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# NOTES ON $(\alpha, \beta)-$GENERALIZED DERIVATIONS OF $*$-PRIME RINGS

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**Abstract**. Let $R$ be a $*-$prime ring with involution $*$ and $F$ be a nonzero $(\alpha, \beta)-$derivation associated with a $(\alpha, \beta)-$derivation $d$ commuting with $*$. We prove classical results of Posner and Herstein for Lie ideals of rings with involution. Also, we discuss the commutativity of $*-$prime rings addmitting a generalized $(\alpha, \beta)-$derivations $F$ satisfying several conditions on Lie ideals.

## 1 Introduction

Let $R$ will be an associative ring with center $Z$. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$. Recall that a ring $R$ is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$ if $[u, r] \in U$, for all $u \in U, r \in R$. An additive mapping $* : R \to R$ is called an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*-$ring. A ring with an involution is said to $*-$prime if $xRy = xRy^* = 0$ or $xRy = x^*Ry = 0$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is $*-$prime but the converse need not hold general. An example due to Oukhtite [15] justifies the above statement that is, $R$ be a prime ring, $S = R \times R^o$ where $R^o$ is the opposite ring of $R$. Define involution $*$ on $S$ as $*(x, y) = (y, x)$. $S$ is $*-$prime, but not prime. This example shows that every prime ring can be injected in a $*-$prime ring and from this point of view $*-$prime rings constitute a more general class of prime rings. In all that follows the symbol $S_{a_*}(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of $R$, i.e. $S_{a_*}(R) = \{x \in R \mid x^* = \pm x\}$.

An additive subgroup $L$ of $R$ is said to be a Lie ideal of $R$ if $[L, R] \subseteq L$. A Lie ideal is said to be a $*$-Lie ideal if $L^* = L$. If $L$ is a Lie (resp. $*$-Lie) ideal of $R$, then $L$ is called a square closed Lie (resp. $*$-Lie) ideal of $R$ if $x^2 \in L$ for all $x \in L$.

Let $\alpha$ and $\beta$ be endomorphisms of $R$. For any $x, y \in R$, set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$. Following [5], an additive mapping $F : R \longrightarrow R$ is called a generalized derivation associated with a derivation $d$ if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. An additive map $d : R \longrightarrow R$ is called an $(\alpha, \beta)$-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. For a fixed $a$, the map $d_a : R \longrightarrow R$ given by $d_a(x) = [a, x]_{\alpha,\beta}$ for all $x \in R$ is an $(\alpha, \beta)$-derivation which is said to be an $(\alpha, \beta)$-inner derivation. An additive mapping $F : R \longrightarrow R$ is called a generalized $(\alpha, \beta)$-inner derivation if $F(x) = a\alpha(x) + \beta(x)b$ holds for some fixed $a, b \in R$ and for all $x \in R$. A simple computation yields that if $F$ is a generalized $(\alpha, \beta)$-inner derivation, then for all $x, y \in R$, we have $F(xy) = F(x)\alpha(y) + \beta(x)d_{(-b)}(y)$, where $d_{(-b)}$ is an $(\alpha, \beta)$-inner derivation. With this viewpoint, an additive map $F : R \longrightarrow R$ is called a generalized $(\alpha, \beta)$-derivation associated with an $(\alpha, \beta)$-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. A $(1, 1)$-generalized derivation is called simply a generalized derivation, where 1 is the identity map on $R$.

Over the past thirty years, there has been ongoing interest concerning the relationship between the commutativity of prime ring $R$ and the behaviour of a special mapping of ring, such as derivation. Recently, some well-known results concerning prime rings have been proved for

$*-$prime ring by Oukhtite et al. (see, [11-17], where further references can be found). A mapping $F$ from $R$ to $R$ is called centralizing on $S$ if $[F(x), x] \in Z$, for all $x \in S$ and is called commuting on $S$ if $[F(x), x] = 0$, for all $x \in S$. In [20], Posner showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative (Posner's second theorem). A number of authors have generalized these results by considering mappings which are only assumed to be centralizing on an appropriate subset of the ring. (see [2], [3], [9], [11], [12], where further references can be found). In [19], Oukhitite et al. generalized Posner's second theorem to rings with involution in the case of characteristic not 2. Our first aim in this paper is to prove this theorem for a $(\alpha, \beta)-$derivation and a Lie ideal of $*-$prime rings.

A famous result due to Herstein [8] states that if $R$ is a prime ring of characteristic not 2 which admits a nonzero derivation $d$ such that $[d(x), a] = 0$ for all $x \in R$, then $a \in Z$. This result proved for a nonzero Lie ideal of R in [6] and for $(\alpha, \beta)-$derivation of a $*-$prime ring in [21]. We shall prove this result for a $(\alpha, \beta)-$derivation and a Lie ideal of $*-$prime rings.

Recently, in [10], Marubayashi et. al. obtained some results for generalized $(\alpha, \beta)$-derivations and Lie ideals of prime rings. In the present paper our aim is extend may results fora $*-$Lie ideal and a generalized $(\alpha, \beta)-$derivation of $*-$prime rings.

In [4], Bell and Kappe proved that if $d$ is a derivation of prime ring $R$ which acts as a homomorphism or anti-homomorphism on a nonzero ideal of $R$, then $d = 0$. Oukhitite proved this result is also true for $*-$prime rings in [15] and Shuliang extended to $(\alpha, \beta)$-derivations in [22]. Finally, we prove this theorem for a $*-$Lie ideal and a $(\alpha, \beta)-$derivation of $*-$prime rings.

Throughout the paper, $R$ will be a $2-$torsion free $*-$prime ring, where $*$ is an involution of $R$, $F$ be a generalized $(\alpha, \beta)$-derivation associated with $(\alpha, \beta)$-derivation $d$ of $R$ and $U$ is a $*$-Lie ideal of $R$.

