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Sum annihilating ideal graph of a commutative ring

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Abstract Let R be a commutative ring with identity which is not an integral domain. An ideal I of a ring R is called an annihilating ideal if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$. In this paper, we consider a simple undirected graph associated with R denoted by $\Omega(R)$ whose vertex set equals the set of all nonzero annihilating ideals of R and two distinct vertices I, J are adjacent if and only if $I + J$ is an annihilating ideal of R.

1 Introduction

The rings considered in this paper are commutative with identity which are not integral domains. The idea of associating a graph to a ring was initiated by Beck in [16] and subsequently several researchers have done interesting and enormous work on zero-divisor graphs of rings. To mention a few, see [9, 10, 14, 22, 24]. For an excellent and inspiring survey of the research work done in the area of zero-divisor graphs in commutative rings, the reader is referred to [5].

Let R be a ring which is not an integral domain. Let $Z(R)$ denote the set of all zero-divisors of R. Recall from [9] that the zero-divisor graph $\Gamma(R)$ of a ring R is a simple undirected graph with vertex set $Z(R)^*$ and two distinct vertices x, y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. Recall from [17] that an ideal I of R is said to be an *annihilating-ideal* if Ir = (0) for some $r \in R^*$. As in [17], we denote by $A(R)$, the set of all annihilating-ideals of R and by $A(R)^*$, the set of all nonzero annihilating-ideals of R. Recall from [17] that the annihilating-ideal graph of a ring R, denoted by $AG(R)$ is a simple undirected graph whose vertex set is $A(R)^*$ and two distinct $I, J \in A(R)^*$ are adjacent in this graph if and only if $IJ = (0)$. The concept of annihilating-ideal graph of a ring was introduced by Behboodi and Rakeei in [17]. Many interesting and inspiring theorems proved in [17, 18] on annihilating-ideal graph of a commutative ring reveal that this graph is also worthy to study just like the zero-divisor graph of a ring. The interplay between the ring theoretic properties of a ring R and the graph theoretic properties of its annihilating ideal graph has also been investigated in [1, 20].

In [6], Anderson and Badawi introduced the concept of the total graph of a commutative ring R, denoted by $T(\Gamma(R))$, as an undirected graph with all the elements of R as vertices and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$ and they have established several illuminating theorems on this graph in [6, 7]. Moreover, this graph has been generalized and investigated in [22]. Recently S. Visweswaran. and H. D. Patel[28] have introduced and investigated the graph $\Omega(R)$ of a commutative ring R. For a non-domain commutative ring R, let $\mathbb{A}^*(R)$ be the set of non-zero ideals with non-zero annihilators. The vertex set of this graph is $A(R)^*$ the set of all nonzero annihilating ideals of R and for distinct $I, J \in A(R)^*$, the vertices I and J are joined by an edge in this graph if and only if $I+J \in A(R)^*$. For convenience we call this graph as *sum annihilating ideal graph* and denote it by $\Omega(R)$. The main aim of this paper is to study some of the properties of $\Omega(R)$. We investigate the interplay between the graph-theoretic properties of $\Omega(R)$ and the ring-theoretic properties of R. For basic definitions on rings, one may refer [21].

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. If $G = K_{1,n}$ where $n \geq 1$, then G is a star graph. A *split graph* is a simple graph in which the vertices can be partitioned into a clique and an independent set. A graph G is said to be *unicyclic* if it contains a unique cycle. For basic definitions on graphs, one may refer[19].

Throughout this paper, we assume that R is a finite commutative ring with identity but not an integral domain, $Z(R)$ its set of zero-divisors, $\mathfrak{N}(R)$ its set of nilpotent elements, $R^{\times}$ its group of units, $\mathbb{F}_q$ denote the field with q elements, and $R^* = R - \{0\}$.

2 Basic Properties of $\Omega(R)$

In this section, we study some fundamental properties of $\Omega(R)$. Especially we identify when the annihilator graph is isomorphic to some well-known graphs. By the definition of $\Omega(R)$, if R is an integral domain, then $\Omega(R)$ is an empty graph.

Remark 2.1. Let R be a finite commutative ring but not a field. Then every non-zero proper ideal is an annihilating ideal of R.

Theorem 2.1. Let R be a finite commutative ring. Then R is a local ring if and only if $\Omega(R)$ is complete.

