d_n = F^n/n!$, for all $n \in \mathbb{N}$, where F is a triple derivation on M.

Claim: $D = \{d_n\}_{n\in\mathbb{N}}$ is a triple higher derivation on M.

We shall use induction on n to prove the claim;
For $n = 0$, $d_0(a\alpha b\beta c) = \dfrac{F^0(a\alpha b\beta c)}{0!} = a\alpha b\beta c.$

For $n = 1$, $d_1(a\alpha b\beta c) = \dfrac{F^1(a\alpha b\beta c)}{1!} = F(a\alpha b\beta c) = F(a)\alpha b\beta c + a\alpha F(b)\beta c + a\alpha b\beta F(c).$
Suppose that, $d_m = \dfrac{F^m}{m!}$ defines a triple higher derivation on M for each $m < n$.
Consider, $d_n(a\alpha b\beta c) = \dfrac{F^n(a\alpha b\beta c)}{n!} = \dfrac{1}{n}\left(F\left(\dfrac{F^{n-1}(a\alpha b\beta c)}{(n-1)!}\right)\right) = \dfrac{1}{n}F(d_{n-1}(a\alpha b\beta c)).$
Applying the hypothesis of induction on d_{n-1}, we have

$$
\begin{aligned}
d_n(a\alpha b\beta c) &= \frac{F}{n}\sum_{p+q+r=n-1} d_p(a)\alpha d_q(b)\beta d_r(c) \\
&= \frac{F}{n}\sum_{p+q+r=n-1} \frac{F^p(a)}{p!}\alpha\frac{F^q(b)}{q!}\beta\frac{F^r(c)}{r!} \\
&= \frac{1}{n}\sum_{p+q+r=n-1}\Bigg(\frac{F^{p+1}(a)}{p!}\alpha\frac{F^q(b)}{q!}\beta\frac{F^r(c)}{r!} + \frac{F^p(a)}{p!}\alpha\frac{F^{q+1}(b)}{q!}\beta\frac{F^r(c)}{r!} \\
&\quad + \frac{F^p(a)}{p!}\alpha\frac{F^q(b)}{q!}\beta\frac{F^{r+1}(c)}{r!}\Bigg) \\
&= \frac{1}{n}\sum_{p+q+r=n-1}\Big(d_{p+1}(a)\alpha d_q(b)\beta d_r(c)(p+1) + d_p(a)\alpha d_{q+1}(b)\beta d_r(c)(q+1) \\
&\quad + d_p(a)\alpha d_q(b)\beta d_{r+1}(c)(r+1)\Big) \\
&= \frac{1}{n}\Bigg\{\sum_{j=0}^{n-1}\Bigg(\sum_{i=0,i<j}^{j} d_{i+1}(a)\alpha d_{j-i}(b)\Bigg)\beta d_{n-1-j}(c)(i+1) \\
&\quad + \sum_{j=0}^{n-1}\Bigg(\sum_{i=0,i<j}^{j} d_i(a)\alpha d_{j-i+1}(b)\Bigg)\beta d_{n-1-j}(c)(j-i+1) \\
&\quad + \sum_{j=0}^{n-1}\Bigg(\sum_{i=0,i<j}^{j} d_i(a)\alpha d_{j-i}(b)\Bigg)\beta d_{n-j}(c)(n-j)\Bigg\} \\
&= \frac{1}{n}\sum_{j=2}^{n-2}\sum_{i=2}^{j-1} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c)i + \frac{1}{n}\sum_{i=2}^{n-1} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)i \\
&\quad - \frac{1}{n}\sum_{j=2}^{n-2}\sum_{i=2}^{j-1} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c) - \frac{1}{n}\sum_{i=2}^{n-2} d_i(a)\alpha d_{n-i-1}(b)\beta d_1(c)
\end{aligned}
$$

$$+\frac{1}{n}\sum_{j=2}^{n-1} d_j(a)\alpha b\beta d_{n-j}(c)j - \frac{1}{n}\sum_{j=2}^{n-1} d_j(a)\alpha b\beta d_{n-j}(c)$$