## 2 Preliminary results

Throughout the present paper $\alpha, \beta$ will denote automorphisms of $R$. We shall use, without explicit mention, the following basic identities:

$$[x, yz] = y[x, z] + [x, y]z$$

$$[xy, z] = [x, z]y + x[y, z]$$

$$[xy, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y,$$

$$[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z),$$

$$(x \circ (yz))_{\alpha,\beta} = (x \circ y)_{\alpha,\beta}\alpha(z) - \beta(y)[x, z]_{\alpha,\beta} = \beta(y)(x \circ z)_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z),$$

$$\text{and } \ ((xy) \circ z)_{\alpha,\beta} = x(y \circ z)_{\alpha,\beta} - [x, \beta(z)]y = (x \circ z)_{\alpha,\beta}y + x[y, \alpha(z)].$$

**Lemma 2.1.** *[17, Lemma 2.3]Let $R$ be a $*-$prime ring with characteristic not two, $U$ be a nonzero $*-$Lie ideal of $R$. If $[U, U] \subseteq Z$, then $U \subseteq Z$.*

**Lemma 2.2.** *[18, Lemma 4]Let $R$ be a $*-$prime ring with characteristic not two, $U$ be a nonzero $*-$Lie ideal of $R$ and $a, b \in R$. If $aUb = *(a)Ub = 0$, then $a = 0$ or $b = 0$ or $U \subseteq Z$.*

**Lemma 2.3.** *[1, Lemma 2.7]Let $R$ be a $2-$torsion free $*-$prime ring and $U$ be a $*-$Lie ideal of $R$. If $a \in R$ such that $[a, U] \subseteq Z$, then either $U \subseteq Z$ or $a \in Z$.*

**Lemma 2.4.** *[21, Lemma 5]Let $R$ be a $*-$prime ring with characteristic not two, $d$ be a nonzero $(\alpha, \beta)-$derivation of $R$ which commutes with $*$ and $U$ be a nonzero $*-$Lie ideal of $R$. If $d(U) = 0$, then $U \subseteq Z$.*

**Lemma 2.5.** *[21, Lemma 6]Let $R$ be a $*-$prime ring, $d$ be a nonzero $(\alpha, \beta)-$derivation of $R$ which commutes with $*$ and $U$ be a nonzero $*-$Lie ideal of $R$. If $a \in R$ and $ad(U) = 0$ (or $d(U)a = 0$), then $a = 0$ or $U \subseteq Z$.*

**Lemma 2.6.** *[21, Lemma 8]Let $R$ be a $*-$prime ring with characteristic not two, $d$ be a nonzero $(\alpha, \beta)-$derivation of $R$ which commutes with $*$ and $U$ be a nonzero $*-$Lie ideal of $R$. If $d(U) \subset Z$, then $U \subseteq Z$.*

Now we prove the following:

**Lemma 2.7.** *Let $R$ be a prime ring with char(R)$\neq$ 2, and let be $U$ a nonzero square-closed Lie ideal of $R$. If $[u,v]_{\alpha,\beta}=0$ for all $u,v\in U$, then $U\subseteq Z(R)$.*

*Proof.* We have

$$[u,v]_{\alpha,\beta}=0 \text{ for all } u,v\in U. \tag{2.1}$$

Replacing $v$ by $[r,u]$ in (2.1), we get $[u,[r,u]]_{\alpha,\beta}=0$ for all $u\in U,\ \ r\in R$. Again replace $r$ by $rs$, to get $[u,[rs,u]]_{\alpha,\beta}=0$ for all $u\in U$ and $r,s\in R$. That is,

$$[u,[r,u]]_{\alpha,\beta}\alpha(s)+\beta([r,u])[u,s]_{\alpha,\beta}+\beta(r)[u,[s,u]]_{\alpha,\beta}+[u,r]_{\alpha,\beta}\alpha([s,u])=0$$

for all $u\in U$ and $r,s\in R$. This implies that

$$\beta([r,u])[u,s]_{\alpha,\beta}+[u,r]_{\alpha,\beta}\alpha([s,u])=0 \text{ for all } u\in U,\ r,s\in R.$$

Now replace $r$ by $v$ in the above expression and use (2.1), to get $\beta([v,u])[u,s]_{\alpha,\beta}=0$ for all $u,v\in U$ and $s\in R$. Again replacing $s$ by $sr$ and using the above expression we find that $\beta([v,u])\beta(s)[u,r]_{\alpha,\beta}=0$ for all $u,v\in U$ and $s\in R$, that is, $\beta([v,u])R[u,r]_{\alpha,\beta}=\{0\}$. For $U\in S_{a_*}(R)$ we have either $\beta([v,u])=0$ or $[u,r]_{\alpha,\beta}=0$ for all $v\in U$, $r\in R$. Using the fact that $u+u^*$ and $u-u^*$ are in $U\cap S_{a_*}(R)$ for all $u\in U$, we easily deduce that $U$ is union of two additive subgroups $A$ and $B$ where

$$A=\{u\in U\mid \beta([v,u])=0,\ for\ all\ v\in U\}$$

and

$$B=\{u\in U\mid [u,r]_{\alpha,\beta}=0,\ for\ all r\in R\}$$

But a group cannot be a union of two its proper subgroups. Hence $U=A$ or $U=B$. If $U=A$, then $\beta([v,u])=0$ for all $u,v\in U$ and hence $[v,u]=0$. Thus by Lemma 2.1, we get the required result. In the second case, replace $u$ by $2vu$ in the expression $[u,r]_{\alpha,\beta}=0$ to obtain $[v,\beta(r)]u=0$ for all $u,v\in U,\ r\in R$. It follows by Lemma 2.2 that $U\subseteq Z(R)$. □

## 3 Results

The following theorem gives a generalization of Herstein's well known result [8, Theorem ] and [21, Theorem 1].

**Theorem 3.1.** *Let $R$ be a $*-$prime ring with characteristic not two, $U$ a nonzero $*-$Lie ideal of $R$ such that $u^2\in U$ for all $u\in U$ and $a\in S_{a_*}(R)$. If $R$ admits a nonzero $(\alpha,\beta)-$derivation $d$ which commutes with $*$ and $[d(U),a]_{\alpha,\beta}=0$, then $a\in Z$ or $U\subseteq Z$.*

*Proof.* Notice that $uv+vu=(u+v)^2-u^2-v^2$, for all $u,v\in U$. Since $u^2\in U$ for all $u\in U$, $uv+vu\in U$. Also $uv-vu\in U$, for all $u,v\in U$. Hence, we get $2uv\in U$, for all $u,v\in U$.