Proof. Suppose that R is a local ring. Then R has a unique maximal ideal, say, $\mathfrak{m}$. Note that any non-zero proper ideal of R is an annihilating ideal of R. For any two non-zero proper ideals I, J in R, $I + J \subseteq \mathfrak{m}$ and so $I + J$ is an annihilating ideal in R. By definition of $\Omega(R)$, I and J are adjacent in $\Omega(R)$ for all non-zero proper ideals I, J in R and hence $\Omega(R)$ is complete.

Conversely, assume that $\Omega(R)$ is complete. Suppose that R is not a local ring. Then R has at least two maximal ideals, say, M_1 and M_2. Note that $M_1 + M_2 = R$. By definition of $\Omega(R)$, $M_1 + M_2$ is not an annihilating ideal of R and so M_1 and M_2 are nonadjacent in $\Omega(R)$, a contradiction. Hence R is a local ring. □

Theorem 2.2. Let R be a finite commutative non-local ring. Then $\Omega(R)$ is totally disconnected if and only if $R \cong F_1 \times F_2$ where F_1 and F_2 are fields.

Proof. Suppose that $\Omega(R)$ is totally disconnected. Then $\Omega(R)$ has no edge. Since R is a finite non-local ring, $R \cong R_1 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. If $n \geq 3$, then $(0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$ and $(0) \times R_2 \times R_3 \times (0) \times \cdots \times (0)$ are adjacent in $\Omega(R)$, a contradiction. Hence $n = 2$.

Suppose $\mathfrak{m}_1 \neq (0)$. Then $(0) \times R_2$ and $\mathfrak{m}_1 \times (0)$ are adjacent in $\Omega(R)$, a contradiction. Hence R_1 and R_2 are fields.

Conversely, if $R \cong F_1 \times F_2$, where F_1 and F_2 are fields, then $\Omega(R) \cong \overline{K}_2$ and hence $\Omega(R)$ is totally disconnected. □

Remark 2.2. Let $(R, \mathfrak{m})$ be a finite local ring. Then $\Omega(R)$ is totally disconnected if and only if $\mathfrak{m}$ is the only non-zero proper ideal of R. Hence in this case diam$(\Omega(R) = \infty$.

Corollary 2.3. Let R be a finite commutative non-local ring. Then diam$(\Omega(R)) = \infty$ if and only if $R \cong F_1 \times F_2$ where F_1 and F_2 are fields.

Proof. If $R \cong F_1 \times F_2$, where F_1 and F_2 are fields, then $\Omega(R) \cong \overline{K}_2$ and hence diam$(\Omega(R)) = \infty$.

Suppose that diam$(\Omega(R)) = \infty$. Since R is a finite non-local ring, $R \cong R_1 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. If $n \geq 3$, then $\Omega(R)$ is connected, a contradiction. Hence $n = 2$ and $R = R_1 \times R_2$

If $\mathfrak{m}_i \neq (0)$ for some i, then $\Omega(R)$ is connected, a contradiction. Hence R_1 and R_2 are fields. □

Theorem 2.4. Let R be a finite commutative ring and $|\Omega(R)| \geq 3$. Then $\Omega(R)$ is unicyclic if and only if
(i) R is a local ring which contains three non-zero proper ideals
(ii) $R = R_1 \times R_2$, where $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1$ as only non-zero proper ideal in R_1 and R_2 is a field.

Proof. Suppose that $\Omega(R)$ is unicyclic. Since R is finite, $R = R_1 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a local ring. If $n \geq 3$, then $(0) \times R_2 \times (0) \times (0) \times \cdots \times (0) - R_1 \times (0) \times (0) \times (0) \times \cdots \times (0) - (0) \times (0) \times R_3 \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \times (0) \times \cdots \times (0)$ and $R_1 \times R_2 \times (0) \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \times (0) \times \cdots \times (0) - R_1 \times (0) \times (0) \times (0) \times \cdots \times (0) - R_1 \times R_2 \times (0) \times (0) \times \cdots \times (0)$ are two distinct cycles in $\Omega(R)$, a contradiction. Hence $n \leq 2$.

If $n = 1$, then by Theorem 2.1, $\Omega(R)$ is complete. Since $\Omega(R)$ is unicyclic and $|\Omega(R)| \geq 3$, R contains three non-zero proper ideals.