$$+\frac{1}{n}\sum_{j=2}^{n-2} d_j(a)\alpha d_{n-j}(b)\beta cj + \frac{1}{n}d_n(a)\alpha b\beta cn$$

$$+\frac{1}{n}d_{n-1}(a)\alpha d_1(b)\beta c(n-1) - \frac{1}{n}d_n(a)\alpha b\beta c - \frac{1}{n}d_{n-1}(a)\alpha d_1(b)\beta c$$

$$-\frac{1}{n}\sum_{j=2}^{n-2} d_j(a)\alpha d_{n-j}(b)\beta c + \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{j} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c)$$

$$-\frac{1}{n}\sum_{j=2}^{n-2}\sum_{i=1}^{j-1} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c)i + \frac{1}{n}\sum_{i=0}^{n-2} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)n$$
$$-\frac{1}{n}\sum_{i=0}^{n-2} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c) - \frac{1}{n}\sum_{i=2}^{n-2} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)i$$

$$-\frac{1}{n}d_1(a)\alpha d_{n-2}(b)\beta d_1(c) + \sum_{i=0}^{n-1} d_i(a)\alpha d_{n-j}(b)\beta c$$

$$-\frac{1}{n}\sum_{i=2}^{n-1} d_i(a)\alpha d_{n-i}(b)\beta ci - \frac{1}{n}d_1(a)\alpha d_{n-1}(b)\beta c$$

$$+\sum_{j=0}^{n-1}\sum_{i=0}^{j} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c) - \frac{1}{n}d_1(a)\alpha b\beta d_{n-1}(c)$$

$$-\sum_{i=0}^{n-1} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c) + \frac{1}{n}\sum_{i=0}^{n-1} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)$$

$$= \sum_{j=0}^{n-1}\sum_{i=0}^{j} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c) + d_n(a)\alpha b\beta c+$$

$$+\sum_{i=0}^{n-1} d_i(a)\alpha d_{n-i}(b)\beta c$$

$$= \sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c).$$

Thus, the family $D = \{d_n\}_{n\in\mathbb{N}}$ where, $d_n = \dfrac{F^n}{n!}$ defines a triple higher derivation on M.

Similarly, if $f : R \to R$ is considered to be a Jordan triple derivation on R then using similar procedure one can find example of a Jordan triple higher derivation on M.

Remark 2.2. It is also to remark that in the above example if we consider $f : R \to R$ as derivation (resp. Jordan derivation), then using similar arguments as given in the above example with necessary variations, one can construct the example of higher derivation (resp. Jordan higher derivation) on M.

It can be easily observe that every triple higher derivation is a Jordan triple higher derivation. But the converse is not true in general. In the present paper we establish the converse of the above statement and proved the following theorem:

Theorem 2.3. *Let M be a prime Γ-ring of characteristic different from two, then every Jordan triple higher derivation on M is a triple higher derivation on M.*

Before proving our main result, let us list some basic facts, which will be used without specific mention to prove our main results.

The proofs of the following results can be seen in [14]. For all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, let $[a, b, c]_{\alpha,\beta} = a\alpha b\beta c - c\alpha b\beta a$.

Lemma 2.4. *([14], Lemma 2.3) If M is a Γ-ring, then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$*

(i) $[a, b, c]_{\alpha,\beta} + [c, b, a]_{\alpha,\beta} = 0$

(ii) $[a + c, b, d]_{\alpha,\beta} = [a, b, d]_{\alpha,\beta} + [c, b, d]_{\alpha,\beta}$

(iii) $[a, b, c + d]_{\alpha,\beta} = [a, b, c]_{\alpha,\beta} + [a, b, d]_{\alpha,\beta}$

(iv) $[a, b + d, c]_{\alpha,\beta} = [a, b, c]_{\alpha,\beta} + [a, d, c]_{\alpha,\beta}$

(v) $[a, b, c]_{\alpha+\beta,\gamma} = [a, b, c]_{\alpha,\gamma} + [a, b, c]_{\beta,\gamma}$

(vi) $[a, b, c]_{\alpha,\beta+\gamma} = [a, b, c]_{\alpha,\beta} + [a, b, c]_{\alpha,\gamma}$.

