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REMARKS ON PROJECTIONS ON PLANAR SETS

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Abstract. In a paper [2] some kinds of $f-$ projections of a planar set E has been defined for a function $f : \mathbf{R} \to \mathbf{R}$ (where $\mathbf{R}$ is set of real numbers) and word 'projection' has been used when f is linear. Some descriptive properties of projections (category and measure) of a planar set has been established. The main result in this paper is answered a question raised by Ceder and Ganguly [2].

1 Introduction

Let us take a subset A of $\mathbf{R}$, where $\mathbf{R}$ is a set of real numbers and let $D(A)$ denote the set of all numbers $|x - y|$ where $x \in A$ and $y \in A$. Then $D(A)$ is called the Distance set of the set A. The investigation of the set of the distances between the points of a measurable set was started by H. Steinhaus in 1920. H. Steinhaus [13] proved that if A is a measurable subset of real line, with positive Lebesgue measure, then Distance set of A, that is, $D(A)$ contains an interval with origin as an left hand end point; and the Difference set of A represented as $A - A = \{x - y : x \in A, y \in A\}$ contains an interval with origin as an interior point. In 1925, Ruziewicz [12] extended Steinhaus's result. An alternative proof of Ruziewicz result was given by D.K.Ganguly and M.Majumdar[4]. Also the category analogue of Ruziewicz result was established by Ganguly and Majumdar [4].

The set A is said to have the property of Baire if A can be expressed as the symmetric difference of the set of non empty open set and set of first category (cf. [10, p.19]). The category analogue of Steinhaus result was established by Piccard [11] when the sets are of second category having the property of Baire. Ceder and Ganguly [2] provided an alternative way of interpreting the Steinhaus result in terms of projection of the planar set and strengthened the Steinhaus result by applying various kind of projections of planar sets. Inspiring by the paper [2] several authors ([1], [5], [7]) were devoted to study the various properties of projections of planar sets.

Terminology:

Let $f : \mathbf{R} \to \mathbf{R}$ be a function and $d \in \mathbf{R}$. The Lebesgue measure on measurable subsets of $\mathbf{R}$ ($\mathbf{R^2}$) is denoted by λ_1 (resp. λ_2). Let $E \subset \mathbf{R^2}$. We define the $f-$projection, $f-$measure projection and $f-$category projection of E denoted by $P(f, E)$, $P_m(f, E)$ and $P_c(f, E)$ respectively as below.

- $P(f, E) = \{d : (f + d) \cap E \neq \emptyset\}$
- $P_m(f, E) = \{d : \lambda_1(dom[(f + d) \cap E]) > 0\}$
- $P_c(f, E) = \{d : dom[(f + d) \cap E]$ is second category$\}$

In general, $P_m(f, E)$ and $P_c(f, E)$ are subsets of $P(f, E)$. We use the word projection to refer to any $f-$projection when f is linear. It is easy to verify with the help of Fubini's theorem that $P_m(f, E)$ fills up all most all of $P(f, E)$ in the sense of measure where E is of full measure (i.e. its complement is a null set.)

In this article we shall prove some descriptive properties of $P(f, A \times B)$ and give an alternative answer based on Martin's axiom to a question raised by Ceder and Ganguly [2] as follows: "It is unknown whether or not a second category set A can be found whose Lebesgue mesure is not zero such that projection of this set on each line has empty interior". Some open questions are also raised in this paper.

2 Some properties of projections of planar set

Proposition 2.1. *Let A and B be measurable subsets of* $\mathbf{R}$ *and $E = A \times B$. If f is continuous and linear mapping from R into R, then*

(i) $P(f, E) = B - f(A)$

(ii) $P(f, E)' = \{c : c + f(A) \subset B'\}$ *where the symbol B' denotes the complement of B*

(iii) $P_m(f, E) = \{c : \lambda_1\{A \cap f_c^{-1}(B)\} > 0\}$ *where $f_c : \mathbf{R} \to \mathbf{R}$ is defined by $f_c(x) = f(x) + c$ for all $x \in \mathbf{R}$.*

Proofs directly follow from the definition of projections.

Corollary 2.2. *If $E = A \times B$ where A is of second category set and B a residual set in R, then $P(f, E) = \mathbf{R}$, where f is continuous and linear.*

Proof. Here f maps a second category set into a second category set and hence $f(A) + c$ is also second category. As B is a residual set hence B' is of first category. This implies that $P(f, E)' = \emptyset$ and hence $P(f, E) = \mathbf{R}$. □

Remark 2.3. The above corollary is no longer true when A and B are sets of second category which follows from the following result established by Ceder and Ganguly [2].

Theorem 2.4. *([2]) There exists a second category set A such that projection of $A \times A$ onto any line with rational slope and rational intercept does not contain any interval.*

Theorem 2.5. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function. If A and B are two compact subsets of $\boldsymbol{R}$, then $P(f, A \times B)$ is compact in* $\mathbf{R}$.