Suppose that $n = 2$. Then $R = R_1 \times R_2$. If $\mathfrak{m}_i \neq (0)$ for $i = 1, 2$, then $\mathfrak{m}_1 \times (0) - \mathfrak{m}_1 \times \mathfrak{m}_2 - (0) \times \mathfrak{m}_2 - \mathfrak{m}_1 \times (0)$ and $\mathfrak{m}_1 \times (0) - (0) \times \mathfrak{m}_2 - \mathfrak{m}_1 \times R_2 - \mathfrak{m}_1 \times (0)$ are two distinct cycles in $\Omega(R)$, a contradiction. Hence $\mathfrak{m}_i = (0)$ for some i.

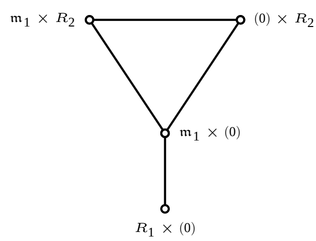

Fig. 2.1: $\Omega(R_1 \times R_2)$

Without loss of generality, we assume that $\mathfrak{m}_2 = (0)$. Then R_2 is a field. Since $|\Omega(R)| \geq 3$, by Corollary 2.3, $\Omega(R)$ is connected and so R_1 is not a field. Suppose that I is any non-zero proper ideal in R_1 with $I \neq \mathfrak{m}_1$. Then $I \times (0) - \mathfrak{m}_1 \times (0) - (0) \times R_2 - I \times (0)$ and $I \times (0) - (0) \times R_2 - \mathfrak{m}_1 \times R_2 - I \times (0)$ are two distinct cycles in $\Omega(R)$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-zero proper ideal in R_1.

Conversely, suppose that (i) and (ii) holds. Then $\Omega(R) \cong K_3$ or $\Omega(R)$ is isomorphic to Fig. 2.1. □

Theorem 2.5. Let R be a finite commutative ring. If $\Omega(R)$ is connected, then $\Omega(R)$ is a tree if and only if R is a local ring which contains two non-zero proper ideals.

Proof. Suppose that R is a local ring which contains two non-zero proper ideals. Then by Theorem 2.1, $\Omega(R) \cong K_2$.

Conversely, assume that $\Omega(R)$ is a tree. Suppose R is a non-local ring. Then $R = R_1 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. If $n \geq 3$ then R contains a cycle, a contradiction. Hence $n = 2$. Since $\Omega(R)$ is connected, R_1 and R_2 are not fields and so R_i is not a field for some i. Since Fig. 2.1 is a subgraph of $\Omega(R_1 \times R_2)$, $\Omega(R)$ contains a cycle, a contradiction. Hence R is a local ring and by Theorem 2.1, $\Omega(R)$ is complete. Thus R contains two non-zero proper ideals. □

3 Hamiltonian nature of $\Omega(R)$

In this section, we discuss about the Hamiltonian property of $\Omega(R)$. In view of Theorem 2.1, $\Omega(R)$ is Hamiltonian when R is a local ring which contains at least three non-zero proper ideals.

If R is finite, then $R = R_1 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 3$. Let $Max(R) = \{M_i : M_i = R_2 \times \cdots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \cdots \times R_n, 1 \leq i \leq n\}$ be the set of all maximal ideals in R and $\mathcal{J}(R)$ be the Jacobson radical of R.

Theorem 3.1. Let R be a finite commutative ring and $|Max(R)| \geq 3$. Then $\Omega(R)$ is Hamiltonian.

Proof. Let $A_i = \{I \subseteq M_i : I$ is a non-zero proper ideal in $R\}$ for $1 \leq i \leq n$. Then $A_i \cap A_j \neq \emptyset$ for all $i \neq j$ and $V(\Omega(R)) = \bigcup\limits_{i=1}^{n} A_i$. Clearly the subgraph $\langle A_i \rangle$ induced by A_i is a complete subgraph of $\Omega(R)$ and also $\langle A_i \cap A_j \rangle$ is a complete subgraph of $\Omega(R)$. Let $I_{i(i+1)} \in A_i \cap A_{i+1}$ for $1 \leq i \leq n-1$ and $I_{n1} \in A_n \cap A_1$.

