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Stability in nonlinear system of neutral difference equations with functional delay

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Abstract In this paper, we use the fixed point theorem to obtain stability results of the zero solution of a nonlinear neutral system of difference equations with functional delay. In the analysis we use the fundamental matrix solution to invert the equation.

1 Introduction

Difference equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$st generation $x(n+1)$ is a function of the nth generation $x(n)$ (see [8]).

The problems of stability of time-delay systems of neutral type have received considerable attention in the last two decades, see [1, 2, 3, 10, 11, 12, 16, 17, 21, 22]. Practical examples of such systems include distributed networks containing losses transmission lines [4], and population ecology [14], vibration of masses attached to an elastic bar [20].

Certainly, the Lyapunov direct method has been, for more than 100 years, the main tool for the study of stability properties of ordinary, functional, partial differential and difference equations (see [6, 9, 10]). The fixed point theory does not only solve the problem on stability but has a significant advantage over Lyapunov's direct method.

In 2005, Y. N. Raffoul in [19] studied the existence of periodic solutions for a system of nonlinear neutral functional difference equations

$$\Delta x(n) = A(n)x(n) + \Delta Q(n, x(n-g(n))) + G(n, x(n), x(n-g(n))), \tag{1.1}$$

and said that the study of the stability of the zero solution of the equation (1.1) remains open.

In the current paper, we study the stability results of the zero solution of (1.1) with the initial condition

$$\psi(n) = x(n), \quad n \in [m(n_0), n_0] \cap \mathbb{Z},$$

where $\psi \in C\left([m(n_0), n_0] \cap \mathbb{Z}, \mathbb{R}^N\right)$ is bounded sequence and for each $n_0 \in \mathbb{Z}^+$,

$$[m(n_0), n_0] = \{n - \tau(n) \leq 0, \ n \geq n_0\}.$$

Here Δ denotes the forward difference operator, $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n), \ n \geq n_0\}$. $A(\cdot)$: is $N \times N$ matrix functions defined in $\mathbb{Z}^+$, $g : \mathbb{Z} \to \mathbb{Z}^+$ is scalar and the function $Q : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ and $G : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous in x. The sets $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the integers and the nonnegative integers, respectively. In the analysis we use the fundamental matrix solution of

$$\Delta x(n) = A(n)x(n), \tag{1.2}$$

to invert the equation (1.1). Then we employ a fixed point Theorem to show the stability of the zero solutions of the equation (1.1).

2 Preliminaries

Let $C\left(\mathbb{Z},\mathbb{R}^N\right)$ is the space of all N-vector continuous functions endowed with the supremum norm

$$\|x(\cdot)\| = \max_{n\in[0,T-1]\cap\mathbb{Z}} |x(n)|,$$

where $|\cdot|$ denotes the infinity norm for $x\in\mathbb{R}^N$. Also, if A is an $N\times N$ matrix valued sequence A, given by $A(n) := [a_{ij}(n)]$, then we define the norm of A by

$$\|A\| := \sup_{n\in[0,T-1]\cap\mathbb{Z}} |A(n)|,$$

where

$$|A(n)| = \max_{1\le i\le N} \sum_{j=1}^{N} |a_{ij}(n)|.$$

Let $\psi \in C\left([m(n_0), n_0]\cap\mathbb{Z},\mathbb{R}^N\right)$ be a given bounded initial sequence. We denote such a solution by $x(n) = x(n,n_0,\psi)$. From the existence theory we can conclude that for each $\psi\in C\left([m(n_0), n_0]\cap\mathbb{Z},\mathbb{R}^N\right)$, there exists a unique solution $x(n) = x(n,n_0,\psi)$ of (1.1) defined on $[n_0,\infty)$. We define $\|\psi\| = \max\{|\psi(n)| : n\in[m(n_0), n_0]\cap\mathbb{Z}\}$.

Throughout this paper it is assumed that the matrix $B(n) = I + A(n)$ is nonsingular, where I is the $N\times N$ identity matrix. Also, if $x(\cdot)$ is a sequence, then the forward operator E is defined as $Ex(n) = x(n+1)$. Now, we recall some definition for fundamental matrix, see also [5, 7, 8].

Definition 2.1. An $N\times N$ matrix function $n\to\Phi(n)$, defined on an open interval J, is called a matrix solution of the homogeneous linear system (1.2) if each of its columns is a (vector) solution.