Now, let $u,v\in U$. Then

$$\begin{aligned}0&=[d(2uv),a]_{\alpha,\beta}=2[d(u)\alpha(v)+\beta(u)d(v),a]_{\alpha,\beta}\\&=2[d(u),a]_{\alpha,\beta}\,\alpha(v)+2d(u)[\alpha(v),\alpha(a)]+2\beta(u)[d(v),a]_{\alpha,\beta}+2[\beta(u),\beta(a)]d(v)\end{aligned}$$

and so

$$d(u)\alpha([v,a])=\beta([a,u])d(v),\ \text{for all } u,v\in U. \tag{3.1}$$

Replacing $v$ by $2vw$ in (3.1) and using (3.1), we arrive at

$$d(u)\alpha(v)\alpha([w,a])=\beta([a,u])\beta(v)d(w),\ \text{for all } u,v,w\in U. \tag{3.2}$$

Let in (3.2) $v$ be $[v,a]$ and again using (3.1), we have

$$\begin{aligned}d(u)\alpha([v,a])\alpha([w,a])&=\beta([a,u])\beta([v,a])d(w)\\ \beta([a,u])d(v)\alpha([w,a])&=\beta([a,u])\beta([v,a])d(w)\end{aligned}$$

and so

$$\beta([a,u])\beta([a,v])d(w)) = \beta([a,u])\beta([v,a])d(w), \text{ for all } u,v,w \in U.$$

That is $2\beta([a,u])\beta([a,v])d(w)) = 0$. Since $R$ is $2-$torsion free, we get

$$\beta([a,u][a,v])d(U) = 0, \text{ for all } u,v \in U.$$

By Lemma 2.5, we arrive at

$$[a,w][a,v] = 0, \text{ for all } u,v \in U.$$

Again replacing $v$ by $2vw$ in the last equation and using this, we have

$$[a,u]U[a,w] = 0, \text{ for all } u,w \in U.$$

Since $U$ is a nonzero $*-$Lie ideal of $R$ and $a \in S_{a_*}(R)$ yields that

$$([a,u])^* U[a,w] = 0, \text{ for all } u,w \in U.$$

By the application of Lemma 2.2 yields that $[a,u] = 0$ for all $u \in U$ or $[a,w] = 0$, for all $w \in U$. Hence $a \in Z$ or $U \subseteq Z$ by Lemma 2.3. This completes the proof. □

**Theorem 3.2.** *Let $R$ be a $*-$prime ring with characteristic not two, $U$ a nonzero square closed $*-$Lie ideal of $R$. If $R$ admits a nonzero $(\alpha,\beta)-$derivation $d$ which commutes with $*$ and $[d(u),u]_{\alpha,\beta} = 0$, for all $u \in U$, then $U \subseteq Z$.*

*Proof.* Suppose, on the contrary that, $U \nsubseteq Z(R)$. Define $B(.,.) : R \times R \to R$ by

$$B(u,v) = [d(u),v]_{\alpha,\beta} + [d(v),u]_{\alpha,\beta} \text{ for all } u,v \in U;$$

and note that by linearizing the condition $[d(u),u]_{\alpha,\beta} = 0$, we get $B(u,v) = 0$ for all $u,v \in U$. It is easily verified that

$$B(uv,w) = B(u,w)\alpha(v) + \beta(u)B(v,w) + d(u)\alpha([v,w]) + \beta([u,w])d(v) \text{ for all } u,v,w \in U,$$

so by taking $w = u$ we obtain $d(u)\alpha([v,u]) = 0$ for all $u,v \in U$. Replacing $v$ by $2vw$ yields $d(u)\alpha(v)\alpha([w,u]) = 0$,

Since $\alpha$ is an automorphism of $R$, we see that

$$\alpha^{-1}(d(u))U[w,u] = 0, \text{ for all } u,w \in U. \tag{3.3}$$

Since $U$ is a nonzero $*-$Lie ideal of $R$ yields that

$$\alpha^{-1}(d(u))U([w,u])^* = 0, \text{ for all } w \in U, u \in U \cap S_{a_*}(R).$$

By Lemma 2.2, we get either $[w,u] = 0$, for all $w \in U$ or $d(u) = 0$ for each $u \in U \cap S_{a_*}(R)$. Let $u \in U$, as $u + (u)^*, u - (u)^* \in U \cap S_{a_*}(R)$ and $[w, u \pm (u)^*] = 0$, for all $v \in U$, or $d(u \pm (u)^*) = 0$. Hence we have $[w,u] = 0$ or $d(u) = 0$, for all $u,w \in U$. We obtain that $U$ is union of two additive subgroups of $U$ such that

$$K = \{u \in U \mid d(u) = 0\}$$

and

$$L = \{u \in U \mid [w,u] = 0, \text{ for all } w \in U\}.$$

Morever, $U$ is the set-theoretic union of $K$ and $L$. But a group can not be the set-theoretic union of two proper subgroups, hence $K = U$ or $L = U$. In the former case, we get $U \subseteq Z$ by Lemma 2.4. In the latter case, $[U,U] = (0)$ . That is $U \subseteq Z$ by Lemma 2.1. This completes the proof.

Using the same techniques with necessary variations, we can prove the following corollary even without the characteristic assumption on the ring.

