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# $\delta-$ideals in Pseudo-complemented Almost Distributive Lattices

N. Rafi, Ravi Kumar Bandaru and S.N. Rao

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**Abstract**. $\delta-$ideals are established in a Pseudo-complemented Almost Distributive lattice in terms of filters and some important properties are derived. $\delta-$ideals are utilized to characterize the stone ADLs.

## 1 Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [7] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of $L$ forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. With this motivation, Swamy, Rao and Nanaji[8] introduced the concept of pseudo-complementation on an ADL. They observed that unlike in a distributive lattice, an ADL $R$ can have more than one pseudo-complementation. If $*, \ \perp$ are two pseudo-complementations on $L$, it was observed that $x^* \vee x^{**}$ is maximal, for all $x \in L$ if and only if $x^{\perp} \vee x^{\perp\perp}$ is maximal, for all $x \in L$. With this motivation, in refswamy stone,the concept of a Stone ADL was introduced as an ADL with a pseudo-complementation $*$ satisfying the condition $x^* \vee x^{**}$ is maximal, for all $x \in L$. They studied the properties of pseudo-complemented ADLs and characterized Stone ADLs algebraically, topologically and by means of prime ideals. In [5], G.C. Rao and S. Ravi Kumar proved that some important results on minimal prime ideal of an ADL. In [6], Sambasiva Rao introduced $\delta-$ideals in Pseudo-complemented Distributive lattices and proved their properties. In this paper, we extend the concept of $\delta$ideals to a Pseudo-complemented ADL in terms of filters. Some properties of these $\delta-$ideals are studied and then proved that the set of all $\delta-$ideals can be made into a complete distributive lattice. We proved that the set of all $\delta-$ideals of a pseudo-complemented ADL forms a complete distributive lattice on its own. We derive a set of equivalent conditions for the class of all $\delta-$ideals to become a sublattice to the lattice of all ideals, which leads to a characterization of Stone ADLs. Derived the image a $\delta-$ideal of Pseudo-complemented ADL under homomorphism is again a $\delta-$ideal. Finally, the set of $\delta-$ideals of a pseudo-complemented ADL is characterized in terms of filter congruences.

## 2 Preliminaries

**Definition 2.1.** [7] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$

7. $x \vee 0 = x$, for all $x, y, z \in L$.

Every non-empty set $X$ can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on $X$ by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where $x_0$ is the zero) and is called a discrete ADL. If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on $L$.

**Theorem 2.2.** *[7] If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*
$(1)$. $a \vee b = a \Leftrightarrow a \wedge b = b$
$(2)$. $a \vee b = b \Leftrightarrow a \wedge b = a$
$(3)$. $\wedge$ *is associative in L*
$(4)$. $a \wedge b \wedge c = b \wedge a \wedge c$
$(5)$. $(a \vee b) \wedge c = (b \vee a) \wedge c$
$(6)$. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
$(7)$. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
$(8)$. $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
$(9)$. $a \leq a \vee b$ *and* $a \wedge b \leq b$
$(10)$. $a \wedge a = a$ *and* $a \vee a = a$
$(11)$. $0 \vee a = a$ *and* $a \wedge 0 = 0$
$(12)$. $If\ a \leq c,\ b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
$(13)$. $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL $L$ satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL $L$ a distributive lattice. That is

**Theorem 2.3.** *[7] Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*
$1)$. $(L, \vee, \wedge, 0)$ *is a distributive lattice*
$2)$. $a \vee b = b \vee a$, *for all* $a, b \in L$
$3)$. $a \wedge b = b \wedge a$, *for all* $a, b \in L$
$4)$. $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

**Theorem 2.4.** *[7] Let L be an ADL and $m \in L$. Then the following are equivalent:*
$1)$. $m$ *is maximal with respect to* $\leq$
$2)$. $m \vee a = m$, *for all* $a \in L$
$3)$. $m \wedge a = a$, *for all* $a \in L$
$4)$. $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [1, 2], a non-empty sub set $I$ of an ADL $L$ is called an ideal of $L$ if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset $F$ of $L$ is said to be a filter of $L$ if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of $L$ is a bounded distributive lattice with least element $\{0\}$ and greatest element $L$ under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of $I$ and $J$ while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal $P$ of $L$ is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal $M$ of $L$ is said to be maximal if it is not properly contained in any proper ideal of $L$. It can be observed that every maximal ideal of $L$ is a prime ideal. Every proper ideal of $L$ is contained in a maximal ideal. For any subset $S$ of $L$ the smallest ideal containing $S$ is given by $(S] :=$

$\{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

**Theorem 2.5.** *[7] For any $x$, $y$ in $L$ the following are equivalent:*
1). $(x] \subseteq (y]$
2). $y \wedge x = x$
3). $y \vee x = y$
4). $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of $L$ is a sublattice of the distributive lattice $I(L)$ of ideals of $L$.

