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A fixed point approach to the stability of a bi-cubic functional equation in 2-Banach spaces

M. Almahalebi, A. Chahbi and S. Kabbaj

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Abstract In this paper, we use the fixed point method to investigate the Generalized Hyers-Ulam-Rassias stability for a bi-cubic functional equation in 2-Banach spaces.

1 Introduction and preliminaries

In the middle of 1960s, S. Gähler [13],[14] introduced the concept of linear 2-normed spaces. We recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.1. let X be a real linear space with $dimX > 1$ and $\|.,.\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(ii) $\|x, y\| = \|y, x\|$,

(iii) $\|\lambda x, y\| = |\lambda|\|x, y\|$,

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called *a 2-norm* on X and the pair $(X, \|.,.\|)$ is called a *linear 2-normed space*. Sometimes the condition (4) called the *triangle inequality*.

Example 1.2. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in E = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_E$ is defined by

$$\|x, y\|_E = |x_1y_2 - x_2y_1| .$$

Definition 1.3. A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.4. A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a *2-Banach space*.

Now, we state the following results as lemma (See [21] for the details).

Lemma 1.5. *Let X be a 2-normed space. Then,*

(i) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ *for all* $x, y, z \in X$,

(ii) if $\|x, z\| = 0$ *for all* $z \in X$, *then* $x = 0$,

(iii) for a convergent sequence x_n in X,

$$\lim_{n\longrightarrow\infty}\|x_n,z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|$$

for all $z \in X$.

The stability problem of functional equations originated from the following question of Ulam [27], [28] in 1940, concerning the stability of group homomorphisms: Let $(G_1,.)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d\big(h(x.y), h(x) * h(y)\big) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d\big(h(x), H(x)\big) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [15] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces E and E' such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$ and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $t \longrightarrow f(tx)$ is continuous in real t for each fixed $x \in E$, then T is linear. The generalizations of this result have been published by Aoki [6] and Rassias [23] for additive mappings and linear mappings, respectively. Since then several stability problems for various functional equations have been investigated by many authors worldwide.

Hyers's method used in [15], which is often called the *direct method*, has been applied for studying the stability of various functional equations but this method sometimes does not work [16]. Nevertheless, there are also other approaches proving the Hyers-Ulam stability, for example: the method of *invariant means* [25], the method of based on *sandwich theorems* [17], the method using the concept of *shadowing* [26] and the *fixed point* method.

In this work, we use the fixed point method which is the second most popular technique of proving the stability of functional equations. Although it was used for the first time by J. A. Baker [7] who applied a variant of Banach's fixed point theorem to obtain the Hyers-Ulam stability of a functional equation in a single variable, most authors follow Radu's approach [22] and make use of a theorem of Diaz and Margolis [12]. In 1996, Isac and Th. M. Rassias [18] provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In 2003 L. Cădariu and V. Radu [8] noticed that a fixed point alternative method is very essential for the solution of the Hyers-Ulam stability problem. Subsequently, this method was applied to investigate the Hyers-Ulam-Rassias stability for Jensen functional equation, as well as for the additive Cauchy functional equation [9] by considering a general control function $\varphi(x,y)$, with appropriate properties. The stability problem of various types of functional equations have been investigated by a number of authors by using the fixed point approach.

Before present the fixed point method, we need to some concepts.

Definition 1.6. [12] Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies the following:
(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.
Then (X,d) is called *a generalized metric space*. (X,d) is called *complete* if every d-Cauchy sequence in X is d-convergent.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

Example 1.7. Let $X := C(\mathbb{R})$ "the space of the continuous functions on $\mathbb{R}$" and let $d : X^2 \longrightarrow [0,\infty]$ given by

$$d(x,y) := \sup_{t\in\mathbb{R}} \big|x(t) - y(t)\big|.$$

Then, the pair $\big(X, d\big)$ is a generalized complete metric space.

Definition 1.8. Let (X, d) be a generalized complete metric space. A mapping $J : X \to X$ satisfies a Lipschitz condition with a constant $L > 0$ "*Lipschitz constant*" if

$$d\Big(J(x), J(y)\Big) \leq Ld(x, y)$$

for all $x, y \in X$. If $L < 1$, then J is called a *strictly contractive operator*.

We remark that the only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.
By these notions, B. Margolis and J. Diaz gave one of the fundamental results of the fixed point theory. For the proof, we refer to [12].