**Corollary 3.3.** *Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. If $R$ admits a nonzero $(\alpha,\beta)$-derivation $d$ such that $[d(x),x]_{\alpha,\beta}=0$ for all $x\in I$, then $R$ is commutative.*

**Theorem 3.4.** *Let $R$ be a $*-$prime ring with characteristic not two, $U$ a nonzero square closed $*-$Lie ideal of $R$. If $R$ admits a nonzero generalized $(\alpha,\beta)-$derivation $F$ associated with nonzero $(\alpha,\beta)$-derivation $d$ which commutes with $*$ such that $[F(u),u]_{\alpha,\beta}=0$ for all $u\in U$ then $U\subseteq Z$.*

*Proof.* Suppose, on the contrary that $U\nsubseteq Z(R)$ and

$$[F(u),u]_{\alpha,\beta}=0 \text{ for all } u\in U. \tag{3.4}$$

Linearizing (3.4) and using (3.4), we obtain

$$[F(u),v]_{\alpha,\beta}+[F(v),u]_{\alpha,\beta}=0 \text{ for all } u,v\in U. \tag{3.5}$$

Replacing $v$ by $2vu$ in (3.5), we get

$$[F(u),vu]_{\alpha,\beta}+[F(v)\alpha(u)+\beta(v)d(u),u]_{\alpha,\beta}=0$$

that is,

$$[F(u),v]_{\alpha,\beta}\alpha(u)+\beta(v)[F(u),u]_{\alpha,\beta}+[F(v),u]_{\alpha,\beta}\alpha(u)+$$

$$F(v)[\alpha(u),\alpha(u)]+\beta(v)[d(u),u]_{\alpha,\beta}+[\beta(v),\beta(u)]d(u)=0. \tag{3.6}$$

Now combining (3.4), (3.5) and (3.6) we find that

$$\beta(v)[d(u),u]_{\alpha,\beta}+[\beta(v),\beta(u)]d(u)=0 \text{ for all } u,v\in U. \tag{3.7}$$

Again replace $v$ by $2wv$ in (3.7) and use (3.7), to get

$$[\beta(w),\beta(u)]\beta(v)d(u)=0 \text{ for all } u,v,w\in U. \tag{3.8}$$

This implies that $[w,u]U\beta^{-1}(d(u))=\{0\}$ for all $u,w\in U$. Since $\beta$ is an automorphism of $R$, we see that

$$[w,u]\,U\beta^{-1}\left(d\left(u\right)\right)=0, \text{ for all } u,w\in U. \tag{3.9}$$

Now using the similar arguments as used in the proof of Theorem 3.2, we get the required result. □

**Theorem 3.5.** *Let $R$ be a $*-$prime ring with characteristic not two, $U$ a nonzero square closed $*-$Lie ideal of $R$. If $R$ admits a nonzero generalized $(\alpha,\beta)-$derivation $F$ associated with nonzero $(\alpha,\beta)$-derivation $d$ which commutes with $*$ such that $[F(u),u]=0$ for all $u\in U$ then $U\subseteq Z$.*

*Proof.* Suppose, on the contrary that $U\nsubseteq Z(R)$ and $F([u,v])=0$ for all $u,v\in U$. Replacing $v$ by $2vu$ in the above expression and using the fact that char$R\neq 2$, we find that

$$0=F([u,vu])=F([u,v]u)=\beta([u,v])d(u).$$

Now, again replace $v$ by $2wv$, to get $\beta([u,w]v)d(u)=0$ for all $u,v,w\in U$. This implies that $[u,w]U\beta^{-1}(d(u))=\{0\}$ for all $u,w\in U$. Now application of similar techniques as used in the proof of Theorem 3.2 after (3.9) yields the required result. □

**Theorem 3.6.** *Let $R$ be a 2-torsion free prime ring and $U$ a nonzero square-closed Lie ideal of $R$. Suppose that $R$ admits a generalized $(\alpha,\beta)$-derivation $F$ with associated $(\alpha,\beta)$-derivation $d$ such that*

$(i)$ *$F([u,v])=[u,v]_{\alpha,\beta}$ for all $u,v\in U$ or*

$(ii)$ *$F([u,v])=-[u,v]_{\alpha,\beta}$ for all $u,v\in U$.*

*If $F=0$ or $d\neq 0$, then $U\subseteq Z(R)$.*

*Proof.* $(i)$ Given that $F$ is a generalized $(\alpha, \beta)$-derivation of $R$ such that $F([u, v]) = [u, v]_{\alpha,\beta}$ for all $u, v \in U$. If $F = 0$, then $[u, v]_{\alpha,\beta} = 0$ for all $u, v \in U$. Thus by Lemma 2.7, we get the required result.

Henceforth, we shall assume that $d \neq 0$. Suppose on the contrary that $U \not\subseteq Z(R)$. For any $u, v \in U$, we have

$$F([u, v]) = [u, v]_{\alpha,\beta}. \tag{3.10}$$

Replacing $v$ by $2wv$ in (3.10) and using the fact that char$(R) \neq 2$, we get

$$F(w[u, v] + [u, w]v) = [u, wv]_{\alpha,\beta} \text{ for all } u, v, w \in U$$

that is,

$$F(w)\alpha([u, v]) + \beta(w)d([u, v]) + F([u, w])\alpha(v) + \beta([u, w])d(v)$$

$$= [u, w]_{\alpha,\beta}\alpha(v) + \beta(w)[u, v]_{\alpha,\beta} \text{ for all } u, v, w \in U. \tag{3.11}$$

Now by application of (3.10), we find that

$$F(w)\alpha([u, v]) + \beta(w)d([u, v]) + \beta([u, w])d(v) = \beta(w)[u, v]_{\alpha,\beta}. \tag{3.12}$$

Replace $v$ by $2vu$ in (3.12), to get

$$F(w)\alpha([u, v]u) + \beta(w)d([u, v]u) + \beta([u, w])(d(v)\alpha(u) + \beta(v)d(u))$$

$$= \beta(w)([u, v]_{\alpha,\beta}\alpha(u) + \beta(v)[u, u]_{\alpha,\beta}).$$

This implies that

$$\{F(w)\alpha([u, v]) + \beta(w)d([u, v]) + \beta([u, w])d(v)\}\alpha(u) + \beta(w)\beta([u, v])d(u)+$$

$$\beta([u, w])\beta(v)d(u) = \beta(w)([u, v]_{\alpha,\beta}\alpha(u) + \beta(w)\beta(v)[u, u]_{\alpha,\beta}) \text{ for all } u, v, w \in U.$$

Now using (3.12), we obtain

$$\beta(w)\beta([u, v])d(u) + \beta([u, w])\beta(v)d(u) = \beta(w)\beta(v)[u, u]_{\alpha,\beta}. \tag{3.13}$$