**Theorem 2.6** ([4]). *Let $I$ be an ideal and $F$ a filter of $L$ such that $I \cap F = \emptyset$. Then there exists a prime ideal $P$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

**Definition 2.7.** [5] A prime ideal of $L$ is called a minimal prime ideal if it is a minimal element in the set of all prime ideals of $L$ ordered by set inclusion.

**Theorem 2.8.** *[5] Let $L$ be an ADL and $P$ a prime ideal of $L$. Then $P$ is a minimal prime ideal of $L$ if and only if $L \setminus P$ is a maximal filter of $L$.*

**Theorem 2.9.** *[5] Let $L$ be an ADL. Then a prime ideal $P$ is minimal if and only if for any $x \in P$, there exist an element $y \notin P$ such that $x \wedge y = 0$.*

**Definition 2.10** ([4]). An equivalence relation $\theta$ on an ADL $L$ is called a congruence relation on $L$ if $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$, for all $(a, b), (c, d) \in \theta$

**Theorem 2.11** ([4]). *An equivalence relation $\theta$ on an ADL $L$ is a congruence relation if and only if for any $(a, b) \in \theta,\ x \in L, (a \vee x, b \vee x), (x \vee a, x \vee b), (a \wedge x, b \wedge x), (x \wedge a, x \wedge b)$ are all in $\theta$*

**Definition 2.12.** [8] Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \longrightarrow a^*$ on $L$ is called a pseudo-complementation on $L$ if, for any $a, b \in L$, it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

**Theorem 2.13.** *[8] Let $L$ be an ADL and $*$ a pseudo-complementation on $L$. Then, for any $a, b \in L$, we have the following:*

*(1)* $0^*$ *is a maximal element*

*(2) If $a$ is a maximal element then $a^* = 0$*

*(3)* $0^{**} = 0$

*(4)* $0^* \wedge a = a$

*(5)* $a^{**} \wedge a = a$

*(6)* $a^{***} = a^*$

*(7)* $a \leq b \Rightarrow b^* \leq a^*$

*(8)* $a^* \wedge b^* = b^* \wedge a^*$

*(9)* $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

*(10)* $a^* \wedge b = (a \wedge b)^* \wedge b^*$.

For any pseudo-complemented ADL $L$, let us denote the set of all elements of the form $x^* = 0$ by $D(L)$. It is easy to prove that $D(L)$ is a filter of an ADL $L$.

## 3 $\delta-$ideals in pseudo-complemented ADLs

The concept of a $\delta-$ideal in a Pseudo-complemented distributive lattice was given by M.S.Rao [6]. In this section we extend the concept of a $\delta-$ideal to a Pseudo-complemented ADL, analogously. Though many results look similar, the proofs are not similar because we do not have the properties like commutativity of $\vee$, commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ in an ADL.
We begin with the following definition.

**Definition 3.1.** Let $L$ be a pseudo-complemented ADL. Then for any filter $F$ of $L$, define the set $\delta(F) = \{x \in L \mid x^* \in F\}$.

Now we have the following results.

**Lemma 3.2.** *Let $L$ be a pseudo-complemented ADL with maximal elements. Then for any filter $F$ of $L$, $\delta(F)$ is an ideal of $L$.*

*Proof.* Since $0^* \in F$, we get that $0 \in \delta(F)$. Hence $\delta(F) \neq \emptyset$. Let $x, y \in \delta(F)$. Then $x^*, y^* \in F$. That implies $x^* \wedge y^* \in F$, since $F$ is a filter of $L$. Therefore $(x \vee y)^* \in F$. Now, let $x \in \delta(F)$ and $r \in L$. Then $x^* \in F$. That implies $x^* \vee r^* \in F$. Now, $(x \wedge r)^* = (x \wedge r)^{***} = (x^{**} \wedge r^{**})^* = (x^* \vee r^*)^{**} \in F$. Therefore $x \wedge r \in \delta(F)$ hence $\delta(F)$ is an ideal of $L$. □

**Lemma 3.3.** *Let $L$ be a pseudo-complemented ADL with maximal elements. For any two filters $F, G$ of $L$, we have the following properties:*

1. $F \cap \delta(F) = \emptyset$,
2. *If* $x \in \delta(F)$ *then* $x^{**} \in \delta(F)$,
3. $F = L$ *if and only if* $\delta(F) = L$,
4. *If* $F \subseteq G$ *then* $\delta(F) \subseteq \delta(G)$,
5. $\delta(F \cap G) = \delta(F) \cap \delta(G)$.