Theorem 1.9. *Let (X, d) be a generalized complete metric space and $J : X \to X$ be strictly contractive mapping with the Lipshitz constant L. Then for each given element $x \in X$, either*

$$d\Big(J^n x, J^{n+1}x\Big) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) $d\Big(J^n x, J^{n+1}x\Big) < \infty$, *for all $n \geq n_0$;*

(ii) the sequence $\big\{J^n x\big\}$ converges to a fixed point y^* of J;

(iii) y^* is the unique fixed point of J in the set $Y = \Big\{y \in X : d\big(J^{n_0}(x), y\big) < \infty\Big\}$;

(iv) $d(y, y^*) \leq \frac{1}{1-L} d\Big(J(y), y\Big)$ *for all $y \in Y$.*

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to symmetric bi-additive function. It is natural that this equation is called *a quadratic functional equation*. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between two real vector spaces X and Y is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all $x \in X$ (see [1],[20]). The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}\big(f(x + y) - f(x - y)\big)$$

The stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \longrightarrow B$, where A is normed space and B Banach space [24]. Cholewa [10] noticed that the Theorem of Skof is still true if relevant domain A is replaced by an abelian group. In the paper [11], Czerwik proved the stability of the equation (1.1).

Jun and Kim [19] introduced the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.2}$$

and they established the general solution and the Hyers-Ulam stability for the functional equation (1.2). They proved that a mapping f between two real vector spaces X and Y is a solution of (1.2) if and only if there exists a unique mapping $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$. Moreover, C is symmetric for each fixed one variable and is additive for fixed two variables. The mapping C is given by

$$C(x, y, z) = \frac{1}{24}\Big(f(x + y + z) + f(x - y - z) - f(x + y - z) - f(x - y + z)\Big)$$

for all $x, y, z \in X$. Obviously, the function $f(x) = cx^3$ satisfies the functional equation (1.2), which is called *a cubic functional equation*.

Let X, Y and Z be vector spaces on $\mathbb{R}$ or $\mathbb{C}$. We say that a mapping $f : X \times Y \longrightarrow Z$ is *sextic* if f satisfies one of the following functional equation

$$f(2x+y,2z+w)+f(2x-y,2z+w)+f(2x+y,2z-w)+f(2x-y,2z-w)=$$

$$4f(x+y,z+w)+4f(x+y,z-w)+24f(x+y,z)+4f(x-y,z+w)+4f(x-y,z-w)$$

$$+24f(x-y,z)+24f(x,z+w)+24f(x,z-w)+144f(x,z) \tag{1.3}$$

for all $x, y \in X$ and all $z, w \in Y$.
It easy to see that the function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x, y) = x^3y^3$ is a sextic mapping which satisfying (1.3).

The purpose of this work is keep continuity of our previous works in [2], [3], [4] and [5]. Indeed, we prove the generalized Hyers-Ulam-Rassias stability of the functional equation (1.3) in 2-Banach spaces by using the fixed point method.

2 Main results

Let X be a vector space, Y a 2-Banach space with $dimY > 1$. For convenience, we use the following abbreviation for a given mapping $f : X \times X \longrightarrow Y$

$$D_f(x,y,z,w):=f(2x+y,2z+w)+f(2x-y,2z+w)+f(2x+y,2z-w)$$

$$+f(2x-y,2z-w)-4f(x+y,z+w)-4f(x+y,z-w)-24f(x+y,z)-4f(x-y,z+w)$$

$$-4f(x-y,z-w)-24f(x-y,z)-24f(x,z+w)-24f(x,z-w)-144f(x,z) \tag{2.1}$$

for all $x, y, z, w \in X$.

Theorem 2.1. *Let $f : X \times X \longrightarrow Y$ be a mapping for which there exists a function $\varphi : X \times X \times X \times X \longrightarrow [0,\infty)$ satisfying*

$$\|D_f(x,y,z,w),t\| \le \varphi(x,y,z,w), \tag{2.2}$$

$$\lim_{n\longrightarrow+\infty} \frac{1}{2^{6n}}\varphi(2^nx,2^ny,2^nz,2^nw)=0 \tag{2.3}$$

and

$$\varphi(x,y,z,w) \le 64L\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}) \tag{2.4}$$

for all $x, y, z, w \in X$, all $t \in Y$ and for some $0 < L < 1$. Then, there exists a unique sextic mapping $S : X \times X \longrightarrow Y$ satisfying (1.3) and

$$\|f(x,z)-S(x,z),t\| \le \frac{1}{256(1-L)}\varphi(x,0,z,0) \tag{2.5}$$

for all $x, z \in X$ and all $t \in Y$.