Again replace $w$ by $2w_1w$ in (3.13), to get

$$\beta(w_1)\beta(w)\beta([u, v])d(u) + \beta(w_1[u, w] + [u, w_1]w)\beta(v)d(u)$$

$$= \beta(w_1)\beta(w)\beta(v)[u, u]_{\alpha,\beta} \text{ for all } u, v, w, w_1 \in U.$$

Application of (3.13) gives that

$$\beta([u, w_1])\beta(w)\beta(v)d(u) = 0 \text{ for all } u, v, w, w_1 \in U,$$

and hence

$$[u, w_1]wU\beta^{-1}(d(u)) = \{0\} \text{ for all } u, w, w_1 \in U.$$

It follows by Lemma 2.2 and using the similar arguments as used after equation (3.9), we get the required result. □

$(ii)$ If $F$ satisfies $F([u, v]) = -[u, v]_{\alpha,\beta}$, then $(-F)$ satisfies the condition $(-F)([u, v]) = [u, v]_{\alpha,\beta}$ for all $u, v \in U$ and hence by part $(i)$, our result follows. □

**Theorem 3.7.** *Let $R$ be a 2-torsion free prime ring and $U$ a square-closed Lie ideal of $R$. Suppose that $R$ admits a generalized $(\alpha, \beta)$-derivation $F$ with associated $(\alpha, \beta)$-derivation $d$ such that*

$(i)$ $F(uv) = \alpha(uv)$ *for all* $u, v \in U$ *or*

$(ii)$ $F(uv) = \alpha(vu)$ *for all* $u, v \in U$.

*If $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

*Proof.* $(i)$ For any $u, v \in U$, we have $F(uv - vu) = F(uv) - F(vu) = \alpha(uv) - \alpha(vu)$, and hence $F([u,v]) = \alpha([u,v])$. If $F = 0$, then $\alpha([u,v]) = 0$ for all $u, v \in U$. Thus $[u,v] = 0$ for all $u, v \in U$ and hence by Lemma 2.1, $U \subseteq Z(R)$.

Henceforth, we shall assume that $d \neq 0$. Suppose on the contrary that $U \nsubseteq Z(R)$. For any $u, v \in U$ we have $F([u,v]) = \alpha([u,v])$. This can be rewritten as

$$F(u)\alpha(v) + \beta(u)d(v) - F(v)\alpha(u) - \beta(v)d(u) = \alpha([u,v]) \tag{3.14}$$

Replacing $v$ by $2vu$ in (3.14) and using the fact that char$(R) \neq 2$, we find that

$$F(u)\alpha(v)\alpha(u) + \beta(u)d(v)\alpha(u) + \beta(u)\beta(v)d(u) - F(v)\alpha(u)\alpha(u)$$

$$-\beta(v)d(u)\alpha(u) - \beta(v)\beta(u)d(u) = \alpha([u,v])\alpha(u) \text{ for all } u, v \in U$$

and hence the application of (3.14) gives that $\beta([u,v])d(u) = 0$ for all $u, v \in U$. Again replace $v$ by $2wv$, to get $\beta([u,w])\beta(v)d(u) = 0$ for all $u, v, w \in U$ and hence

$$[u,w]U\beta^{-1}(d(u)) = \{0\} \text{ for all } u, w \in U.$$

The last expression is same as equation(3.8) and hence the result follows.

□

$(ii)$ Using similar techniques with necessary variations, we get the required result. □

If the commutator is replaced by the anti-commutator in Theorems 3.5 & 3.6, then we see that the conclusion of these theorems hold good.

**Theorem 3.8.** *Let $R$ be a 2-torsion free prime ring and $U$ a nonzero square-closed Lie ideal of $R$. Suppose that $R$ admits a generalized $(\alpha, \beta)$-derivation $F$ with associated $(\alpha, \beta)$-derivation $d$ such that $F(uov) = 0$ for all $u, v \in U$. If $d \neq 0$, then $U \subseteq Z(R)$.*

*Proof.* Suppose on the contrary that $U \nsubseteq Z(R)$, Replacing $v$ by $2vu$ in our hypothesis, we obtain

$$0 = F(u \circ vu) = F((u \circ v)u) = \beta(u \circ v)d(u) \text{ for all } u, v \in U.$$

Now replace $v$ by $2wv$ and use the above relation, to get $\beta([u,w]v)d(u) = 0$ for all $u, v, w \in U$. This implies that $[u,w]U\beta^{-1}(d(u)) = \{0\}$ for all $u, w \in U$. Now, application of similar arguments as used after (3.8) in Theorem 3.4 yields the required result. □

**Theorem 3.9.** *Let $R$ be a 2-torsion free prime ring and $U$ a square-closed Lie ideal of $R$. Suppose that $R$ admits a generalized $(\alpha, \beta)$-derivation $F$ with associated $(\alpha, \beta)$-derivation $d$ such that*

$(i)$ *$F(u \circ v) = (u \circ v)_{\alpha,\beta}$ for all $u, v \in U$ or*

$(ii)$ *$F(u \circ v) = -(u \circ v)_{\alpha,\beta}$ for all $u, v \in U$.*

*If $F = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

*Proof.* $(i)$ If $F = 0$, then we have

$$(u \circ v)_{\alpha,\beta} = 0 \text{ for all } u, v \in U. \tag{3.15}$$

Replacing $v$ by $2vw$ in (3.11) and using (3.11), we get $\beta(v)[u,w]_{\alpha,\beta} = 0$ for all $u, v, w \in U$. Now replace $v$ by $[v,r]$, to get $\beta([v,r])[u,w]_{\alpha,\beta} = 0$. Again replacing $r$ by $rs$ in the above expression, we find that $\beta([v,r])R[u,w]_{\alpha,\beta} = \{0\}$ for all $u, v, w \in U$, $r \in R$. Notice that the arguments given in the last paragraph of the proof of Lemma 2.7 are stil valid in the present situation, and hence repeating the same process, we get the required result.