*Proof.* (1) Suppose $F \cap \delta(F) \neq \emptyset$. Choose $x \in F \cap \delta(F)$. Then $x \in F$ and $x \in \delta(F)$. That implies $x \in F$ and $x^* \in F$. Since $F$ is a filter, we get $x^* \wedge x \in F$ and hence $0 \in F$, which is a contradiction. Therefore $F \cap \delta(F) = \emptyset$.
(2) Let $x \in \delta(F)$. Then $x^* \in F$. Since $x^{***} = x^*$, we get that $x^{***} \in F$. Hence $x^{**} \in \delta(F)$.
(3) Assume that $F = L$. Then, we can choose $0 \in F$. That implies $0^{**} \in F$. Therefore $0^* \in \delta(F)$ and hence $\delta(F) = L$. Conversely, assume that $\delta(F) = L$. Then, choose any maximal element $m$ of $L$ such that $m \in \delta(F)$. That implies $m^* \in F$. Therefore $0 \in F$ and hence $F = L$.
(4) Suppose $F \subseteq G$. Let $x \in \delta(F)$. Then $x^* \in F \subseteq G$. Therefore $x \in \delta(G)$.
(5) Clearly, we have $\delta(F \cap G) \subseteq \delta(F) \cap \delta(G)$. Let $x \in \delta(F) \cap \delta(G)$. Then $x^* \in F$ and $x^* \in G$. That implies $x^* \in F \cap G$. Hence $x \in \delta(F \cap G)$. Therefore $\delta(F) \cap \delta(G) \subseteq \delta(F \cap G)$. Thus $\delta(F \cap G) = \delta(F) \cap \delta(G)$. □

We introduce the concept of $\delta-$ideals in a pseudo-complemented ADL.

**Definition 3.4.** Let $L$ be a pseudo-complemented ADL. An ideal $I$ of $L$ is called a $\delta-$ideal if $I = \delta(F)$, for some filter $F$ of $L$.

**Example 3.5.** Consider a discrete ADL $A = \{0, a\}$ and a distributive lattice $B = \{0', a', b', c', 1\}$ whose Hasse diagram is given in the following Figure-1.

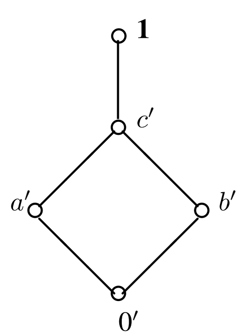


**Figure 1**

Take
$R = A \times B = \{(0,0'),(0,a'),(0,b'),(0,c'),(0,1),(a,0'),(a,a'),(a,b'),(a,c'),(a,1)\}$. Then $(R,\vee,\wedge,\bar{0})$ is an ADL with zero $\bar{0} = (0,0')$ under point-wise operations. Consider $I = \{(0,0'),(0,a'),(0,b'),(0,c'),(0,1)\}$ and
$F = \{(a,0'),(a,a'),(a,b'),(a,c'),(a,1)\}$. Clearly, $I$ is an ideals of $R$ and $F$ is a filter of $R$. Now $\delta(F) = \{x \in R \mid x^* \in F\} = \{(0,0'),(0,a'),(0,b'),(0,c'),(0,1)\} = I$. Therefore $I$ is $\delta-$ideal of $R$.

Every $\delta-$ideal is an ideal but converse is need not be true. For this, we have the following example.

**Example 3.6.** In a distributive lattice $B$ as shown in the above figure-1, take $J = \{0', a', b', c'\}$ and $F_1 = \{b', c', 1\}$. Clearly $J$ is an ideal of $B$ and $F_1$ is a filter of $B$. But $J$ is not a $\delta-$ideal of $B$. Suppose $J = \delta(F)$. Then $0' = c'^* \in F$. Hence $F = B$, which is a contradiction. Therefore $J$ is not a $\delta-$ideal of $B$.