Lemma 2.5. *([14], Lemma 2.6) Let M be a 2-torsion free semiprime Γ-ring and suppose that $a, b \in M$. If $a\Gamma m\Gamma b + b\Gamma m\Gamma a = 0$ for all $m \in M$, then $a\Gamma m\Gamma b = b\Gamma m\Gamma = 0$.*

Now we prove the following:

Lemma 2.6. *Let M be a Γ-ring and $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple higher derivation on M. Then for all $a, b, c \in M$, and for all $\alpha, \beta \in \Gamma$, we have*

$$d_n(a\alpha b\beta c + c\alpha b\beta a) = \sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c) + \sum_{p+q+r=n} d_p(c)\alpha d_q(b)\beta d_r(a).$$

Proof. Since $d_n(a\alpha b\beta a) = \sum\limits_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c)$. Linearizing on a we get, $d_n((a+c)\alpha b\beta(a+c)) = \sum\limits_{p+q+r=n} d_p(a+c)\alpha d_q(b)\beta d_r(a+c)$. Computing and canceling the like terms from both sides, we prove the lemma. □

Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple higher derivation of a Γ-ring M. Then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ we define

$$G^n_{\alpha,\beta}(a, b, c) = d_n(a\alpha b\beta c) - \sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c) \text{ for all } n \in \mathbb{N}.$$

Lemma 2.7. *Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple derivation of a Γ-ring M. Then for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$ and for all $n \in \mathbb{N}$, we have*

(i) $G^n_{\alpha,\beta}(a, b, c) + G^n_{\alpha,\beta}(c, b, a)$=0,

(ii) $G^n_{\alpha,\beta}(a + c, b, e) = G^n_{\alpha,\beta}(a, b, e) + G^n_{\alpha,\beta}(c, b, e)$,

(iii) $G^n_{\alpha,\beta}(a, b, c + e) = G^n_{\alpha,\beta}(a, b, c) + G^n_{\alpha,\beta}(a, b, e)$,

(iv) $G^n_{\alpha,\beta}(a, b + c, e) = G^n_{\alpha,\beta}(a, b, e) + G^n_{\alpha,\beta}(a, c, e)$,

(v) $G^n_{\alpha+\gamma,\beta}(a, b, c) = G^n_{\alpha,\beta}(a, b, c) + G^n_{\gamma,\beta}(a, b, c)$,

(vi) $G^n_{\alpha,\beta+\gamma}(a, b, c) = G^n_{\alpha,\beta}(a, b, c) + G^n_{\alpha,\gamma}(a, b, c)$.

Proof. Proof of part (i) is obvious by Lemma 2.6, while the proofs of parts (ii)-(vi) can be obtained easily by using additivity of d_n. □

Lemma 2.8. *Let M is a 2-torsion free semiprime Γ-ring. If $G^n_{\alpha,\beta}(a, b, c)\gamma x\delta[a, b, c]_{\alpha,\beta} = 0$, then $G^n_{\alpha,\beta}(a, b, c)\gamma x\delta[u, v, w]_{\alpha,\beta} = 0$, for all $a, b, c, u, v, w, x \in M$, $\alpha, \beta, \gamma, \delta \in \Gamma$ and for all $n \in \mathbb{N}$.*

Proof. Replacing a by $a+u$ in the hypothesis we get $G^n_{\alpha,\beta}(a+u,b,c)\gamma x\delta[a+u,b,c]_{\alpha,\beta}=0$. Hence using Lemma 2.4, we find that $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}+G^n_{\alpha,\beta}(u,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}=0$. Now consider; $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}\gamma x\delta G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}=$
$-\,G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}\gamma x\delta G^n_{\alpha,\beta}(u,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}=0$ using hypothesis. Since M is semiprime Γ-ring, then $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}=0$. Similarly, replacing b by $b+v$ and c by $c+w$ and using semiprimeness of M, we get the required result. □