Therefore, we shall assume that $d \neq 0$. Suppose on the contrary that $U \nsubseteq Z(R)$. For any $u, v \in U$, we have $F(u \circ v) = (u \circ v)_{\alpha,\beta}$. This can be rewritten as

$$F(u)\alpha(v) + \beta(u)d(v) + F(v)\alpha(u) + \beta(v)d(u) = (u \circ v)_{\alpha,\beta}. \tag{3.16}$$

Replacing $v$ by $2vu$ in (3.16), we find that

$$F(u)\alpha(v)\alpha(u) + \beta(u)d(v)\alpha(u) + \beta(u)\beta(v)d(u) + F(v)\alpha(u)\alpha(u) + \beta(v)d(u)\alpha(u)+$$
$$\beta(v)\beta(u)d(u) = (u \circ v)_{\alpha,\beta}\alpha(u) - \beta(v)[u,u]_{\alpha,\beta} \text{ for all } u, v \in U.$$

Thus an application (3.16), gives that $\beta(u \circ v)d(u) + \beta(v)[u,u]_{\alpha,\beta} = 0$ for all $u, v \in U$. Again replace $v$ by $2wv$, to get $\beta([u,w])\beta(v)d(u) = 0$ i.e., $[u,w]U\beta^{-1}(d(u)) = \{0\}$ for all $u, w \in U$. Now application of similar arguments as used after (3.8) in the proof of Theorem 3.4 yields the required result.

$(ii)$ Use similar arguments as above. □

In view of these results we get the following corollary:

**Corollary 3.10.** *Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. Suppose that $R$ admits a generalized $(\alpha,\beta)$-derivation $F$ with associated $(\alpha,\beta)$-derivation $d$ such that any one of the following holds:*

$(i)$ $[F(x),x]_{\alpha,\beta} = 0$ *for all $x \in I$,*

$(ii)$ $F([x,y]) - [x,y]_{\alpha,\beta} = 0$ *for all $x, y \in I$, or* $F([x,y]) + [x,y]_{\alpha,\beta} = 0$ *for all $x, y \in I$,*

$(iii)$ $F(x \circ y) - (x \circ y)_{\alpha,\beta} = 0$ *for all $x, y \in I$, or* $F(x \circ y) + (x \circ y)_{\alpha,\beta} = 0$ *for all $x, y \in I$.*

*If $F = 0$ or $d \neq 0$, then $R$ is commutative.*

**Theorem 3.11.** *Let $R$ be a 2-torsion free $*$-prime ring and $F : R \longrightarrow R$ be a generalized $(\alpha,\beta)$-derivation with associated nonzero $(\alpha,\beta)$-derivation $d$. Suppose that $\beta$ and $d$ commutes with $*$. If $U$ is a nonzero $*$-square closed Lie ideal of $R$ such that $F([u,v]) = [F(u),v]_{\alpha,\beta}$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

*Proof.* We have

$$F([u,v]) = [F(u),v]_{\alpha,\beta}, \text{ for all } u, v \in U. \tag{3.17}$$

Replacing $u$ by $[u, ru]$ in (3.17) we have

$$F([[u,ru],v]) = [F([u,ru]),v]_{\alpha,\beta}, \text{ for all } u, v \in U \text{ and} r \in R. \tag{3.18}$$

$$F([[u,r]u,v]) = [F([u,r]u),v]_{\alpha,\beta}, \text{ for all } u, v \in U \text{ and} r \in R. \tag{3.19}$$

This implies that $F([u,r][u,v] + [[u,r],v]u) = [F[u,r]\alpha(u) + \beta([u,r])d(u), v]_{\alpha,\beta}$ for all $u, v \in U$ and $r \in R$. Using the hypothesis we obtain $\beta([u,r])d[u,v] = \beta[u,r][d(u),v]_{\alpha,\beta}$, for all $u, v \in U$ and $r \in R$. This gives us $\beta[u,r][d(v),u]_{\alpha,\beta} = 0$, for all $u, v \in U$ and$r \in R$. Replacing $r$ by $sr$ for some s in R we get

$$\beta[u,sr][d(v),u]_{\alpha,\beta} = 0, \text{ for all } u, v \in U \text{ and s,r} \in R.$$

This implies that

$$\beta([u,R])R[d(v),u]_{\alpha,\beta} = 0, \text{ for all } u, v \in U. \tag{3.20}$$

If $u \in S_*(R) \cap U$, then $\beta([u,R])R[d(v),u]_{\alpha,\beta} = \beta * ([u,R])R[d(v),u]_{\alpha,\beta} = *\beta([u,R])R[d(v),u]_{\alpha,\beta} = 0$ . Thus, for some $u \in S_*(R) \cap U$ either $\beta([u,R]) = 0$ *or* $[d(v),u]_{\alpha,\beta} =$

0. But for any $u \in U, u - u^*, u + u^* \in S_*(R) \cap U$. Therefore, for some $u \in U$ either $\beta([u - u^*, R]) = 0$ *or* $[d(v), u - u^*]_{\alpha,\beta} = 0$. If $\beta([u - u^*, R]) = 0$ then from equation (3.20) we obtain that $\beta([u, R])R[d(v), u]_{\alpha,\beta} = *\beta([u, R])R[d(v), u]_{\alpha,\beta}$ for all $u \in U$ hence either $\beta([u, R]) = 0$ *or* $[d(v), u]_{\alpha,\beta} = 0$. Let $A = \{u \in U \mid \beta[u, R] = 0\}$ *and* $B = \{u \in U \mid [d(v), u]_{\alpha,\beta} = 0\}$. Then it can be seen that $L$ and $K$are two additive subgroups of $U$ whose union is $U$. Using Brauer's trick we have either $A = U$ or $B = U$. If $A = U$, then $[u, R] = 0$ for all $u \in U$ $r \in R$ and hence we get $U \subseteq Z(R)$ on the other hand if $B = U$, then $[d(v), u]_{\alpha,\beta} = 0$ for all $u, v \in U$. In particular $[d(u), u]_{\alpha,\beta} = 0$ for all $u \in U$. Thus by Theorem 3.2 we get the required result. □

**Theorem 3.12.** *Let $R$ be a 2-torsion free $*$-prime ring and $F : R \longrightarrow R$ be a generalized $(\alpha, \beta)$-derivation with associated nonzero $(\alpha, \beta)$-derivation $d$. Suppose that $\beta$ and $d$ commutes with $*$. If $U$ is a nonzero $*$-square closed Lie ideal of $R$ such that $F(u \circ v) = (F(u) \circ v)_{\alpha,\beta}$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

