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NEW RESULTS RELATED TO THE CONVEXITY OF INTEGRAL OPERATOR

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Abstract. Let $\mathbb{U}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{H}[\mathbb{U}]$ be the class of holomorphic functions in $\mathbb{U}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} := \{1, 2, 3, ...\}$, let

$$\mathcal{H}[a, n] = \left\{ f \in \mathcal{H}[\mathbb{U}] : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...(z \in \mathbb{U}) \right\},$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{H}[\mathbb{U}] : f(z) = z + a_{n+1} z^{n+1} + ...(z \in \mathbb{U}) \right\}.$$

In the present paper, the author determines the sufficient condition for the function $f \in \mathcal{A}_n$ defined on the open unit disk $\mathbb{U}$ such that image of f under the integral operator

$$\mathcal{I}^n_{\mu,\beta}(f)(z) = \left(\frac{\beta + \frac{\mu}{n}}{z^{\frac{\mu}{n}}} \int_0^z t^{\frac{\mu}{n}-1} f^{\beta}(t) dt \right)^{\frac{1}{\beta}}$$

is convex univalent function. We also determine the sufficient condition for the function class $\mathcal{H}(1, 1)$. Our result extends the corresponding previously known results.

1 Introduction

Let $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{H}[\mathbb{U}]$ denote the class of holomorphic functions in $\mathbb{U}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} := \{1, 2, 3, ...\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[\mathbb{U}] : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...(z \in \mathbb{U})\},$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[\mathbb{U}] : f(z) = z + a_{n+1} z^{n+1} + ...(z \in \mathbb{U})\}, \tag{1.1}$$

with $\mathcal{A}_1 = \mathcal{A}$.

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be starlike or convex if f maps $\mathbb{U}$ conformally onto the domains, respectively, starlike with respect to origin and convex. The analytic characterization of these classes are

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \left(\frac{zf'(z)}{f(z} \right) > \alpha \quad (0 \le \alpha < 1; z \in \mathbb{U}) \right\}, \tag{1.2}$$

and

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (0 \le \alpha < 1; z \in \mathbb{U}) \right\}. \tag{1.3}$$

Note that the relation $f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha)$ and $f(z) \in \mathcal{S}^*(\alpha) \iff \int_0^z \frac{f(t)}{t} dt \in \mathcal{K}(\alpha)$ are well-known from Alexander theorem (see [2]). In particular, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are the starlike and convex classes respectively.

Finding sufficient conditions for univalence, starlikeness, convexity of integral, derivative and other operators is an important topic of research in Geometric Function Theory. In recent years, several authors have investigated sufficient conditions for the univalence, starlikeness and convexity of various linear and non-linear integral operators. Analogous to the integral operator defined by Oros [6] (also see [7]) on the normalized analytic functions, we defined the following integral operator on the space of holomorphic functions for the class $\mathcal{A}_n$.

Definition 1.1. Let $\mu,\ \beta$ be real numbers such that $\beta > 0$ and $\frac{\mu}{n\beta} \geq 1$. Let $f(z) \in \mathcal{A}_n$ be such that $\frac{f(z)}{z} \neq 0$ in $\mathbb{U}$. Define the integral operator $\mathcal{I}^n_{\mu,\beta} : \mathcal{A}_n \longrightarrow \mathcal{A}_n$ by

$$\mathcal{I}^n_{\mu,\beta}(f)(z) = \left(\frac{\beta + \frac{\mu}{n}}{z^{\frac{\mu}{n}}} \int_0^z t^{\frac{\mu}{n}-1} f^{\beta}(t) dt \right)^{\frac{1}{\beta}} . \tag{1.4}$$

Here the powers are taken as the principal value. The totality of the function $\mathcal{I}^n_{\mu,\beta}(f)(z)$ is single-valued analytic and belong to the class $\mathcal{A}_n$.

Note that for $n = \beta = 1$, the integral operator $\mathcal{I}^1_{\mu,\ 1}(f)(z) = L_\mu(f)(z)$, the Bernardi operator (see [1]).

In order to prove our main results, we need the following lemma.

Lemma 1.2. *(see [3, 4, 5]) Let $\psi : \mathbb{C}^2 \times \mathbb{U} \longrightarrow \mathbb{C}$ satisfying the condition*

$$\Re\psi(is, t;\ z) \leq 0, \quad (z \in \mathbb{U})$$

for $s, t \in \mathbb{R},\ t \leq \frac{-n}{2}(1 + s^2)$.

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} ...$ satisfies

$$\Re\{p(z), zp'(z); z\} > 0$$

then

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

More general forms of this lemma can be found in [5].