Now we are well equipped to prove our result:

Proof of Theorem 2.3. Given that the family $D=\{d_n\}_{n\in\mathbb{N}}$ of additive mappings on M satisfying $d_n(a\alpha b\beta a)=\sum\limits_{p+q+r=n}d_p(a)\alpha d_q(b)\beta d_r(a)$ for all $a,b\in M$ and $\alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$. Now we compute $\Delta=d_n(a\alpha(b\beta c\gamma x\delta c\alpha b)\beta a+c\alpha(b\beta a\gamma x\delta a\alpha b)\beta c)$ by using the above definition, we get

$$\begin{aligned}\Delta &= \sum_{p+i+v=n}d_p(a)\alpha d_i(b\beta c\gamma x\delta c\alpha b)\beta d_v(a)\\ &\quad+\sum_{p+i+v=n}d_p(c)\alpha d_i(b\beta a\gamma x\delta a\alpha b)\beta d_v(c)\\ &= \sum_{p+q+j+u+v=n}d_p(a)\alpha d_q(b)\beta d_j(c\gamma x\delta c)\alpha d_u(b)\beta d_v(a)\\ &\quad+\sum_{p+q+j+u+v=n}d_p(c)\alpha d_q(b)\beta d_j(a\gamma x\delta a)\alpha d_u(b)\beta d_v(c)\\ &= \sum_{p+q+r+s+t+u+v=n}d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_s(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a)\\ &\quad+\sum_{p+q+r+s+t+u+v=n}d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_s(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c).\end{aligned}$$

On the other hand, $\Delta=d_n((a\alpha b\beta c)\gamma x\delta(c\alpha b\beta a)+(c\alpha b\beta a)\gamma x\delta(a\alpha b\beta c))$ and using Lemma 2.6, we get

$$\Delta=\sum_{i+s+j=n}d_i(a\alpha b\beta c)\gamma d_s(x)\delta d_j(c\alpha b\beta a)+\sum_{i+s+j=n}d_i(c\alpha b\beta a)\gamma d_s(x)\delta d_j(a\alpha b\beta c).$$

On comparing the above two equalities we get,

$$\begin{aligned}&\sum_{i+s+j=n}d_i(a\alpha b\beta c)\gamma d_s(x)\delta d_j(c\alpha b\beta a)+\sum_{i+s+j=n}d_i(c\alpha b\beta a)\gamma d_s(x)\delta d_j(a\alpha b\beta c)\\ &=\sum_{p+q+r+s+t+u+v=n}d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_s(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a)\\ &+\sum_{p+q+r+s+t+u+v=n}d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_s(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c).\end{aligned}\tag{2.1}$$

In (2.1), put $n=1$ and cancel the like terms from both sides of this equality and then arrange them, to obtain

$$G^1_{\alpha,\beta}(a,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}+[a,b,c]_{\alpha,\beta}\gamma x\delta G^1_{\alpha,\beta}(a,b,c)=0.\tag{2.2}$$

In view of Lemma 2.5 and the above equation, we obtain

$$G^1_{\alpha,\beta}(a,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}=[a,b,c]_{\alpha,\beta}\gamma x\delta G^1_{\alpha,\beta}(a,b,c)=0.$$

Also using Lemma 2.8 we find that

$$G^1_{\alpha,\beta}(a,b,c)\gamma x\delta[u,v,w]_{\alpha,\beta}=0,\ \text{for all } a,b,c,u,v,w\in M \text{ for all } \alpha,\beta,\gamma,\delta\in\Gamma.$$