*Proof.* We have $F(u \circ v) = (F(u) \circ v)_{\alpha,\beta}$ for all $u, v \in U$,$r \in R$. Replacing $u$ by $[u, ru]$ in the above expression we find that

$$F(([u, r] \circ v)u + [u, r][u, v]) = (F[u, r]\alpha(u) \circ v)_{\alpha,\beta} + (\beta([u, r])d(u) \circ v)_{\alpha,\beta}$$

for all $u, v \in U, r \in R$. Thus we obtain,

$$F([u, r] \circ v)\alpha(u) + \beta([u, r] \circ v)d(u) + F([u, r])\alpha([u, v]) + \beta([u, r])d[u, v] =$$

$$(F[u, r] \circ v)_{\alpha,\beta}\alpha(u) + F([u, r])[\alpha(u), \alpha(v)] + (\beta[u, r] \circ v)_{\alpha,\beta}d(u) + \beta([u, r])[d(u), \alpha(v)]$$

for all $u, v \in U, r \in R$. Using our hypothesis we find that

$$\beta([u, r] \circ v)d(u) + \beta([u, r])d[u, v] =$$

$$(\beta[u, r] \circ v)_{\alpha,\beta}d(u) + \beta([u, r])[d(u), \alpha(v)] \text{ for all } u, v \in U, r \in R.$$

Hence, we obtain, $\beta([u, r])[d(v), u]_{\alpha,\beta} = 0$ for all $u, v \in U, r \in R$. Replacing $r$ by $rs$ for some $s$ in $R$ we get $\beta([u, R])R[d(v), u]_{\alpha,\beta} = 0$ for all $u, v \in U$. The last expression is same as the equation (3.20) and hence the result follows. □

**Theorem 3.13.** *Let $R$ be a 2-torsion free $*$-prime ring and $F : R \longrightarrow R$ be a generalized $(\alpha, \beta)$-derivation with associated nonzero $(\alpha, \beta)$-derivation $d$. Suppose that $\beta$ and $d$ commutes with $*$. If $U$ is a nonzero $*$-square closed Lie ideal of $R$ such that $F[u, v] = [F(u), v]_{\alpha,\beta} + [d(v), u]_{\alpha,\beta}$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

*Proof.* **Proof** We have $F([u, v]) = [F(u), v]_{\alpha,\beta} + [d(v), u]_{\alpha,\beta}$ for all $u, v \in U$. Now Replacing $u$ by $[u, ru]$ we get

$$F([[u, ru], v]) = [F([u, ru]), v]_{\alpha,\beta} + [d(v), [u, ru]]_{\alpha,\beta} \text{ for all } u, v \in U, r \in R$$

The last expression can be rewritten as

$$F([u, r][u, v] + [[u, r], v]u) = [F([u, r])\alpha(u), v]_{\alpha,\beta} + [\beta([u, r])d(u), v]_{\alpha,\beta}$$

$$+\beta([d(v), u]_{)} + [d(v), [u, r]]_{\alpha,\beta}\alpha(u) \text{ for all } u, v \in U, r \in R.$$

Hence

$$F[u, r]\alpha[u, v] + \beta[u, r]d[u, v] + F[[u, r], v]\alpha(u) + \beta[[u, r], v]d(u) =$$

$$F[u, r][\alpha(u), v]_{\alpha,\beta} + [F[u, r], \beta(v)]\alpha(u) + \beta[u, r][d(u), v]_{\alpha,\beta}+$$

$$[\beta[u, r], \beta(v)]d(u) + \beta[u, r][d(v), u]_{\alpha,\beta} + [d(v), [u, r]]_{\alpha,\beta}\alpha(u) \text{ for all } u, v \in U, r \in R.$$

We find that, $\beta([u, r])\beta(u)d(v) = \beta([u, r])d(v)\alpha(u)$ for all $u, v \in U, r \in R$. This gives us $\beta([u, r])[d(v), u]_{\alpha,\beta} = 0$ for all $u, v \in U, r \in R$. Now application of similar arguments as used after (3.20) in the proof of Theorem 3.11 yields the required result. □

**Theorem 3.14.** *Let $R$ be a 2-torsion free $*$-prime ring and $F: R \longrightarrow R$ be a generalized $(\alpha,\beta)$-derivation with associated nonzero $(\alpha,\beta)$-derivation $d$. Suppose that $\beta$ and $d$ commutes with $*$. If $U$ is a nonzero $*$-square closed Lie ideal of $R$ such that $F(u\circ v) = (F(u)\circ v)_{\alpha,\beta} + (d(v)\circ u)_{\alpha,\beta}$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

*Proof.* We have

$$F(u\circ v) = (F(u)\circ v)_{\alpha,\beta} + (d(v)\circ u)_{\alpha,\beta} \text{ for all } u, v \in U.$$

Replacing $u$ by $[u, ru]$ in the above expression we obtain

$$F([u,r]u\circ v) = (F([u,r]u)\circ v)_{\alpha,\beta} + (d(v)\circ [u,r]u)_{\alpha,\beta} \text{ for all } u, v \in U, r \in R.$$

This gives us,

$$\begin{aligned} F(([u,r]\circ v)u + [u,r][u,v]) &= ((F[u,r]\alpha(u) + \beta[u,r]d(u))\circ v)_{\alpha,\beta} \\ &+((d(v)\circ [u,r])u + [u,r][d(v),u])_{\alpha,\beta} \text{ for all } u, v \in U, r \in R. \end{aligned}$$