Definition 2.2. A set of N solutions of the homogeneous linear difference equation (1.2), all defined on the same open interval J, is called a fundamental set of solutions on J if the solutions are linearly independent functions on J.

Definition 2.3. A matrix solution is called a fundamental matrix solution if its columns form a fundamental set of solutions. In addition, a fundamental matrix solution $t\to\Phi(t)$ is called the principal fundamental matrix solution at $t_0\in J$ if $\Phi(t_0) = I$, where I denotes the $N\times N$ identity matrix.

Definition 2.4. The state transition matrix for the homogeneous linear system (1.2) on the open interval J is the family of fundamental matrix solutions $t\to\Phi(t,r)$ parameterized by $r\in J$ such that $\Phi(r,r) = I$.

Throughout this paper, $\Phi(n)$ will denote a fundamental matrix solution of the homogeneous (unperturbed) linear problem (1.2). First, we have to transform (1.1) into an equivalent equation that possesses the same basic structure and properties to define a fixed point mapping.

Lemma 2.5. *$x(\cdot)$ is a solution of the equation (1.1) if and only if*

$$\begin{aligned} x(n) &= Q(n, x(n-\tau(n))) + \Phi(n,n_0)\left[x(n_0) - Q(n_0, x(n_0-\tau(n_0)))\right] \\ &\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\left(I + A(s)B^{-1}(s)\right)\left[G(s, x(s), x(s-\tau(s)))\right. \\ &\quad \left. + A(s)Q(s, x(s-\tau(s)))\right]. \end{aligned} \tag{2.1}$$

Proof. Let x be a solution of (1.1) and $\Phi(n,n_0)$ is a fundamental matrix of (1.2). Rewrite the equation (1.1) as

$$\begin{aligned} &\Delta\left[x(n) - Q(n, x(n-\tau(n)))\right] \\ &= A(n)\left[x(n) - Q(n, x(n-\tau(n)))\right] \\ &+ A(n)Q(n, x(n-\tau(n))) + G(n, x(n), x(n-\tau(n))). \end{aligned}$$

Since $\Phi(n,n_0)\Phi^{-1}(n,n_0)=I$, it follows that

$$\begin{aligned}0 &= \Delta\left[\Phi(n,n_0)\Phi^{-1}(n,n_0)\right]\\ &= A(n)\Delta\Phi(n,n_0)E\Phi(n,n_0)+\Phi(n,n_0)\Delta\Phi^{-1}(n,n_0)\\ &= A(n)\Phi(n,n_0)\Phi^{-1}(n,n_0)B^{-1}(n)+\Phi(n,n_0)\Delta\Phi^{-1}(n,n_0).\end{aligned}$$

This implies

$$\Delta\Phi^{-1}(n,n_0)=-\Phi^{-1}(n,n_0)A(n)B^{-1}(n).$$

If $x(\cdot)$ is a solution of (1.1) with $x(n_0)=x_0$, then

$$\begin{aligned}&\Delta\left[\Phi^{-1}(n,n_0)(x(n)-Q(n,x(n-\tau(n))))\right]\\ &=\Delta\Phi^{-1}(n,n_0)E[x(n)-Q(n,x(n-\tau(n)))]\\ &+\Phi^{-1}(n,n_0)\Delta[x(n)-Q(n,x(n-\tau(n)))]\\ &=-\Phi^{-1}(n,n_0)A(n)B^{-1}(n)\left[B^{-1}(n)(x(n)-Q(n,x(n-\tau(n))))\right.\\ &+\left.A(n)Q(n,x(n-\tau(n)))+G(n,x(n),x(n-\tau(n)))\right]\\ &+\Phi^{-1}(n,n_0)A(n)[x(n)-Q(n,x(n-\tau(n)))]\\ &+\Phi^{-1}(n,n_0)[A(n)Q(n,x(n-\tau(n)))+G(n,x(n),x(n-\tau(n)))].\end{aligned}$$

An summation of the above equation from n_0 to $n-1$ yields

$$\begin{aligned}&x(n)\\ &=Q(n,x(n-\tau(n)))+\Phi(n,n_0)(x(n_0)-Q(n_0,x(n_0-\tau(n_0))))\\ &+\Phi(n,n_0)\sum_{s=n_0}^{n-1}\Phi^{-1}(s,n_0)\left(I+A(s)B^{-1}(s)\right)[G(s,x(s),x(s-\tau(s)))\\ &+A(s)Q(s,x(s-\tau(s)))]\\ &=Q(n,x(n-\tau(n)))+\Phi(n,n_0)(x(n_0)-Q(n_0,x(n_0-\tau(n_0))))\\ &+\sum_{s=n_0}^{n-1}\Phi(n,s)\left(I+A(s)B^{-1}(s)\right)[G(s,x(s),x(s-\tau(s)))\\ &+A(s)Q(s,x(s-\tau(s)))].\end{aligned}$$