Therefore using primeness of M, we get either $G^1_{\alpha,\beta}(a,b,c)=0$ or $\delta[u,v,w]_{\alpha,\beta}=0$, for all $a,b,c,u,v,w\in M$ for all $\alpha,\beta,\gamma,\delta\in\Gamma$. If we suppose that $[u,v,w]_{\alpha,\beta}=0$ for all $u,v,w\in M$ and $\alpha,\beta\in\Gamma$, then we have $u\alpha v\beta w=w\alpha v\beta u$. Hence by Lemma 2.6 we find that,

$$d_1(a\alpha b\beta c+a\alpha b\beta c)=\sum_{p+q+r=1}d_p(a)\alpha d_q(b)\beta d_r(c)+\sum_{p+q+r=1}d_p(a)\alpha d_q(b)\beta d_r(a).$$

Since M is 2-torsion free, we get $d_1(a\alpha b\beta c)=\sum\limits_{p+q+r=n}d_p(a)\alpha d_q(b)\beta d_r(c)$ or we may say that for $n=1$, $D=\{d_n\}n\in\mathbb{N}$ is a triple higher derivation. On the other hand if $G^1_{\alpha,\beta}(a,b,c)=0$,

then $d_1(a\alpha b\beta c) = \sum\limits_{p+q+r=1} d_p(a)\alpha d_q(b)\beta d_r(c)$, which again implies that $D = \{d_n\}n \in \mathbb{N}$ is triple higher derivation.
Now let the result holds for $n-1$, i.e., $G^{n-1}_{\alpha,\beta}(a\alpha b\beta c) = 0$ for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$.
Also (2.1) can be rewritten as

$$
\begin{aligned}
& \sum_{i+j=n} d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a) + \sum_{i+j=n-1} d_i(a\alpha b\beta c)\gamma d_1(x)\delta d_j(c\alpha b\beta a) \\
& +\ldots + \sum_{i+j=1} d_i(a\alpha b\beta c)\gamma d_{n-1}(x)\delta d_j(c\alpha b\beta a) + \sum_{i+j=0} d_i(a\alpha b\beta c)\gamma d_n(x)\delta d_j(c\alpha b\beta a) \\
& + \sum_{i+j=n} d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c) + \sum_{i+j=n-1} d_i(c\alpha b\beta a)\gamma d_1(x)\delta d_j(a\alpha b\beta c) \\
& +\ldots + \sum_{i+j=1} d_i(c\alpha b\beta a)\gamma d_{n-1}(x)\delta d_j(a\alpha b\beta c) + \sum_{i+j=0} d_i(c\alpha b\beta a)\gamma d_n(x)\delta d_j(a\alpha b\beta c) \\
= & \sum_{p+q+r+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& + \sum_{p+q+r+t+u+v=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_1(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& +\ldots + \sum_{p+q+r+t+u+v=1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_{n-1}(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& + \sum_{p+q+r+t+u+v=0} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_n(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& + \sum_{p+q+r+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c) \\
& + \sum_{p+q+r+t+u+v=n-1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_1(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c) \\
& +\ldots + \sum_{p+q+r+t+u+v=1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_{n-1}(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c) \\
& + \sum_{p+q+r+t+u+v=0} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_n(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
& \sum_{i+j=n} d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a) + \sum_{i+j=n-1} d_i(a\alpha b\beta c)\gamma d_1(x)\delta d_j(c\alpha b\beta a) \\
& +\ldots + d_1(a\alpha b\beta c)\gamma d_{n-1}(x)\delta c\alpha b\beta a + a\alpha b\beta c\gamma d_{n-1}(x)\delta d_1(c\alpha b\beta a) \\
& +a\alpha b\beta c\gamma d_n(x)\delta c\alpha b\beta a + \sum_{i+j=n} d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c) \\
& + \sum_{i+j=n-1} d_i(c\alpha b\beta a)\gamma d_1(x)\delta d_j(a\alpha b\beta c) + \ldots + d_1(c\alpha b\beta a)\gamma d_{n-1}(x)\delta a\alpha b\beta c \\
& +c\alpha b\beta a\gamma d_{n-1}(x)\delta d_1(a\alpha b\beta c) + c\alpha b\beta a\gamma d_n(x)\delta a\alpha b\beta c \\
= & \sum_{p+q+r+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& + \sum_{p+q+r+t+u+v=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_1(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& +\ldots + d_1(a)\alpha b\beta c\gamma d_{n-1}(x)\delta c\alpha b\beta a + a\alpha d_1(b)\beta c\gamma d_{n-1}(x)\delta c\alpha b\beta a \\
& +a\alpha b\beta d_1(c)\gamma d_{n-1}(x)\delta c\alpha b\beta a + a\alpha b\beta c\gamma d_{n-1}(x)\delta d_1(c)\alpha b\beta a \\
& +a\alpha b\beta c\gamma d_{n-1}(x)\delta c\alpha d_1(b)\beta a + a\alpha b\beta c\gamma d_{n-1}(x)\delta c\alpha b\beta d_1(a) \\
& +a\alpha b\beta c\gamma d_n(x)\delta c\alpha b\beta a + \sum_{p+q+r+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c) \\
& + \sum_{p+q+r+t+u+v=n-1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_1(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c) \\
& +\ldots + d_1(c)\alpha b\beta a\gamma d_{n-1}(x)\delta a\alpha b\beta c + c\alpha d_1(b)\beta a\gamma d_{n-1}(x)\delta a\alpha b\beta c \\
& +c\alpha b\beta d_1(a)\gamma d_{n-1}(x)\delta a\alpha b\beta c + c\alpha b\beta a\gamma d_{n-1}(x)\delta d_1(a)\alpha b\beta c \\
& +c\alpha b\beta a\gamma d_{n-1}(x)\delta a\alpha d_1(b)\beta c + c\alpha b\beta a\gamma d_{n-1}(x)\delta a\alpha b\beta d_(c) \\
& +c\alpha b\beta a\gamma d_n(x)\delta a\alpha b\beta c.
\end{aligned}
$$