2 Main results

In the following theorem we determine the sufficient condition such that, for a function $f \in \mathcal{A}_n$, the image under the integral operator defined in (1.4) is convex.

Theorem 2.1. *Let $f \in \mathcal{A}_n,\ \frac{\mu}{n\beta} \geq 1$ and $n \in \mathbb{N}$. If*

$$\Re\left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) > -\frac{\beta}{2\mu} \quad (z \in \mathbb{U}), \tag{2.1}$$

where

$$\phi(z) = \frac{f^{\beta}(z)}{\left(\mathcal{I}^n_{\mu,\beta}(f)(z)\right)^{\beta-1}},$$

then the function $\mathcal{I}^n_{\mu,\beta}(f)(z)$ given by (1.4) is convex.

Proof. Let the function $f \in \mathcal{A}_n$ be given by (1.1). Then from (1.4), we have

$$\begin{aligned}
\mathcal{I}^n_{\mu,\ \beta}(f)(z) &= \left[\frac{\beta + \frac{\mu}{n}}{z^{\frac{\mu}{n}}} \int_0^z t^{\frac{\mu}{n}-1} \left(t + a_{n+1}t^{n+1} + a_{n+2}t^{n+2} + ...\right)^{\beta} dt \right]^{\frac{1}{\beta}} \\
&= \left[\frac{\beta + \frac{\mu}{n}}{z^{\frac{\mu}{n}}} \int_0^z \left(t^{\beta+\frac{\mu}{n}-1} + \beta a_{n+1} t^{n+\beta+\frac{\mu}{n}-1} + \beta a_{n+2} t^{n+\beta+\frac{\mu}{n}} + ...\right) dt \right]^{\frac{1}{\beta}} \\
&= z + \frac{\beta + \frac{\mu}{n}}{n + \beta + \frac{\mu}{n}} a_{n+1} z^{n+1} + ... \\
&= z + b_{n+1} z^{n+1} + ... \qquad (2.2)
\end{aligned}$$

Thus $\mathcal{I}^n_{\mu,\beta}(f) \in \mathcal{A}_n$. From (1.4), we have

$$z^{\frac{\mu}{n}} \left(\mathcal{I}^n_{\mu,\beta}(f)(z)\right)^{\beta} = \left(\beta + \frac{\mu}{n}\right) \int_0^z t^{\frac{\mu}{n}-1} f^{\beta}(t)dt. \tag{2.3}$$

By differentiating both sides of (2.3) with respect to z and simplifying, we have

$$\frac{\mu}{n} \left(\mathcal{I}^n_{\mu,\beta}(f)(z)\right) + \beta z \left(\mathcal{I}^n_{\mu,\beta}(f)(z)\right)' = \left(\beta + \frac{\mu}{n}\right) \phi(z) \quad (z \in \mathbb{U}), \tag{2.4}$$

where

$$\phi(z) = \frac{f^{\beta}(z)}{(\mathcal{I}^n_{\mu,\beta}(f)(z))^{\beta-1}}. \tag{2.5}$$

Now differentiating both sides of (2.4) with respect to z and using (2.5) in the resulting equation, we obtain

$$\frac{\mu}{n} \left(\mathcal{I}^n_{\mu,\beta}(f)(z)\right)' + \beta \left(\mathcal{I}^n_{\mu,\beta}(f)(z)\right)' \left(1 + \frac{z\left(\mathcal{I}^n_{\mu,\beta}(f)(z)\right)''}{(\mathcal{I}^n_{\mu,\beta}(f)(z))'}\right) = \left(\beta + \frac{\mu}{n}\right) \phi'(z) \tag{2.6}$$

Let

$$p(z) = 1 + \frac{z(\mathcal{I}^n_{\mu,\beta}(f)(z))''}{(\mathcal{I}^n_{\mu,\beta}(f)(z))'} \quad (z \in \mathbb{U}). \tag{2.7}$$

Now, clearly $p(0) = 1,\ p(z) = 1 + p_n z^n + \ldots$.