The converse implication is easily obtained and the proof is complete. □

If $x:[n_0,\infty)\cap\mathbb{Z}\to\mathbb{R}^N$ is a given solution of (1.1), then discussing the behavior of another solution y of this equation relative to the solution x, i.e. discussing the behavior of the difference $y-x$ is equivalent to studying the behavior of the solution $z=y-x$ of the equation

$$\begin{aligned}\Delta z(n)=A(n)&[y(n)-x(n)]\\ &+\Delta[Q(n,z(n-\tau(n))+x(n-\tau(n)))-Q(n,x(n-\tau(n)))]\\ &+G(n,z(n)+x(n),z(n-\tau(n))+x(n-\tau(n)))\\ &-G(n,x(n),x(n-\tau(n))),\end{aligned}$$

relative to the trivial solution $z\equiv 0$. Thus we may, without loss in generality, assume that (1.1) has the trivial solution as a reference solution, i.e.

$$Q(n,0)=G(n,0,0)\equiv 0,$$

an assumption we shall henceforth make.

In this paper we assume that, for $n\in\mathbb{Z}$, x, y, z, $w\in\mathbb{R}^N$, the functions $Q(n,x)$ and $G(n,x,y)$ are globally Lipschitz continuous in x and in x and y, respectively. That, there are positive constants k_1, k_2, k_3 such that

$$|Q(n,x)-Q(n,y)|\leq k_1\|x-y\|, \tag{2.2}$$

$$|G(n,x,y) - G(n,z,w)| \leq k_2 \|x - z\| + k_3 \|y - w\| . \tag{2.3}$$

3 Main results

Our aim here is to give a necessary and sufficient condition for asymptotic stability of the zero solution of (1.1). Stability definitions may be found in [6] and [8], for example. By the Lemma 2.5, let a mapping $\mathcal{H}$ given by

$$\begin{aligned}(\mathcal{H}\varphi)(n) &= Q(n, \varphi(n-\tau(n))) + \Phi(n, n_0)[\psi(n_0) - Q(n_0, \psi(n_0 - \tau(n_0)))] \\ &\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\left(I + A(s) B^{-1}(s)\right)[G(s, \varphi(s), \varphi(s-\tau(s))) \\ &\quad + A(s) Q(s, \varphi(s-\tau(s)))] ,\end{aligned} \tag{3.1}$$

and define the space $\mathcal{S}_\psi$ by

$$\begin{aligned}\mathcal{S}_\psi = \{&\varphi : \mathbb{R} \to \mathbb{R}^N, \varphi(n) = \psi(n) \text{ if } m(n_0) \leq n \leq n_0, \varphi(n) \to 0 \text{ as } n \to \infty, \\ &\varphi \in C \text{ is bounded}\} .\end{aligned} \tag{3.2}$$

Then, $(\mathcal{S}_\psi, \|\cdot\|)$ is a complete metric space where $\|\cdot\|$ is the supremum norm.

Theorem 3.1. *Assume (2.2) and (2.3) hold. Further assume that*

$$\Phi(n) \to 0 \text{ as } n \to \infty, \tag{3.3}$$

$$n - \tau(n) \to \infty \text{ as } n \to \infty, \tag{3.4}$$

and there is $\alpha > 0$ such that

$$k_1 + \sum_{s=n_0}^{n-1} |\Phi(n,s)| \left(1 + \|A\| \left\|B^{-1}\right\|\right)(k_2 + k_3 + \|A\| k_1) \leq \alpha < 1, \; n \geq n_0, \tag{3.5}$$

hold. Then every solution $x(n, n_0, \psi)$ of (1.1) with small initial sequence ψ, is bounded and asymptotically stable. Moreover, the zero solution is stable at n_0.