Now since $d_{n-1}(a\alpha b\beta c) = \sum\limits_{p+q+r=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)$ for all $a,b,c \in M,\ \alpha,\beta \in \Gamma$ and for all $n \in \mathbb{N}$, the above expression reduces to

$$
\begin{aligned}
& \sum_{i+j=n} d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a) + \sum_{i+j=n} d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c) \\
= & \sum_{p+q+r+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& + \sum_{p+q+r+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c).
\end{aligned}
$$

It can also be written as

$$\begin{aligned}
& d_n(a\alpha b\beta c)\gamma x\delta c\alpha b\beta a + a\alpha b\beta c\gamma x\delta d_n(c\alpha b\beta a) + \sum_{0<i,j\leq n-1}^{i+j=n} d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a) \\
& +d_n(c\alpha b\beta a)\gamma x\delta a\alpha b\beta c + c\alpha b\beta a\gamma x\delta d_n(a\alpha b\beta c) + \sum_{0<i,j\leq n-1}^{i+j=n} d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c) \\
= & \sum_{p+q+r=0,t+u+v=n} a\alpha b\beta c\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& + \sum_{p+q+r=n,t+u+v=0} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta c\alpha b\beta a \\
& \sum_{0<p+q+r,t+u+v\leq n-1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a) \\
& + \sum_{p+q+r=0,t+u+v=n} c\alpha b\beta a\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c) \\
& + \sum_{p+q+r=n,t+u+v=0} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta a\alpha b\beta c \\
& + \sum_{0<p+q+r,t+u+v\leq n-1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c).
\end{aligned}$$

On using $d_{n-1}(a\alpha b\beta c) = \sum_{p+q+r=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)$ for all $a,b,c\in M,\ \alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$, we get,

$$G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta} + [a,b,c]_{\alpha,\beta}\gamma x\delta G^n_{\alpha,\beta}(a,b,c) = 0,$$

for all $a,b,c\in M,\ \alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$.