Now we start with the vertex M_1, traverse all vertices in $\big\langle A_1 - \{I_{i(i+1)}, I_{n1} : 1 \leq i \leq n-1\}\big\rangle$ through a spanning path in $\big\langle A_1 - \{I_{i(i+1)}, I_{n1} : 1 \leq i \leq n-1\}\big\rangle$, pass on to I_{12}, traverse vertices in $\big\langle A_2 - \{I_{i(i+1)}, I_{n1} : 2 \leq i \leq n-1\}\big\rangle$ through a spanning path in $\big\langle A_2 - \{I_{i(i+1)}, I_{n1} : 2 \leq i \leq n-1\}\big\rangle$, pass on to I_{23}. Continue this process through $\big\langle A_3 - \{I_{i(i+1)}, I_{n1} : 3 \leq i \leq n-1\}\big\rangle$, $\big\langle A_3 - \{I_{i(i+1)}, I_{n1} : 3 \leq i \leq n-1\}\big\rangle$, $\big\langle A_4 - \{I_{i(i+1)}, I_{n1} : 4 \leq i \leq n-1\}\big\rangle$, ..., $\langle A_n - \{I_{n1}\}\rangle$ to get a Hamiltonian path at I_{n1}. From this Hamiltonian path together with the edge joining M_1 and I_{n1} gives a required Hamiltonian cycle in $\Omega(R)$. Hence $\Omega(R)$ is Hamiltonian. □

Proof of the following is analogous .

Corollary 3.2. Let R be a finite commutative ring and $|Max(R)| = 2$. If the condition (ii) in Theorem 2.4 does not hold, then $\Omega(R)$ is Hamiltonian.

4 Genus of $\Omega(R)$

In this section, we characterize all commutative rings R for which $\Omega(R)$ is planar. Also we determine all isomorphism classes of finite commutative rings with identity whose $\Omega(R)$ has genus one.

Let S_k denote the sphere with k handles, where k is a nonnegative integer, that is, S_k is an oriented surface of genus k. The genus of a graph G, denoted $g(G)$, is the minimal integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A graph G with genus 0 is called a planar graph where as a graph G with genus 1 is called as a toroidal graph. Further note that if H is a subgraph of a graph G, then $g(H) \leq g(G)$. For details on the notion of embedding a graph in a surface, see [29]. First let us summarize certain results on the genus of a graph.

Lemma 4.1. [29] $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ if $n \geq 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.

Lemma 4.2. [29] $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ if $m, n \geq 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,4}) = 2$ if $m = 7, 8, 9, 10$.

First let us characterize finite commutative rings R for which genus of $AG(R)$ is zero.

Theorem 4.3. Let $R \cong R_1 \times \cdots \times R_n$ be a finite commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring but not a field and $n \geq 1$. Then $\Omega(R)$ is planar if and only if R is a local ring and R contains at most four non-zero proper ideals.

Proof. Assume that $\Omega(R)$ is planar. Suppose $n \geq 2$. Let $A = \{\mathfrak{m}_1 \times 0, 0 \times \mathfrak{m}_2, \mathfrak{m}_1 \times \mathfrak{m}_2, R_1 \times 0, \mathfrak{m}_1 \times R_2, R_1 \times \mathfrak{m}_2\} \subseteq V(\Omega(R))$. Then the subgraph induced by A in $\Omega(R)$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $n = 1$, R is local and by Theorem 2.1, $\Omega(R)$ is complete. Since $\Omega(R)$ is planar, R contains at most four non-zero proper ideals.

Conversely, suppose R is a local ring which contains at most four non-zero proper ideals. Then by Theorem 2.1, $\Omega(R) \cong K_n$, where $1 \leq n \leq 4$ and hence $\Omega(R)$ is planar. □

Theorem 4.4. Let $R \cong F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each F_i is a field and $n \geq 2$. Then $\Omega(R)$ is planar if and only if $n = 2$ or 3.

Proof. Suppose $\Omega(R)$ is planar. Suppose $n \geq 4$. Let $A = \{0 \times F_2 \times F_3 \times \cdots \times F_n, 0 \times 0 \times F_3 \times \cdots \times F_n, 0 \times F_2 \times 0 \times \cdots \times F_n, 0 \times F_2 \times F_3 \times \cdots \times F_n, 0 \times 0 \times 0 \times F_4 \times \cdots \times F_n\} \subseteq V(\Omega(R))$. Then the subgraph induced by A in $\Omega(R)$ contains K_5 as a subgraph, a contradiction. Hence $n \leq 3$.