Now, we have the following.

**Lemma 3.7.** *Let $L$ be a pseudo-complemented ADL. For any $x \in L$, $(x^*]$ is a $\delta-$ideal of $L$.*

*Proof.* Let $a \in (x^*]$. Then $x^* \wedge a = a$. Now $a \wedge x = x^* \wedge a \wedge x = 0$. That implies $a \wedge x = 0$ and hence $a^* \wedge x = x$. That implies $a^* \vee x = a^*$. So that $a^* \in [x)$. Therefore $a \in \delta([x))$. Thus $(x^*] \subseteq \delta([x))$. Let $a \in \delta([x))$. Then $a^* \in [x)$. That implies $a^* \vee x = a^*$ and hence $a^* \wedge x = x$. Therefore $a \wedge x = a \wedge a^* \wedge x = 0$. Thus $x^* \wedge a = a$ and hence $a \in (x^*]$. So that we have $\delta([x)) \subseteq (x^*]$. Therefore $(x^*]$ is a $\delta-$ideal of $L$. □

**Lemma 3.8.** *Let $L$ be a pseudo-complemented ADL. Every prime ideal without dense element is a $\delta-$ideal.*

*Proof.* Let $P$ be a prime ideal of $L$ with $P \cap D(L) = \emptyset$. Let $x \in P$. Since $x \vee x^*$ is a dense element of $L$ which is not in $P$, we get that $x^* \notin P$. That implies $x^* \in L \setminus P$. Therefore $x \in \delta(L \setminus P)$. Hence $P \subseteq \delta(L \setminus P)$. Conversely, let $x \in \delta(L \setminus P)$. Then $x^* \in L \setminus P$. Since $P$ is prime and $x \wedge x^* = 0 \in P$, which implies that $x \in P$. Therefore $\delta(L \setminus P) \subseteq P$ and hence $P = \delta(L \setminus P)$. Thus $P$ is a $\delta-$ideal of $L$. □

**Lemma 3.9.** *Let $L$ be a pseudo-complemented ADL. Every minimal prime ideal of $L$ is a $\delta-$ideal.*

*Proof.* Let $P$ be a minimal prime ideal of $L$. We prove that $P$ is a $\delta-$ideal of $L$. For this it is enough to prove that $P \cap D(L) = \emptyset$. Suppose $x \in P \cap D(L)$. Then $x \in P$ and $x \in D(L)$. Since $x \in D(L)$, we have $x^* = 0$. Since $x \in P$ and $P$ is a minimal prime ideal of $L$, there exists $y \notin P$ such that $x \wedge y = 0$. That implies $x^* \wedge y = y$ and hence $0 \wedge y = y$. Therefore $y = 0 \in P$, which is a contradiction. Thus $P \cap D(L) = \emptyset$. By above lemma we get that, $P$ is a $\delta-$ideal. □

**Lemma 3.10.** *Let $L$ be a pseudo-complemented ADL. A proper $\delta-$ideal contains no dense element.*

*Proof.* Let $I$ be a proper $\delta-$ideal of $L$. Then $I = \delta(F)$, for some filter $F$. We prove that $\delta(F) \cap D(L) = \emptyset$. Suppose $x \in \delta(F) \cap D(L)$. Then we get $0 = x^* \in F$, which is a contradiction. Therefore $\delta(F) \cap D(L) = \emptyset$. □

Let us denote the set of all $\delta-$ideals of $L$ by $\mathfrak{I}^\delta(L)$. Then by Example 3.6, it can be observed that $\mathfrak{I}^\delta(B)$ is not a sublattice of $\mathfrak{I}(B)$ of all ideals of $B$. Consider $F = \{b', c', 1\}$ and $G = \{a', c', 1\}$. Clearly $F$ and $G$ are filters of $B$. Now $\delta(F) = \{0', a'\}$ and $\delta(G) = \{0', b'\}$. But $\delta(F) \vee \delta(G) = \{0', a', b', c'\}$ is not a $\delta-$ideal of $B$, because $c' \in \delta(F) \vee \delta(G)$ is a dense element. In the following theorem, we prove that $\mathfrak{I}^\delta(B)$ forms a complete distributive lattice.