Proof. Let the mapping $\mathcal{H}$ defined by (3.1). Since Q, G, are continuous, it is easy to show that $\mathcal{H}$ is. Let ψ be a small given initial sequence with $\|\psi\| < \delta$ $(\delta > 0)$. Since $\varphi \in \mathcal{S}_\psi$ then there exist a positive constant K, such that $\|\varphi\| \leq K$, this and the condition (3.5) implies

$$\begin{aligned}|(\mathcal{H}\varphi)(n)| &\leq |Q(n, \varphi(n-\tau(n)))| \\ &\quad + |\Phi(n,n_0)| [|\psi(n_0)| + |Q(n_0, \psi(n_0 - \tau(n_0)))|] \\ &\quad + \sum_{s=n_0}^{n-1} |\Phi(n,s)| \left(1 + \|A\| \left\|B^{-1}\right\|\right) \\ &\quad \times [|G(s, \varphi(s), \varphi(s-\tau(s)))| + |A(s)| |Q(s, \varphi(s-\tau(s)))|] \\ &\leq k_1 K + \|\Phi\| \delta (1 + k_1) \\ &\quad + K \sum_{s=n_0}^{n-1} |\Phi(n,s)| \left(1 + \|A\| \left\|B^{-1}\right\|\right)(k_2 + k_3 + \|A\| k_1) \\ &\leq \|\Phi\| \delta (1 + k_1) + \alpha K,\end{aligned}$$

which implies $\mathcal{H}\varphi$ is bounded, for the right δ. Next we show that $(\mathcal{H}\varphi)(n) \to 0$ as $n \to \infty$. The first term on the right side of (3.1) tends to zero, by condition (3.4). Also, the second term on the right side tends to zero, because of (3.3) and the fact that $\varphi \in \mathcal{S}_\psi$. Let $\epsilon > 0$ be given, then there exists a $n_1 > n_0$ such that for $n > n_1$, $|\varphi(n-\tau(n))| < \epsilon$. By the condition (3.3), there exists a $n_2 > n_1$ such that for $n > n_2$ implies that

$$|\Phi(n,n_2)| < \frac{\epsilon}{\alpha K}.$$

Thus for $n > n_2$, we have

$$\begin{aligned}
&\sum_{s=n_0}^{n-1} |\Phi(n,s)| \left(1+\|A\| \left\|B^{-1}\right\|\right) [k_2 |\varphi(s)| \\
&+ k_3 |\varphi(s-\tau(s))| + \|A\| k_1 |\varphi(s-\tau(s))|] \\
&\leq K \sum_{s=n_0}^{n_1-1} |\Phi(n,s)| \left(1+\|A\| \left\|B^{-1}\right\|\right) (k_2+k_3+\|A\| k_1) \\
&+ \epsilon \sum_{s=n_1}^{n-1} |\Phi(n,s)| \left(1+\|A\| \left\|B^{-1}\right\|\right) (k_2+k_3+\|A\| k_1) \\
&\leq K |\Phi(n,n_2)| \sum_{s=n_0}^{n_1-1} |\Phi(n_2,s)| \left(1+\|A\| \left\|B^{-1}\right\|\right) (k_2+k_3+\|A\| k_1) + \alpha\epsilon \\
&\leq \alpha K |\Phi(n,n_2)| + \alpha\epsilon < \alpha\epsilon + \epsilon.
\end{aligned}$$

Hence, $(\mathcal{H}\varphi)(n) \to 0$ as $n \to \infty$. It is natural now to prove that $\mathcal{H}$ is contraction under the supremum norm. Let , $\varphi_1, \varphi_2 \in \mathcal{S}_\psi$. Then

$$\begin{aligned}
|(\mathcal{H}\varphi_1)(n) - (\mathcal{H}\varphi_2)(n)| &\leq |Q(n,\varphi_1(n-\tau(n))) - Q(n,\varphi_2(n-\tau(n)))| \\
&+ \sum_{s=n_0}^{n-1} |\Phi(n,s)| \left(1+\|A\| \left\|B^{-1}\right\|\right) [k_2 \|\varphi_1-\varphi_2\| \\
&+ k_3 \|\varphi_1-\varphi_2\| + k_1 \|A\| \|\varphi_1-\varphi_2\|] \\
&\leq \alpha \|\varphi_1-\varphi_2\|.
\end{aligned}$$

Hence, the contraction mapping principle implies, $\mathcal{H}$ has a unique fixed point in $\mathcal{S}_\psi$ which solves (1.1), bounded and asymptotically stable. The stability of the zero solution of (1.1) follows simply by replacing K by ϵ. □

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