From (2.6), we obtain

$$(\mathcal{I}^n_{\mu,\beta}(f)(z))' \left(p(z) + \frac{\mu}{n\beta}\right) = \left(1 + \frac{\mu}{n\beta}\right) \phi'(z). \tag{2.8}$$

Since $(\mathcal{I}^n_{\mu,\beta}(f)(z))' \neq 0,\ p(z) + \frac{\mu}{n\beta} \neq 0,\ \phi'(z) \neq 0$, so differentiating logarithmically on both sides of (2.8) and simplifying, we get

$$1 + \frac{z(\mathcal{I}^n_{\mu,\beta}(f)(z))''}{(\mathcal{I}^n_{\mu,\beta}(f)(z))'} + \frac{zp'(z)}{p(z) + \frac{\mu}{n\beta}} = 1 + \frac{z\phi''(z)}{\phi'(z)} \quad (z \in \mathbb{U}). \tag{2.9}$$

Using (2.7) in (2.9), we have

$$p(z) + \frac{zp'(z)}{p(z) + \frac{\mu}{n\beta}} = 1 + \frac{z\phi''(z)}{\phi'(z)} \quad (z \in \mathbb{U}). \tag{2.10}$$

Making use of (2.1) in (2.10), we obtain

$$\Re\left(p(z) + \frac{zp'(z)}{p(z) + \frac{\mu}{n\beta}}\right) > -\frac{\beta}{2\mu} \quad (z \in \mathbb{U}),$$

which is equivalent to

$$\Re\left(p(z) + \frac{zp'(z)}{p(z) + \frac{\mu}{n\beta}} + \frac{\beta}{2\mu}\right) > 0 \quad (z \in \mathbb{U}). \tag{2.11}$$

Let $\psi : \mathbb{C}^2 \times \mathbb{U} \longrightarrow \mathbb{C}$ given by

$$\psi(p(z), zp'(z); z) = p(z) + \frac{zp'(z)}{p(z) + \frac{\mu}{n\beta}} + \frac{\beta}{2\mu}. \tag{2.12}$$

Using (2.11) in (2.12), we have

$$\Re\psi(p(z), zp'(z); z) > 0 \quad (z \in \mathbb{U}).$$

In order to prove our result, we use Lemma 1.2. For that we calculate

$$\begin{aligned}
\Re\psi(is,t;z) =&\Re\left[is+\frac{t}{is+\frac{\mu}{n\beta}}+\frac{\beta}{2\mu}\right]\\
=&\Re\left[is+\frac{\beta}{2\mu}+\frac{t(\frac{\mu}{n\beta}-is)}{(\frac{\mu}{n\beta}+is)(\frac{\mu}{n\beta}-is)}\right]\\
\le&\frac{\beta}{2\mu}-\frac{\mu(1+s^2)}{2\beta\left(\frac{\mu^2}{n^2\beta^2}+s^2\right)}\\
=&-\frac{\mu^2(1-\frac{1}{n^2})+s^2(\mu^2-\beta^2)}{2\mu\beta\left(\frac{\mu^2}{n^2\beta^2}+s^2\right)}\\
\le&0 \qquad (n\in\mathbb{N};\ \frac{\mu}{\beta}\ge\frac{\mu}{n\beta}\ge 1).
\end{aligned}$$

Hence, by application of Lemma 1.2 we get

$$\Re\{p(z)\}>0 \quad (z\in\mathbb{U}),$$

which implies that $\mathcal{I}^n_{\mu,\beta}(f)(z)$ is convex. This complete the proof of Theorem 2.1. □

Remark 2.2. Taking $\beta=1$ in Theorem 2.1, we get the result of Oros (see [7], Theorem 1).

Remark 2.3. Taking $n=1,\ \beta=1$ in Theorem 2.1, we get the result due to Oros (see [6], Theorem 2.1).

Next theorem gives sufficient condition for a function $f\in\mathcal{H}(1,1)$ such that the image of f under the integral operator defined in (2.13) below is convex.

Theorem 2.4. *Let $f\in\mathcal{H}(1,1),\ \frac{\mu}{\beta}\ge 1$ and*

$$\mathcal{I}_{\mu,\beta}(f)(z)=\left(\frac{\mu}{z^\mu}\int_0^z t^{\mu-1}f^\beta(t)dt\right)^{\frac{1}{\beta}} \qquad (z\in\mathbb{U}). \tag{2.13}$$

If

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right)>-\frac{\beta}{2\mu} \qquad (z\in\mathbb{U}), \tag{2.14}$$

then the function defined by (2.13) is convex.