Proof. Let us consider the set $M := \{g : X \times X \longrightarrow Y\}$ and introduce a generalized metric on M as follows:

$$d(g,h)=inf\,\{\alpha\in[0,\infty):\|g(x,z)-h(x,z),t\|\le\alpha\varphi(x,0,z,0)\}$$

for all $x, z \in X$ and all $t \in Y$ where, as usual, $inf\emptyset = +\infty$. It is easy to show that (M, d) is complete (see for example [9]). Now, we consider the linear mapping $J : M \to M$ such that

$$Jg(x,z):=\frac{1}{64}g(2x,2z)$$

for all $g \in M$ and all $x, z \in X$. Given $g, h \in M$, let $\alpha \in [0, \infty)$ be an arbitrary constant with $d(g,h) \leq \alpha$, that is $\| g(x,z) - h(x,z), t \| \leq \alpha\varphi(x,0,z,0)$ for all $x, z \in X$ and all $t \in Y$. So we have

$$\|Jg(x,z) - Jh(x,z), t\| = \frac{1}{64}\|g(2x,2z) - h(2x,2z), t\|$$

$$\leq \frac{1}{64}\alpha\varphi(2x,0,2z,0) \leq \alpha L\varphi(x,0,z,0)$$

for all $g \in M$, all $x, z \in X$ and all $t \in Y$.
Hence, we see that $d(Jg, Jh) \leq Ld(g,h)$, for any $g, h \in M$. So J is a strictly contractive operator. Putting $y = 0$ and $w = 0$ in (2.2), we have

$$\|\frac{1}{64}f(2x,2z) - f(x,z), t\| \leq \frac{1}{256}\varphi(x,0,z,0) \tag{2.6}$$

for all $x, z \in X$ and all $t \in Y$.
Thus, we get that

$$d(f, Jf) \leq \frac{1}{256}$$

for all $f \in M$. By Theorem 2.1, there exists a unique mapping $S : X \times X \longrightarrow Y$ satisfying the following:

(i) S is fixed point of J, that is, $S(2x,2z) = 64S(x,z)$ for all $x, z \in X$. The S is a unique fixed point of J in the set $B = \{g \in M : d(f,g) < \infty\}$. This implies that S is a unique mapping such that there exists
$\alpha \in (0, \infty)$ such that

$$\|f(x,z) - S(x,z), t\| \leq \alpha\varphi(x,0,z,0)$$

for all $x, z \in X$and all $t \in Y$.

(ii) $d(J^n, S) \longrightarrow 0$ as $n \longrightarrow \infty$, which implies the equality

$$\lim_{n\to+\infty} J^n f(x,z) = \lim_{n\to+\infty} \frac{f(2^n x, 2^n z)}{2^{6n}} = S(x) \tag{2.7}$$

for all $x, z \in X$.

(iii)

$$d(f,S) \leq \frac{1}{1-L}d(f,Jf) \leq \frac{1}{256(1-L)}\varphi(x,0,z,0),$$

which implies the inequality (2.5).

It follows from (2.2), (2.3) and (2.7), that

$$\|D_S(x,y,z,w), t\| = \lim_{n\to+\infty} \frac{1}{2^{6n}}\|D_f(2^n x, 2^n y, 2^n z, 2^n w), t\|$$

$$\leq \lim_{n\to+\infty} \frac{1}{2^{6n}}\varphi(2^n x, 2^n y, 2^n z, 2^n w) = 0$$

for all $x, y, z, w \in X$ and all $t \in Y$. Hence, $S : X \times X \longrightarrow Y$ is a sextic mapping, as desired. □

Corollary 2.2. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|.,.\|_Y)$ be a 2-Banach space. Let θ and p be nonnegative real numbers with $p < 6$ and let $f : X \times X \longrightarrow Y$ be a mapping fulfilling*

$$\|D_f(x,y,z,w), t\|_Y \leq \theta\left(\|x\|_X^p + \|y\|_X^p + \|z\|_X^p + \|w\|_X^p\right)$$

for all $x, y, z, w \in X$ and all $t \in Y$. Then there exists a unique sextic mapping $S : X \times X \longrightarrow Y$ such that

$$\|f(x,z) - S(x,z), t\|_Y \leq \frac{\theta}{256 - 2^{p+2}}\left(\|x\|_X^p + \|z\|_X^p\right)$$

foe all $x, z \in X$ and all $t \in Y$.