Suppose $n = 2$. Then $R \cong F_1 \times F_2$ and by Theorem 2.1, $\Omega(R) = \overline{K}_2$. Suppose $n = 3$. Then $R \cong F_1 \times F_2 \times F_3$. Then $V(\Omega(R)) = \{0 \times F_2 \times F_3, F_1 \times 0 \times F_3, F_1 \times F_2 \times 0, 0 \times 0 \times F_3, 0 \times F_2 \times 0, F_1 \times 0 \times 0\}$.

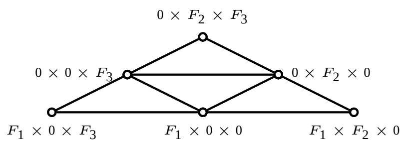

Fig 4.1: $\Omega(F_1 \times F_2 \times F_3)$

Converse follows from Fig. 4.1. □

Theorem 4.5. Let $R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity but not a field, where each $(R_i, \mathfrak{m}_i)$ is a local ring and F_j is a field. Then $\Omega(R)$ is planar if and only if $n = 1$, $m = 1$ and R_1 contains exactly one proper ideal.

Proof. Assume that $\Omega(R)$ is planar. Suppose $n \geq 2$. Then by Theorem 4.3, $\Omega(R)$ is non-planar, a contradiction. Hence $n = 1$. Suppose $m \geq 2$. Let $A = \{I \subseteq \mathfrak{m}_1 \times F_1 \times \cdots \times F_m, I \neq (0), I \text{ is an ideal }\}$. Then $|A| \geq 7$ and so the subgraph induced by A in $\Omega(R)$ contains K_7 as a subgraph, a contradiction. Hence $m = 1$ and $R = R_1 \times F_1$.

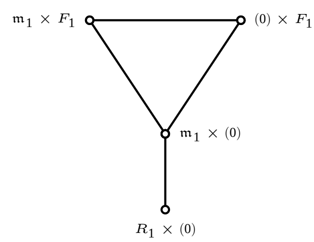

Fig. 4.2: $\Omega(R_1 \times F_1)$

Suppose R_1 contains two proper ideals. Let $I_1, \mathfrak{m}_1$ be two proper ideals with $\mathfrak{m}_1 \neq I_1$. Let $B = \{I \subseteq \mathfrak{m}_1 \times F_1, I \neq 0, I \text{ is an ideal }\}$. Then $|B| \geq 5$ and $\langle B \rangle \cong K_5$ so that $\Omega(R)$ contains K_5 as a subgraph, a contradiction. Hence R_1 contains a unique proper ideal.

Conversely, suppose $n = m = 1$ and R_1 contains unique proper ideal $\mathfrak{m}_1$. Then $V(\Omega(R)) = \{\mathfrak{m}_1 \times F_1, 0 \times F_1, \mathfrak{m}_1 \times 0, R_1 \times 0\}$ and hence $\Omega(R)$ is isomorphic to Fig 4.2. □

Theorem 4.6. Let R be a finite local ring but not a field. Then $g(\Omega(R)) = 1$ if and only if R contains at most n non-zero proper ideals, where $5 \leq n \leq 7$.

Proof. Assume that $g(\Omega(R)) = 1$. Then by Theorem 4.3, R contains at least 5 proper ideals. Since R is local, by Theorem 2.1, $\Omega(R)$ is complete and hence R contains at most n non-zero proper ideals, where $5 \leq n \leq 7$.

Conversely, suppose R contains at most n non-zero proper ideals, where $5 \leq n \leq 7$. Note that $\Omega(R)$ is complete so that $g(\Omega(R)) = 1$. □

Theorem 4.7. Let $R \cong R_1 \times \cdots \times R_n$ be a finite commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring but not a field and $n \geq 2$. Then $g(\Omega(R)) = 1$ if and only if $n = 2$ and each R_i contains exactly one non-zero proper ideal.

Proof. Assume that $g(\Omega(R)) = 1$. Suppose $n \geq 3$. Let $A = \{I \subseteq M_1 : I \neq 0, I \text{ is an ideal }\} \subseteq V(\Omega(R))$. Then $|A| \geq 17$ and so the subgraph induced by A in $\Omega(R)$ contains K_{17} as a subgraph so that $g(\Omega(R)) \geq 4$, a contradiction. Hence $n = 2$ and so $R \cong R_1 \times R_2$.