**Theorem 3.11.** *Let $L$ be a pseudo-complemented ADL. Then the set $\mathfrak{I}^\delta(L)$ forms a complete distributive lattice on its own.*

*Proof.* For any two filters $F, G$ of $L$, define two binary operations $\sqcap$ and $\sqcup$ as $\delta(F) \sqcap \delta(G) = \delta(F \cap G)$ and $\delta(F) \sqcup \delta(G) = \delta(F \vee G)$. Clearly, we have $\delta(F \cap G)$ is the infimum of $\delta(F)$ and $\delta(G)$ in $\mathfrak{I}^{\delta}(L)$. Also $\delta(F) \sqcup \delta(G)$ is a $\delta-$ideal of $L$. Clearly $\delta(F), \delta(G) \subseteq \delta(F \vee G) = \delta(F) \sqcup \delta(G)$. Let $\delta(H)$ be any $\delta-$ideal of $L$ such that $\delta(F) \subseteq \delta(H)$ and $\delta(G) \subseteq \delta(H)$, where $H$ is a filter of $L$. Now we prove that $\delta(F \vee G) \subseteq \delta(H)$. Let $x \in \delta(F \vee G)$. Then $x^* \in F \vee G$ and hence $x^* = f \wedge g$, for some $f \in F$, $g \in G$. Since $f \in F$ and $g \in G$, we get that $f^* \in \delta(F) \subseteq \delta(H)$ and $g^* \in \delta(G) \subseteq \delta(H)$. That implies $f^* \vee g^* \in \delta(H)$ and hence $(f^* \vee g^*)^{**} \in \delta(H)$. So that $(f^{**} \wedge g^{**})^* \in \delta(H)$. That implies $x^{**} \in \delta(H)$. Therefore $x \in \delta(H)$. Hence $\delta(F) \sqcup \delta(G) = \delta(F \vee G)$ is the supremum of both $\delta(F)$ and $\delta(G)$ in $(\mathfrak{I}^{\delta}(L), \sqcap, \sqcup)$ is a lattice. Distributivity of $\delta-$ideals can be easily followed by using the above operations of $\mathfrak{I}^{\delta}(L)$. It is clear that $\mathfrak{I}^{\delta}(L)$ is a partially ordered set with respect to set-inclusion. Then by the extension of the property of Lemma 3.3(5), we can obtain that $\mathfrak{I}^{\delta}(L)$ is a complete lattice. Therefore $\mathfrak{I}^{\delta}(L)$ is a complete distributive lattice. □

Now we prove that $\mathcal{A}^*(L) = \{(x^*] \mid x \in L\}$ is a Boolean algebra.

**Theorem 3.12.** *For any pseudo-complemented ADL $L$, $\mathcal{A}^*(L) = \{(x^*] \mid x \in L\}$ is a sublattice of the lattice $\mathfrak{I}^{\delta}(L)$ of all $\delta-$ideals of L and hence is a Boolean algebra. Moreover, the mapping $x \mapsto (x^*]$ is a dual homomorphism from L onto $\mathcal{A}^*(L)$.*

*Proof.* Let $(x^*], (y^*] \in \mathcal{A}^*(L)$ for some $x, y \in L$. Then clearly $(x^*] \cap (y^*] \in \mathcal{A}^*(L)$. Again, $(x^*] \sqcup (y^*] = \delta([x)) \sqcup \delta([y)) = \delta([x) \vee [y)) = \delta([x \wedge y)) = ((x \wedge y)^*] \in \mathcal{A}^*(L)$. Therefore $\mathcal{A}^*(L)$ is a sublattice of $\mathfrak{I}^{\delta}(L)$ and hence a distributive lattice. Clearly $(0^{**}]$ and $(0^*]$ are the least and greatest elements of $\mathcal{A}^*(L)$. Now for any $x \in L$, $(x^*] \cap (x^{**}] = (0]$ and $(x^*] \sqcup (x^{**}] = \delta([x)) \sqcup \delta([x^*)) = \delta([x) \vee [x^*)) = \delta([x \wedge x^*)) = \delta([0)) = \delta(L) = L$. Hence $(x^{**}]$ is the complement of $(x^*]$ in $\mathcal{A}^*(L)$. Therefore $(\mathcal{A}^*(L), \sqcup, \cap)$ is a bounded distributive lattice in which every element is complemented. The remaining part can be proved easily. □

We have the following result.