Now on using the same method as used after (2.2), we find that, either $G^n_{\alpha,\beta}(a,b,c) = 0$ or $[a,b,c]_{\alpha,\beta} = 0$ for all $a,b,c\in M,\ \alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$. If $G^n_{\alpha,\beta}(a,b,c) = 0$, then by the definition of $G^n_{\alpha,\beta}(a,b,c)$, $D=\{d_n\}_{i\in\mathbb{N}}$ becomes a triple higher derivation. Whereas if $[a,b,c]_{\alpha,\beta} = 0$, in view of Lemma 2.6 and using torsion restriction on M again $D=\{d_n\}_{i\in\mathbb{N}}$ becomes a triple higher derivation. Hence required result is proved. □

Let $D=\{d_n\}_{n\in\mathbb{N}}$ be a Jordan higher derivation of a Γ-ring M. Then for all $a,b\in M$ and $\alpha\in\Gamma$ we define

$$F^n_{\alpha,\beta}(a,b) = d_n(a\alpha b) - \sum_{p+q=n} d_p(a)\alpha d_q(b) \text{ for all } n\in\mathbb{N}.$$

It can be easily seen that every higher derivation on a Γ-ring M is a triple higher derivation on M. But the converse is not true in general. In the theorem, given below, we have obtain the converse part for prime Γ-ring M.

Theorem 2.9. *Any triple higher derivation of a prime Γ-ring M of characteristic different from two is a higher derivation on M.*

Proof. Given that $D=\{d_n\}_{n\in\mathbb{N}}$ is a triple higher derivation on M, i.e.,

$$d_n(a\alpha b\beta c) = \sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c).$$

For each $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ and for any $n\in\mathbb{N}$.
Now consider

$$\begin{aligned}
\Delta &= d_n(a\alpha(b\gamma x\delta a)\alpha b) \\
&= \sum_{p+i+t=n} d_p(a)\alpha d_i(b\gamma x\delta a)\alpha d_t(b) \\
&= \sum_{p+q+r+s+t=n} d_p(a)\alpha d_q(b)\gamma d_r(x)\delta d_s(a)\alpha d_t(b).
\end{aligned}$$

Again,

$$\Delta = d_n((a\alpha b)\gamma x\delta(a\alpha b)) = \sum_{i+r+j=n} d_i(a\alpha b)\gamma d_r(x)\delta d_j(a\alpha b).$$

Comparing the above two expressions so obtained for Δ, we obtain

$$\sum_{i+r+j=n} d_i(a\alpha b)\gamma d_r(x)\delta d_j(a\alpha b) = \sum_{p+q+r+s+t=n} d_p(a)\alpha d_q(b)\gamma d_r(x)\delta d_s(a)\alpha d_t(b). \tag{2.3}$$

Hence, for $n = 1$ the above equation becomes,

$$\begin{aligned} & d_1(a\alpha b)\gamma x\delta a\alpha b + a\alpha b\gamma d_1(x)\delta a\alpha b + a\alpha b\gamma x\delta d_1(a\alpha b) \\ = \; & d_1(a)\alpha b\gamma x\delta a\alpha b + a\alpha d_1(b)\gamma x\delta a\alpha b + a\alpha b\gamma d_1(x)\delta a\alpha b \\ & + a\alpha b\gamma x\delta d_1(a)\alpha b + a\alpha b\gamma x\delta a\alpha d_1(b). \end{aligned}$$

This yields that, $F^1_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b + a\alpha b\gamma x\delta F^1_{\alpha,\beta}(a,b) = 0$. On using Lemma 2.5 we have $F^1_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b = 0$ for all $a,b,x \in M$ and $\alpha,\beta \in \Gamma$, which further on linearizing becomes $F^1_{\alpha,\beta}(a,b)\gamma x\delta c\alpha d$ for all $a,b,x,c,d \in M$ and $\alpha,\beta \in \Gamma$. Again since M is prime, we get $F^1_{\alpha,\beta}(a,b) = 0$ for all $a,b,x \in M$ and $\alpha,\beta \in \Gamma$, or we can say that $d_1(a\alpha b) = \sum_{i+j=1} d_i(a)\alpha d_j(b)$ for all $a,b \in M$ and $\alpha,\beta \in \Gamma$. Let the result hold for $n-1$, i.e.,

$$d_{n-1}(a\alpha b) = \sum_{i+j=n-1} d_i(a)\alpha d_j(b) \text{ for all } a,b \in M \text{ and } \alpha,\beta \in \Gamma. \tag{2.4}$$