Suppose R_1 contains two proper ideals. Let $I, \mathfrak{m}_1$ be two non-zero proper ideals of R_1 with $I \neq \mathfrak{m}_1$. Let $B = \{I \subseteq M_1; I \neq 0, I \text{ is an ideal }\}$. Then $|B| \geq 8$ and so the subgraph induced by B in $\Omega(R)$ contains K_8 as a subgraph. Thus $g(\Omega(R)) \geq 2$, a contradiction. Hence each R_i contains exactly one non-zero proper ideal.

Conversely, assume that $n = 2$ and each R_i contains exactly one non-zero proper ideal. Then $|V(\Omega(R))| = 7$ and so $\Omega(R)$ is a subgraph of K_7. Since $g(K_7) = 1$, $g(\Omega(R)) = 1$. □

Theorem 4.8. Let $R \cong F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each F_i is a field and $n \geq 4$. Then $g(\Omega(R)) > 1$.

Proof. As in the proof of Theorem 4.4, $\Omega(R)$ is non-planar and so $g(\Omega(R)) \geq 1$. Suppose $n \geq 5$. Let $A = \{I \subseteq 0 \times F_2 \times \cdots \times F_n : I \neq 0, I \text{ is an ideal }\}$. Then $|A| \geq 8$ and so the subgraph induced by A in $\Omega(R)$ contains K_8 as subgraph so that $g(\Omega(R)) \geq 2$. Hence $g(\Omega(R)) > 1$.

Suppose $n = 4$. Let $B = \{F_1 \times F_2 \times F_3 \times 0, F_1 \times F_2 \times 0 \times 0, 0 \times F_2 \times F_3 \times 0, F_1 \times 0 \times 0 \times 0, F_1 \times 0 \times F_3 \times 0, 0 \times 0 \times F_3 \times F_4, 0 \times 0 \times 0 \times F_4, 0 \times F_2 \times 0 \times F_4, 0 \times F_2 \times F_3 \times F_4, \} \subseteq V(\Omega(R))$. Then the graph induced by B in $\Omega(R)$ contains H as a subgraph, where $H = 2K_4 + K_1$. Since $g(H) > 1$, $g(\Omega(R)) > 1$. □

Theorem 4.9. Let $R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity but not a field, where each $(R_i, \mathfrak{m}_i)$ is a local ring, F_j is a field and $n, m \geq 1$. Then $g(\Omega(R)) = 1$ if and only if $n = m = 1$ and R_1 contains k non-zero proper ideals, where $k = 2, 3$.

Proof. Assume that $g(\Omega(R)) = 1$. Suppose $n \geq 2$. Let $A = \{I \subseteq M_1 : I \neq 0, I \text{ is an ideal }\}$. Then the subgraph induced by A in $\Omega(R)$ contains K_{11} as a subgraph and so $g(\Omega(R)) > 1$, a contradiction. Hence $n = 1$.

Suppose $n = 1$ and $m \geq 3$. Then $n + m \geq 4$. Clearly $\Omega(F_1 \times F_2 \times F_3 \times F_4)$ is a subgraph of $\Omega(R)$. But by Theorem 4.8, $g(\Omega(F_1 \times F_2 \times F_3 \times F_4)) > 1$, $g(\Omega(R)) > 1$, a contradiction. Hence $m = 1$ or 2.

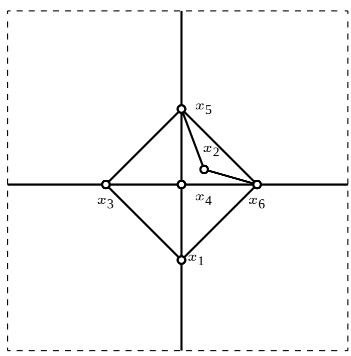

Fig. 4.3: A planar of embedding of $\Omega(R_1 \times F_1)$

Suppose $m = 2$. Then $R = R_1 \times F_1 \times F_2$. Let $B = \{I \subseteq R : I \neq 0 \text{ and } I \neq R, I \text{ is an ideal }\} \subseteq V(\Omega(R))$. Then $|B| \geq 10$, the subgraph induced by B in $\Omega(R)$ contains K_{10} as a subgraph and so $g(\Omega(R)) > 1$, a contradiction. Hence $m = 1$ and so $R = R_1 \times F_1$.