**Lemma 3.13.** *Every proper $\delta-$ideal is contained in a minimal prime ideal.*

*Proof.* Let $I$ be a proper $\delta-$ideal of $L$. Then $I = \delta(F)$, for some filter $F$ of $L$. Clearly $\delta(F) \cap D(L) = \emptyset$. Then there exists a prime ideal $P$ of $L$ such that $\delta(F) \subseteq P$ and $P \cap D(L) = \emptyset$. Let $x \in P$. We have always $x \wedge x^* = 0$. Suppose $x^* \in P$. Then $x \vee x^* \in P \cap D(L)$, which is a contradiction. That means, for any $x \in P$ there exist $x^* \notin P$ such that $x^* \wedge x = 0$. By the theorem 2.9, we have $P$ is a minimal prime ideal of $L$. □

The concept of Stone ADL was introduced by U.M. Swamy, G.C. Rao and G. Nanaji Rao in [9]. Now we have the following Stone ADL definition.

**Definition 3.14.** Let $L$ be an ADL with a pseudo-complementation $*$. Then $L$ is called a Stone ADL if, for any $x \in L$, $x^* \vee x^{**} = 0^*$.

It was already observed that $\mathfrak{I}^{\delta}(L)$ is not a sublattice of the ideal lattice $\mathfrak{I}(L)$. In the following theorem, we establish some equivalent conditions for $\mathfrak{I}^{\delta}(L)$ to become a sublattice of $\mathfrak{I}(L)$, which leads to a characterization of Stone ADL.

**Theorem 3.15.** *Let $L$ be a pseudo-complemented ADL with maximal elements. Then the following are equivalent:*

1. *L is a Stone ADL,*
2. *For any $x, y \in L$, $(x \wedge y)^* = x^* \vee y^*$,*
3. *For any two filters $F, G$ of $L$, $\delta(F) \vee \delta(G) = \delta(F \vee G)$,*
4. *$\mathfrak{I}^{\delta}(L)$ is a sublattice of $\mathfrak{I}(L)$.*

*Proof.* (1) ⇒ (2): Assume that $L$ is a Stone ADL. Then $x^* \vee x^{**} = 0^*$, for all $x \in L$. Let $x, y \in L$. Then $(x \wedge y)^* = (x \wedge y)^{***} = (x^{**} \wedge y^{**})^* = x^{***} \vee y^{***}$, since $x^{**}, y^{**}$ are complemented elements in $[0, 0^*]$. Therefore $(x \wedge y)^* = x^* \vee y^*$.
(2) ⇒ (3): Assume the condition (2). Let $F, G$ be two filters of $L$. We have always $\delta(F) \vee \delta(G) \subseteq \delta(F \vee G)$. Conversely, let $x \in \delta(F \vee G)$. Then $x^* \in F \vee G \Rightarrow x^* = f \wedge g$ for some

$f \in F, g \in G \Rightarrow x^{**} = (f \wedge g)^* \Rightarrow x^{**} = f^* \vee g^* \Rightarrow x^{**} = f^* \vee g^* \in \delta(F) \vee \delta(G)$ since $f^{**} \in F, g^{**} \in G \Rightarrow x \in \delta(F) \vee \delta(G)$. Hence $\delta(F \vee G) \subseteq \delta(F) \vee \delta(G)$. Therefore $\delta(F) \vee \delta(G) = \delta(F \vee G)$.
(3) $\Rightarrow$ (4): It is obvious.
(4) $\Rightarrow$ (1): Assume that $\mathfrak{I}^{\delta}(L)$ is a sublattice of $\mathfrak{I}(L)$. Let $x \in L$. By lemma 3.7, $(x^*]$ and $(x^{**}]$ are both $\delta-$ideals of $L$. Suppose $x^* \vee x^{**} \neq 0^*$. Then by our assumption, $(x^*] \vee (x^{**}]$ is a proper $\delta-$ideal of $L$. Hence there exists a minimal prime ideal $P$ such that $(x^*] \vee (x^{**}] \subseteq P$. Since $P$ is minimal, we get that $x^{**} \notin P$, which is a contradiction. Therefore $L$ is a Stone ADL. □

Unlike in rings, if $f$ is a a homomorphism of an ADL $L$ with 0 into another ADL $L'$ with $0'$ such that $ker\ f = \{x \in L \mid f(x) = 0\} = \{0\}$ and $f$ is onto, then $f$ is need not be an isomorphism. It may be seen in the following example.