Proof. We get the desired result from Theorem 2.1 by taking

$$\varphi(x,y,z,w) = \theta\left(\|x\|_X^p + \|y\|_X^p + \|z\|_X^p + \|w\|_X^p\right)$$

for all $x, y, z, w \in X$ and choosing $L = 2^{p-6}$. □

Corollary 2.3. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|.,.\|_Y)$ be a 2-Banach space. Let θ and p be nonnegative real numbers with $p < 3$ and let $f : X \times X \longrightarrow Y$ be a mapping fulfilling*

$$\|D_f(x,y,z,w), t\|_Y \le \theta\left(\|x\|_X^p . \|z\|_X^p + \|y\|_X^p . \|w\|_X^p\right)$$

for all $x, y, z, w \in X$ and all $t \in Y$. Then there exists a unique sextic mapping $S : X \times X \longrightarrow Y$ such that

$$\|f(x,z) - S(x,z), t\|_Y \le \frac{\theta}{256 - 2^{2p+2}}\left(\|x\|_X^p . \|z\|_X^p\right)$$

foe all $x, z \in X$ and all $t \in Y$.

Proof. We get the desired result from Theorem 2.1 by taking

$$\varphi(x,y,z,w) = \theta\left(\|x\|_X^p . \|z\|_X^p + \|y\|_X^p . \|w\|_X^p\right)$$

for all $x, y, z, w \in X$ and choosing $L = 2^{2p-6}$. □

Corollary 2.4. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|.,.\|_Y)$ be a 2-Banach space. Let θ, p and q be nonnegative real numbers with $p + q < 6$ and let $f : X \times X \longrightarrow Y$ be a mapping fulfilling*

$$\|D_f(x,y,z,w), t\|_Y \le \theta\left(\|x\|_X^p . \|z\|_X^q + \|y\|_X^p . \|w\|_X^q\right)$$

for all $x, y, z, w \in X$ and all $t \in Y$. Then there exists a unique sextic mapping $S : X \times X \longrightarrow Y$ such that

$$\|f(x,z) - S(x,z), t\|_Y \le \frac{\theta}{256 - 2^{p+q+2}}\left(\|x\|_X^p . \|z\|_X^q\right)$$

foe all $x, z \in X$ and all $t \in Y$.

Proof. We get the desired result from Theorem 2.1 by taking

$$\varphi(x,y,z,w) = \theta\left(\|x\|_X^p . \|z\|_X^q + \|y\|_X^p . \|w\|_X^q\right)$$

for all $x, y, z, w \in X$ and choosing $L = 2^{p+q-6}$. □

In a similar way of the proof of Theorem 2.1 we can prove the following theorem

Theorem 2.5. *Let $f : X \times X \longrightarrow Y$ be a mapping for which there exists a function $\varphi : X \times X \times X \times X \longrightarrow [0, \infty)$ satisfying*

$$\|D_f(x,y,z,w), t\| \le \varphi(x,y,z,w), \tag{2.8}$$

$$\lim_{n\to+\infty} 2^{6n}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) = 0 \tag{2.9}$$

and

$$\varphi(x,y,z,w) \le \frac{L}{64}\varphi(2x, 2y, 2z, 2w) \tag{2.10}$$

for all $x, y, z, w \in X$, all $t \in Y$ and for some $0 < L < 1$. Then, there exists a unique sextic mapping $S : X \times X \longrightarrow Y$ satisfying (1.3) and

$$\|f(x,z) - S(x,z), t\| \le \frac{L}{256(1-L)}\varphi(x, 0, z, 0), \tag{2.11}$$

for all $x, z \in X$ and all $t \in Y$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1. Let us consider the linear mapping $J : M \to M$ such that

$$Jg(x, z) := \frac{1}{64} g(2x, 2z)$$

for all $g \in M$ and all $x, z \in X$. Putting $y = 0$ and $w = 0$ in (2.8), we have

$$\|f(2x, 2z) - 64f(x, z), t\| \leq \frac{1}{4}\varphi(x, 0, z, 0)$$

and so

$$\|f(x, z) - 64f(\frac{x}{2}, \frac{z}{2}), t\| \leq \frac{L}{256}\varphi(x, 0, z, 0)$$

for all $x, z \in X$ and all $t \in Y$. Hence, we get that

$$d(f, Jf) \leq \frac{L}{256}$$

for all $f \in M$. The rest of the proof is similar to the proof of Theorem 2.1. □

Remark 2.6. For the cases $p > 6$, $p > 3$ and $p + q > 6$, we can obtain similar results to Corollaries 2.2, 2.3 and 2.4, respectively.

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Author information

M. Almahalebi, A. Chahbi and S. Kabbaj, Department of Mathematics, Faculty of sciences, University of Ibn Tofail, BP 133 Kenitra, Morocco.
E-mail: muaadh1979@hotmail.fr ; ab_1980@live.fr ; samkabbaj@yahoo.fr

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