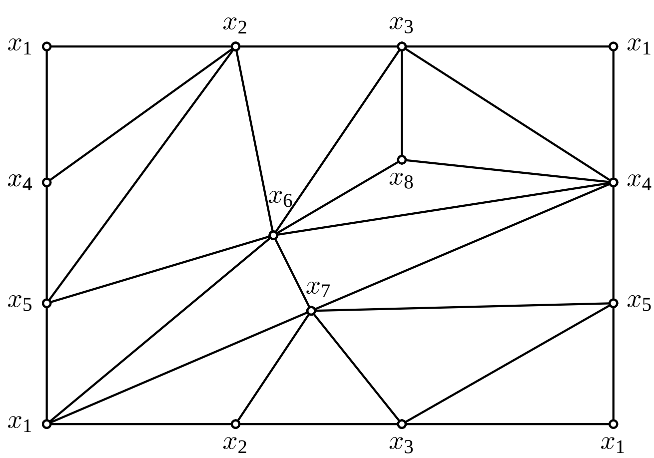

Fig. 4.4: A planar embedding of $\Omega(R_1 \times F_1)$ in S_1

Suppose R_1 contains at least 4 proper ideals. Let $\mathfrak{m}_1, I_1, I_2, I_3$ be four proper ideals in R_1 with $\mathfrak{m}_1 \neq I_1 \neq I_2 \neq I_3$. Let $C = \{I \subseteq M_1 : I \neq 0, I \text{ is an ideal }\}$. Then $|C| \geq 9$, the subgraph induced by C in $\Omega(R)$ contains K_9 as a subgraph and so $g(\Omega(R)) > 1$, a contradiction. By Theorem 4.5, R_1 contains k non-zero proper ideals, where $k = 2, 3$.

Conversely, suppose R_1 contains two non-zero proper ideals. Then $V(\Omega(R)) = \{x_1 = 0 \times F_1, x_2 = R_1 \times 0, x_3 = \mathfrak{m}_1 \times 0, x_4 = I \times 0, x_5 = \mathfrak{m}_1 \times F_1, x_6 = I \times F_1\}$, K_5 is a subgraph of $\Omega(R)$ and so $g(\Omega(R)) \geq 1$. However, we can draw $\Omega(R)$ on the surface of a torus, see Fig. 4.3. Hence $g(\Omega(R)) = 1$.

suppose R_1 contains three non-zero proper ideals. Then $V(\Omega(R)) = \{x_1 = 0 \times F_1, x_2 = \mathfrak{m}_1 \times 0, x_3 = I_1 \times 0, x_4 = I_2 \times 0, x_5 = I_2 \times F_1, x_6 = I_1 \times F_1, x_7 = \mathfrak{m}_1 \times F_1, x_8 = R_1 \times 0\}$ and by Theorem 4.5, $g(\Omega(R)) \geq 1$. However, we can draw $\Omega(R)$ on the surface of a torus, see Fig. 4.4. Hence $g(\Omega(R)) = 1$. □

5 Isomorphism Properties of $\Omega(R)$

Consider the question: If R and S are two rings with $\Omega(R) \cong \Omega(S)$, then do we have $R \cong S$? The following example shows that the above question is not valid in general.

Example 5.1. Let $R = \mathbb{Z}_{25} \times \mathbb{Z}_{13}$ and $S = \mathbb{Z}_9 \times \mathbb{Z}_{29}$. Then $\Omega(R) \cong \Omega(S)$(see. Fig. 6.6). But R and S are not isomorphic.