Thus,

$$\begin{aligned} &F([u,r]\circ v)\alpha(u) + \beta([u,r]\circ v)d(u) + F[u,r]\alpha[u,v] + \beta[u,r]d[u,v] = \\ &(F[u,r]\circ v)_{\alpha,\beta}\alpha(u) + F[u,r]\alpha([u,v]) + (\beta[u,r]\circ v)_{\alpha,\beta}d(u) + \\ &\beta[u,r][d(u),\alpha(v)] + (d(v)\circ [u,r])_{\alpha,\beta}\alpha(u) + \beta[u,r][d(v),u]_{\alpha,\beta} \end{aligned}$$

for all $u, v \in U, r \in R$. By our hypothesis we find that

$$\begin{aligned} \beta[u,r]\beta(v)d(u) + \beta[u,r]d[u,v] &= \beta[u,r]\alpha(v)d(u) + \beta[u,r][d(u),\alpha(v)] \\ &+\beta[u,r][d(v),u]_{\alpha,\beta} \text{ for all } u, v \in U, r \in R. \end{aligned}$$

This gives us $\beta([u,r])[d(v),u]_{\alpha,\beta} = 0$ for all $u, v \in U, r \in R$. Replacing $r$ by $rs$ for some $s$ in $R$ we get $\beta([u,R])R[d(v),u]_{\alpha,\beta} = 0$ for all $u, v \in U$. Now repeating the same process as used after expression (3.20) in the proof of Theorem 3.11,we get the required result. □

**Theorem 3.15.** *Let $R$ be a $*-$prime ring with characteristic not two, $U$ a nonzero $*-$Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$ and $d$ a nonzero $(\alpha,\beta)-$derivation of $R$ which commutes with $*$ and $\beta * = * \beta$. If $d$ acts as a homomorphism on $U$, then $d = 0$ or $U \subseteq Z$.*

*Proof.* Assume that $d$ acts as a homomorphism on $U$, then we have

$$d(u)d(v) = d(u)\alpha(v) + \beta(u)d(v), \text{ for all } u, v \in U. \tag{3.21}$$

Replacing $u$ by $2uw$ in (3.21) and using $R$ is $2-$torsion free, we get

$$d(uw)d(v) = d(u)d(w)\alpha(v) + \beta(u)\beta(w)d(v). \tag{3.22}$$

On the other hand, we have

$$d(uw)d(v) = d(u)d(w)d(v) = d(u)d(wv) = d(u)d(w)\alpha(v) + d(u)\beta(w)d(v). \tag{3.23}$$

Combining (3.22) and (3.23), we arrive at

$$(\beta(u) - d(u))\beta(w)d(v) = 0,$$

and so,

$$(u - \beta^{-1}(d(u)))U\beta^{-1}(d(v)) = 0, \text{ for all } u, v \in U.$$

Using $*d = d*, \beta * = * \beta$, we have

$$*(u - \beta^{-1}(d(u))U\beta^{-1}(d(v)) = 0, \text{ for all } u, v \in U.$$

By Lemma 2.2, we get $u - \beta^{-1}(d(u)) = 0$, for all $u \in U$ or $d(v) = 0$, for all $v \in U$. If $d(v) = 0$, for all $v \in U$, then we get $U \subseteq Z$ by Lemma 2.4.
Now, we assume $u - \beta^{-1}(d(u)) = 0$, and so $\beta(u) = d(u)$, for all $u \in U$. Hence

$$d(uv) = d(u)d(v) = d(u)\alpha(v) + \beta(u)d(v) = d(u)\alpha(v) + d(u)d(v),$$

and so

$$d(U)\alpha(v) = 0, \text{ for all } v \in U.$$

By the application of Lemma 2.5 yields that $U \subseteq Z$. □

**Theorem 3.16.** *Let $R$ be a $*-$prime ring with characteristic not two, $U$ a nonzero $*-$Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$ and $d$ a nonzero $(\alpha, \beta)-$derivation of $R$ which commutes with $*$ and $\beta* = *\beta$. If $d$ acts as an anti-homomorphism on $U$, then $d = 0$ or $U \subseteq Z$.*

*Proof.* Assume that $d$ acts as an anti-homomorphism on $U$, then we have

$$d(v)d(u) = d(u)\alpha(v) + \beta(u)d(v), \text{ for all } u, v \in U. \tag{3.24}$$

Replacing $u$ by $2uv$ in (3.24) and using $R$ is $2-$torsion free, we get

$$d(v)d(uv) = d(v)d(u)\alpha(v) + \beta(u)\beta(v)d(v). \tag{3.25}$$

On the other hand, we have

$$d(v)d(uv) = d(v)d(u)\alpha(v) + d(v)\beta(u)d(v). \tag{3.26}$$

Combining (3.25) and (3.26), we arrive at

$$d(v)\beta(u)d(v) = \beta(u)\beta(v)d(v), \text{ for all } u, v \in U. \tag{3.27}$$

Substituting $2uw$ for $u$ in (3.27) and using (3.27), we arrive at

$$[d(v), \beta(u)]\beta(w)d(v) = 0.$$

Hence we get

$$[\beta^{-1}(d(v)), u]U\beta^{-1}(d(v)) = 0, \text{ for all } u, v \in U.$$

Using $*d = d*, \beta* = *\beta$, we have

$$*([\beta^{-1}(d(v)), u])U\beta^{-1}(d(v)) = 0, \text{ for all } v \in U \cap S_{a_*}(R), u \in U.$$

By Lemma 2.2, we get either $[\beta^{-1}(d(v)), u] = 0$ for all $u \in U$ or $d(v) = 0$ for each $v \in U \cap S_{a_*}(R)$. But $d(v) = 0$ also implies that $[\beta^{-1}(d(v)), u] = 0$ for all $u \in U, v \in U \cap S_{a_*}(R)$. Let $v \in U$, as $v + *(v), v - *(v) \in U \cap S_{a_*}(R)$ and $[\beta^{-1}(d(v \pm *(v))), u] = 0$, for all $u \in U$, and so $[\beta^{-1}(d(*(v))), u] = 0$. Hence we obtain that $[\beta^{-1}(d(*(v))), u] = 0$, for all $u, v \in U$. By Lemma 2.3, we get $\beta^{-1}(d(*(v))) \in Z$, for all $v \in U$ or $U \subseteq Z$. Now, we assume $\beta^{-1}(d(*(v))) \in Z$, for all $v \in U$. Since $*d = d*$ and $\beta$ is an automorphism of $R$, we have $d(U) \subset Z$. Hence $d$ acts as a homomorphism on $U$, and so $U \subseteq Z$ by Theorem 3.8. □

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