**Example 3.16.** Let $L = \{0, a, b\}$ and $L' = \{0', c\}$ be two discrete ADLs. Define a mapping $f : L \longrightarrow L'$ by $f(0) = 0'$ and $f(a) = f(b) = c$. Then clearly, $f$ is a homomorphism from $L$ into $L'$ and also $f$ is onto. Also $Ker\ f = \{0\}$. But f is not one-one. Hence $f$ is not an isomorphism.

However, we have the following.

**Lemma 3.17.** *Let L and $L'$ be two pseudo-complemented ADLs with pseudo-complementation $*$ and $f : L \longrightarrow L'$ an onto homomorphism. If $Ker\ f = \{0\}$, then $f(x^*) = \{f(x)\}^*$ for all $x \in L$.*

*Proof.* We have always $f(x) \wedge f(x^*) = f(x \wedge x^*) = f(0) = 0$. Suppose $f(x) \wedge f(t) = 0$ for some $t \in L$. Then $f(x \wedge t) = 0$ and hence $x \wedge t \in ker\ f = \{0\}$. Thus $x \wedge t = 0$. Hence $x^* \wedge t = t$, which yields $f(x^*) \wedge f(t) = f(x^* \wedge t) = f(t)$. Therefore $f(x^*)$ is the pseudo-complement of $f(x)$ in $L'$. □

In the following, we prove that the image of a $\delta-$ideal of L under the above homomorphism is again a $\delta-$ideal.

**Theorem 3.18.** *Let $L,\ L'$ be two pseudo-complemented ADLs with maximal elements, pseudo-complementation $*$ and $f : L \longrightarrow L'$ an onto homomorphism such that $Ker\ f = \{0\}$. If $I$ is a $\delta-$ideal of $L$, then $f(I)$ is a $\delta-$ideal of $L'$.*

*Proof.* Let $I$ be a $\delta-$ideal of $L$. Then $I = \delta(G)$ for some filter $G$ of $L$. It is clear that $f(G)$ is a filter in $L'$. Now, it is enough to show that $f\{\delta(G)\} = \delta\{f(G)\}$. Let $a \in f\{\delta(G)\}$. Then $a = f(x)$, for some $x \in \delta(G)$. Hence $x^* \in G$. Now $f(x) \wedge f(x^*) = f(x \wedge x^*) = f(0) = 0$. Hence $\{f(x)\}^* \wedge f(x^*) = f(x^*) \in f(G)$. Thus $\{f(x)\}^* \in f(G)$. Therefore $a = f(x) \in \delta\{f(G)\}$. Therefore $f\{\delta(G)\} \subseteq \delta\{f(G)\}$. Conversely, let $y \in \delta\{f(G)\}$. Since $f$ is on-to, there exists $x \in L$ such that $y = f(x)$. Then $\{f(x)\}^* \in f(G)$. Hence $\{f(x)\}^* = f(a)$ for some $a \in G$. Now $f(x) \wedge \{f(x)\}^* = 0 \Rightarrow f(x) \wedge f(a) = 0 \Rightarrow f(x \wedge a) = 0 \Rightarrow x \wedge a \in Ker\ f = \{0\} \Rightarrow x^* \wedge a = a \in G \Rightarrow x^* \in G \Rightarrow x \in \delta(G) \Rightarrow y = f(x) \in f\{\delta(G)\}$. Thus $\delta\{f(G)\} \subseteq f\{\delta(G)\}$. Therefore $\delta\{f(G)\} = f\{\delta(G)\}$. □

The concept of filter congruences introduced by S. Ramesh in an ADL[3]. We have the following definition.

**Definition 3.19.** Let $F$ be a filter of an ADL $L$. Define $\theta_F := \{(a, b) \in L \mid a \wedge x = b \wedge x,$ for some $x \in F\}$.

The following result can be verified easily.

**Theorem 3.20.** *For any filter $F$ of an ADL L, $\theta_F$ is a congruence on $L$.*

We now prove the following.