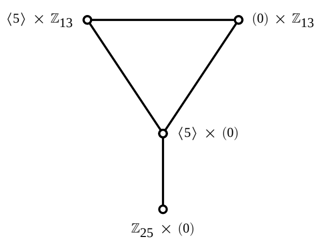

Fig. 6.6: $\Omega(\mathbb{Z}_{25} \times \mathbb{Z}_{13}) \cong \Omega(\mathbb{Z}_9 \times \mathbb{Z}_{29})$

Theorem 5.2. Let $R = \prod_{i=1}^{n} R_i \times \prod_{j=1}^{m} F_j$ and $S = \prod_{i=1}^{n} R'_i \times \prod_{j=1}^{m} F'_j$ be finite commutative rings with $n + m \geq 2$, where each $(R_i, \mathfrak{m}_i)$ and $(R'_i, \mathfrak{m}'_i)$ are local rings which are not fields each F_i and F'_j are fields. Let k_i be the number of ideals in R_i and k'_i be the number of ideals in R'_i. Then $\Omega(R) \cong \Omega(S)$ if and only if $k_i = k'_i$ for all i, $1 \leq i \leq n$.

Proof. If $R \cong S$, then the result is obvious. Assume that $R \ncong S$. Suppose $k_i = k'_i$ for all i, $1 \leq i \leq n$. Then $|V(\Omega(R))| = |V(\Omega(S))|$. Let $\mathbb{I}_j(R_j) = \{I_{1j} = (0), I_{2j} = \mathfrak{m}_j, I_{3j}, \ldots, I_{k_j j} = R_j\}$ be the set of ideals in R_j and $\mathbb{I}'_j(R'_j) = \{I'_{1j} = (0), I'_{2j} = \mathfrak{m}_j, I'_{3j}, \ldots, I'_{k_j j} = R'_j\}$ be the set of ideals in R'_j. Then the map $I_{tj} \to I'_{tj}$ is a bijection from $\mathbb{I}_j(R_j)$ onto $\mathbb{I}'_j(R'_j)$. Define $\psi : V(\Omega(R)) \longrightarrow V(\Omega(S))$ by $\psi(\prod_{i=1}^{n} I_{ti} \times \prod_{j=1}^{m} J_j) = \prod_{i=1}^{n} I'_{ti} \times \prod_{j=1}^{m} J'_j$ where

$$J'_j = \begin{cases} F'_j & \text{if } J_j = F_j \\ (0) & \text{if } J_j = (0) \end{cases}$$

Then ψ is well-defined and bijective. Let $I = \prod_{i=1}^{n} I_i \times \prod_{j=1}^{m} J_j$ and $J = \prod_{i=1}^{n} A_i \times \prod_{j=1}^{m} B_j$ be two non-zero ideals in R. Suppose I and J are adjacent in $\Omega(R)$. Then $I + J$ is an annihilating ideal of R and so $I_i + A_i \subseteq \mathfrak{m}_i$ for some i or $J_j + B_j = (0)$ for some j. From this, $I_i, A_i \subseteq \mathfrak{m}_i$ or $J_j = (0)$ and $B_j = (0)$.

Let $\psi(I) = \prod_{i=1}^{n} I'_i \times \prod_{j=1}^{m} J'_j$ and $\psi(J) = \prod_{i=1}^{n} A'_i \times \prod_{j=1}^{m} B'_j$. By definition of ψ, $I'_i + A'_i \subseteq \mathfrak{m}'_i$ for some i or $J'_j + B'_j \neq (0)$ for some j and so $\psi(I) + \psi(J) = S$. Hence $\psi(I)$ and $\psi(J)$ are adjacent in $\Omega(S)$. Similarly one can prove that ψ preserves non-adjacency also. Hence $\Omega(R) \cong \Omega(S)$.

Conversely, assume that $\Omega(R) \cong \Omega(S)$. Suppose $k_i \neq k'_i$ for some i. Then $|V(\Omega(R))| \neq |V(\Omega(S))|$, a contradiction. Hence $k_i = k'_i$ for all i. □

Corollary 5.3. Let $R_1 = \prod_{i=1}^{n} F_i$ and $R_2 = \prod_{j=1}^{n} F'_i$, where F_i and F'_j are fields and $n \geq 2$. Then $\Omega(R_1) \cong \Omega(R_2)$.

Corollary 5.4. Let $R = \prod_{i=1}^{n} R_i$ and $S = \prod_{i=1}^{n} R'_i$ be finite commutative rings with $n \geq 2$, where each $(R_i, \mathfrak{m}_i)$ and $(R_i, \mathfrak{m}'_i)$ are local rings which are not field. Let k_i be the number of ideals in R_i and k'_i be the number of ideals in R'_i. Then $\Omega(R) \cong \Omega(S)$ if and only if $k_i = k'_i$ for all i, $1 \leq i \leq n$.

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