(2.3) can be rewritten as

$$\begin{aligned} & \sum_{i+j=n} d_i(a\alpha b)\gamma x\delta d_j(a\alpha b) + \sum_{i+j=n-1} d_i(a\alpha b)\gamma d_1(x)\delta d_j(a\alpha b) \\ & + \ldots + \sum_{i+j=1} d_i(a\alpha b)\gamma d_{n-1}(x)\delta d_j(a\alpha b) + \sum_{i+j=0} d_i(a\alpha b)\gamma d_n(x)\delta d_j(a\alpha b) \\ = \; & \sum_{p+q+s+t=n} d_p(a)\alpha d_q(b)\gamma x\delta d_s(a)\alpha d_t(b) + \sum_{p+q+s+t=n-1} d_p(a)\alpha d_q(b)\gamma d_1(x)\delta d_s(a)\alpha d_t(b) \\ & + \sum_{p+q+s+t=1} d_p(a)\alpha d_q(b)\gamma d_{n-1}(x)\delta d_s(a)\alpha d_t(b) \\ & + \sum_{p+q+s+t=0} d_p(a)\alpha d_q(b)\gamma d_n(x)\delta d_s(a)\alpha d_t(b). \end{aligned}$$

On using (2.4) we get, $\sum_{i+j=n} d_i(a\alpha b)\gamma x\delta d_j(a\alpha b) = \sum_{p+q+s+t=n} d_p(a)\alpha d_q(b)\gamma x\delta d_s(a)\alpha d_t(b)$. Also

$$\begin{aligned} & d_n(a\alpha b)\gamma x\delta(a\alpha b) + a\alpha b\gamma x\delta d_n(a\alpha b) + \sum_{0<i,j\leq n-1} d_i(a\alpha b)\gamma x\delta d_j(a\alpha b) \\ = \; & \sum_{s+t=n} a\alpha b\gamma x\delta d_s(a)\alpha d_t(b) + \sum_{p+q=n} d_p(a)\alpha d_q(b)\gamma x\delta a\alpha b \\ & + \sum_{0<p+q,s+t\leq n-1} d_p(a)\alpha d_q(b)\gamma x\delta d_s(a)\alpha d_t(b), \end{aligned}$$

and again using (2.4) we have,

$$d_n(a\alpha b)\gamma x\delta(a\alpha b) + a\alpha b\gamma x\delta d_n(a\alpha b) = \sum_{s+t=n} a\alpha b\gamma x\delta d_s(a)\alpha d_t(b) + \sum_{p+q=n} d_p(a)\alpha d_q(b)\gamma x\delta a\alpha b,$$

or, $F^n_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b + a\alpha b\gamma x\delta F^n_{\alpha,\beta}(a,b) = 0$. On using Lemma 2.5 we have $F^n_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b = 0$ for all $a,b,x \in M$ and $\alpha,\beta \in \Gamma$, which further becomes $F^n_{\alpha,\beta}(a,b)\gamma x\delta c\alpha d = 0$ for all $a,b,x,c,d \in M$ and $\alpha,\beta \in \Gamma$. Again since M is prime, we get $F^n_{\alpha,\beta}(a,b) = 0$ for all $a,b \in M$ and $\alpha,\beta \in \Gamma$, or we can we say $d_n(a\alpha b) = \sum_{i+j=n} d_i(a)\alpha d_j(b)$ for all $a,b \in M$ and $\alpha \in \Gamma$ and for each $n \in \mathbb{N}$. Therefore $D = \{d_n\}_{n\in\mathbb{N}}$ becomes a higher derivation on M. □

In view of Theorems 2.3 and 2.9 one can easily conclude the following:

Theorem 2.10. *Any Jordan triple higher derivation of a prime Γ-ring M of characteristic different from two is a higher derivation on M.*

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Received: October 23, 2014.

Accepted: July 7, 2015.