**Lemma 3.21.** *Let L be a pseudo-complemented ADL with maximal elements. Then for any ideal $I$ of $L$, $F_I = \{x \in L \mid x^* \wedge a^* = 0,$ for some $a \in I\}$ is a filter of $L$.*

*Proof.* Clearly $0^* \in F_I$. Let $x, y \in F_I$. Then $x^* \wedge a^* = 0$ and $y^* \wedge b^* = 0$, for some $a, b \in I$. Hence $x^{**} \wedge a^* = a^*$ and $y^{**} \wedge b^* = b^*$. Now $(x \wedge y)^{**} \wedge (a \vee b)^* = x^{**} \wedge y^{**} \wedge a^* \wedge b^* = a^* \wedge b^*$. Thus $(x \wedge y)^* \wedge (a \vee b)^* = (x \wedge y)^* \wedge a^* \wedge b^* = [(x \wedge y)^* \wedge a^*] \wedge b^* = [(x \wedge y) \vee a]^* \wedge b^* = [(x \vee a) \wedge (y \vee a)]^* \wedge b^* = [(x \vee a) \wedge (y \vee a)]^{***} \wedge b^* = [(x \vee a)^{**} \wedge (y \vee a)^{**}]^* \wedge b^* = [(x^* \wedge a^*)^* \wedge (y \vee a)^{**}]^* \wedge b^* = [0^* \wedge (y \vee a)^{**}]^* \wedge b^* = (y \vee a)^{***} \wedge b^* = y^* \wedge b^* \wedge a^* = 0 \wedge a^* = 0$. Therefore $x \wedge y \in F_I$. Let $x \in F_I$ and $s \in L$. Then $x^* \wedge a^* = 0$, for some $a \in I$. Now $(x \vee s)^* \wedge a^* \leq x^* \wedge a^* = 0$. Thus $x \vee s \in F_I$. Therefore $F_I$ is a filter of $L$. □

We conclude this paper with the following theorem.

**Theorem 3.22.** *For any ideal $I$ of a pseudo-complemented ADL $L$, the following conditions are equivalent:*

1. *$I$ is a $\delta-$ideal,*
2. *$I = Ker\ \theta(F_I)$,*
3. *$I = Ker\ \theta(F)$, for some filter $F$ of $L$.*

*Proof.* (1) ⇒ (2): Assume that $I$ is a $\delta-$ideal of $L$. Then $I = \delta(F)$ for some filter $F$ of $L$. Let $x \in I$. Since $x^{**} \wedge x^* = 0$, we can get $x^* \in F_I$. Since $x \wedge x^* = 0$ and $x^* \in F_I$, we get that $x \in Ker\ \theta(F_I)$. Therefore $I \subseteq Ker\ \theta(F_I)$. Conversely, let $x \in Ker\ \theta(F_I)$. Then $(x, 0) \in \theta(F_I)$. Thus $x \wedge f = 0$, for some $f \in F_I$. Then $f^* \wedge x = x$. Since $f \in F_I$, we get that $f^* \wedge a^* = 0$ for some $a \in I$. That implies $a^{**} \wedge f^* = f^*$. Since $a \in= \delta(F)$, we have $a^* \in F$. That implies $a^{***} \in F$. Therefore $a^{**} \in \delta(F) = I$ and hence $f^* \in \delta(F) = I$. Thus $x \in \delta(F) = I$ Therefore $I = Ker\ \theta(F_I)$.
(2)⇒ (3): Obvious.
(3) ⇒(1): Assume that $I = Ker\ \theta(F)$ for some filter $F$ of $L$. Let $x \in I = Ker\ \theta(F)$. Then $x \wedge f = 0$ for some $f \in F$. Hence $x^* \wedge f = f \in F$. Thus $x^* \in F$, which yields that $x \in \delta(F)$. Therefore $I \subseteq \delta(F)$. Conversely, let $x \in \delta(F)$. Then $x^* \in F$. Since $x \wedge x^* = 0$ and $x^* \in F$, we get $(x, 0) \in \theta(F)$. Thus $x \in Ker\ \theta(F) = I$. Therefore $I$ is a $\delta-$ideal of $L$. □

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### Author information


N. Rafi, Department of Mathematics, Bapatla Engineering College, Bapatla, Andhra Pradesh, India-522 101, India.
E-mail: `rafimaths@gmail.com`

Ravi Kumar Bandaru, Department of Engg. Mathematics, GITAM University, Hyderabad Campus, Andhra Pradesh, India-502 329, India.
E-mail: `ravimaths83@gmail.com`

S.N. Rao, Department of Engineering Mathematics, Andhra University, Visakhapatnam, Andhra Pradesh, India - 530003, India.
E-mail: sivakotinageswarao78@gmail.com

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