Proof. Let $f\in\mathcal{H}(1,1)$ so that f is of the form

$$f(z)=1+a_1z+a_2z^2+... \quad (z\in\mathbb{U}).$$

From (2.13), we have

$$\begin{aligned}
\mathcal{I}_{\mu,\beta}(f)(z)=&\left(\frac{\mu}{z^\mu}\int_0^z t^{\mu-1}(1+a_1t+a_2t^2+....)^\beta dt\right)^{\frac{1}{\beta}}\\
=&\left[\frac{\mu}{z^\mu}\int_0^z t^{\mu-1}\left(1+a_1\beta t+\left(\beta a_2+\frac{\beta(\beta-1)}{2!}a_1^2\right)t^2+...\right)dt\right]^{\frac{1}{\beta}}\\
=&\left[1+\frac{a_1\beta\mu}{\mu+1}z+\left(\beta a_2+\frac{\beta(\beta-1)}{2}a_1^2\right)\frac{\mu}{\mu+2}z^2+...\right]^{\frac{1}{\beta}}\\
=&1+\frac{a_1\mu}{\mu+1}z+....\\
=&1+b_1z+b_2z^2+...
\end{aligned} \tag{2.15}$$

Thus, $\mathcal{I}_{\mu,\beta}(f) \in \mathcal{H}(1,1)$. From (2.13), we have

$$z^{\mu}\left(\mathcal{I}_{\mu,\beta}(f)(z)\right)^{\beta} = \mu \int_0^z t^{\mu-1} f^{\beta}(t)dt. \tag{2.16}$$

By differentiating both sides of (2.16) with respect to z, we obtain

$$\mu\left(\mathcal{I}_{\mu,\beta}(f)(z)\right) + z\beta\left(\mathcal{I}_{\mu,\beta}(f)(z)\right)' = \mu\chi(z), \tag{2.17}$$

where

$$\chi(z) = \frac{f^{\beta}(z)}{\left(\mathcal{I}_{\mu,\beta}(f)(z)\right)^{\beta-1}}. \tag{2.18}$$

Differentiating both sides of (2.17) with respect to z and making use of (2.18) in the resulting equation, we get

$$\left(\mathcal{I}_{\mu,\beta}(f)(z)\right)'\left(q(z) + \frac{\mu}{\beta}\right) = \frac{\mu}{\beta}\chi'(z), \tag{2.19}$$

where

$$q(z) = 1 + \frac{z\left(\mathcal{I}_{\mu,\beta}(f)(z)\right)''}{\left(\mathcal{I}_{\mu,\beta}(f)(z)\right)'} \tag{2.20}$$

Clearly, $\left(\mathcal{I}_{\mu,\beta}(f)(z)\right)' \neq 0,\ q(z) + \frac{\mu}{\beta} \neq 0,\ \chi'(z) \neq 0.$ By logarithmic differentiation of (2.19) with the help of (2.20) gives

$$q(z) + \frac{zq'(z)}{q(z)+\frac{\mu}{\beta}} = 1 + \frac{z\chi''(z)}{\chi'(z)} \tag{2.21}$$

Using (2.14) in (2.21) yields

$$\Re\left[q(z) + \frac{zq'(z)}{q(z)+\frac{\mu}{\beta}}\right] > -\frac{\beta}{2\mu},$$

which is equivalent to

$$\Re\left[q(z) + \frac{zq'(z)}{q(z)+\frac{\mu}{\beta}} + \frac{\beta}{2\mu}\right] > 0. \tag{2.22}$$

Let $\psi : \mathbb{C}^2 \times \mathbb{U} \longrightarrow \mathbb{C}$ by

$$\psi(q(z), zq'(z); z) = q(z) + \frac{zq'(z)}{q(z)+\frac{\mu}{\beta}} + \frac{\beta}{2\mu}. \tag{2.23}$$

From (2.22) and (2.23), we have

$$\Re\psi(q(z), zq'(z); z) > 0 \quad (z \in \mathbb{U}). \tag{2.24}$$

Now

$$\begin{aligned}
\Re\psi(is, t; z) =& \Re\left(is + \frac{t}{is+\frac{\mu}{\beta}} + \frac{\beta}{2\mu}\right)\\
=& \Re\left(is + \frac{\beta}{2\mu} + \frac{t(\frac{\mu}{\beta} - is)}{\frac{\mu^2}{\beta^2} + s^2}\right)\\
\leq& \frac{\beta}{2\mu} - \frac{\mu(1+s^2)}{2\beta(\frac{\mu^2}{\beta^2} + s^2)}\\
=& \frac{(1-\frac{\mu^2}{\beta^2})s^2}{2\frac{\mu}{\beta}(\frac{\mu^2}{\beta^2} + s^2)}\\
\leq& 0 \qquad \left(\frac{\mu}{\beta} \geq 1\right).
\end{aligned}$$

Therefore, the result follows by application of Lemma 1.2. This complete the proof of Theorem 2.4. □

Remark 2.5. Letting $\beta = 1$ in Theorem 2.4 we obtain the result due to Oros (see [7], Theorem 2).

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