Proof. Since A and B are compact, $P(f, A \times B)$ is bounded. Let l be the limit point of $P(f, A \times B)$. Then there exists a sequence $\{l_n\}$ of the elements of $P(f, A \times B)$ such that $l_n \to l$ as $n \to \infty$. Now for each $l_n \in P(f, A \times B)$ there exist $x_n \in A$ and $y_n \in B$ such that $y_n = f(x_n) + l_n$. Since A and B are compact, we get subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{x_{n_k}\} \to p \in A$ and $\{y_{n_k}\} \to q \in B$ as $k \to \infty$. Also $y_{n_k} = f(x_{n_k}) + l_{n_k}$. Taking limit as $k \to \infty$ we have $q = f(p) + l$. Thus $l \in P(f, A \times B)$. Hence the theorem. □

Theorem 2.6. *Let $f : \mathbf{R} \to \mathbf{R}$ be a linear map. If A and B are non empty open subsets of $\mathbf{R}$, then $P(f, A \times B)$ is a non empty open set.*

Proof. Let $d \in P(f, A \times B)$ and let $\epsilon(> 0)$ be given. Then there exists $(x, y) \in A \times B$ such that $y = f(x) + d$. Sine A and B are open sets we can find two open intervals $(x - \epsilon, x + \epsilon)$ and $(y - \epsilon, y + \epsilon)$ such that $(x - \epsilon, x + \epsilon) \subseteq A$ and $(y - \epsilon, y + \epsilon) \subseteq B$. Since $y = f(x) + d \in B$, $y + \frac{\epsilon}{2} = f(x) + d + \frac{\epsilon}{2} \in B$ and $y - \frac{\epsilon}{2} = f(x) + d - \frac{\epsilon}{2} \in B$. Thus for each $c \in (d - \frac{\epsilon}{2}, d + \frac{\epsilon}{2})$, $f(x) + c \in B$. This shows that $(d - \frac{\epsilon}{2}, d + \frac{\epsilon}{2}) \subseteq P(f, A \times B)$. Hence the theorem. □

Theorem 2.7. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function. If A and B are two compact subsets of $\boldsymbol{R}$, then $P_m(f, A \times B)$ is a Borel set of additive class one in* $\mathbf{R}$.

Proof. By Theorem 2.5 $P(f, A \times B)$ is compact. Let $\alpha = \inf P(f, A \times B)$ and $\beta = \sup P(f, A \times B)$. Let us define a function $g : [\alpha, \beta] \to \mathbf{R}$ by $g(d) = \lambda_1\{x \in A : (f(x) + d) \cap B \neq \emptyset\}$. Then $P_m(f, A \times B) = \{d \in [\alpha, \beta] : g(d) > 0\}$. Let $\{d_k\}$ be a sequence in $[\alpha, \beta]$ such that $d_k \to d \in [\alpha, \beta]$ as $k \to \infty$. Let us choose sets $Z_k = \{x \in A : (f(x) + d_k) \cap B \neq \emptyset\}$ for every $k = 1, 2, \ldots\ldots$ We first show that

$$LSZ_k \subset \{x \in A : (f(x) + d) \cap B \neq \emptyset\} \text{ ....(2.7.1)}$$

where LSZ_k is the limit superior of the sequence $\{Z_k\}$ of sets as defined in [8] and shown to be the set

$$LSZ_k = \{z : \text{there exists a subsequence } \{Z_i\} \text{ of } Z_k \text{ and } z_i \in Z_i \text{ for each } i, \text{ such that } z_i \to z \text{ as } i \to \infty\}.$$

Also $LSZ_k \supset \overline{\lim} Z_k$ ([8]). Let $p \in LSZ_k$. Then there is a subsequence $\{Z_i\}$ of $\{Z_k\}$ such that $x_i \in Z_i$ for each i and $x_i \to p$ as $i \to \infty$. Now $x_i \in Z_i$ implies that $(f(x_i)+d_i) \cap B \neq \emptyset$ for each i. Since f is continuous and $[\alpha, \beta]$ closed and bounded, $f(x_i) + d_i$ converge to $(f(p) + d) \in B$. Hence $p \in \{x \in A : (f(x) + d) \cap B \neq \emptyset\}$. Also,

$$g(d) = \lambda_1\{x \in A : \{(f(x) + d) \cap B\} \neq \emptyset\}$$

$$\geq \lambda_1(LSZ_k) \text{ , (Using (2.7.1))}$$

$$\geq \lambda_1(\overline{\lim} Z_k)$$

$$\geq \overline{\lim}\lambda_1(Z_k) \text{ , (Using Fatou's lemma)}$$

$$= \overline{\lim} g(d_k) \ .$$

Thus g is an upper semicontinuous function. Hence, there is a a decreasing sequence $\{g_n\}$ of upper semicontinuous functions defined over $[\alpha, \beta]$ such that $g(x) = \lim g_n(x)$ for all $x \in [\alpha, \beta]$ ([6]). Then $P_m(f, A \times B) = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} \{d \in [\alpha, \beta] : g_n(d) \geq \frac{1}{m}\}$ ([6]). Since each g_n is upper semicontinuous, each of the set $\{d \in [\alpha, \beta] : g_n(d) \geq \frac{1}{m}\}$ is closed. It follows that $P_m(f, A \times B)$ is an F_σ set i.e. Borel set of additive class one ([8]). □

We now strengthen the Theorem 1 of Ceder and Ganguly [2] as follows:

Theorem 2.8. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function not identically* 0*. If A and B measurable subsets of* $\mathbf{R}$ *with finite positive Lebesgue measure, then $P_m(f, A \times B)$ is an open subset of $P(f, A \times B)$.*

Proof. Let us define a function $h : \mathbf{R} \to \mathbf{R}$ by $h(d) = \lambda_1\{f^{-1}(B - d) \cap A\}$. Then h is a continuous function (cf. [3]). Now $P_m(f, A \times B) = \{d : h(d) > 0\}$. Also $d \in P_m(f, A \times B)$ implies that $h(d) > 0$. Since h is continuous there is a neighbourhood I_d of the point d such that $h(x) > 0$ for all $x \in I_d$. Then $P_m(f, A \times B)$ contains I_d. Hence the result. □

Corollary 2.9. *([2]) Let A and B be linear sets of finite positive measure. Suppose f be non-zero linear continuous function. Then $P_m(f, A \times B)$ contains an open interval.*

Theorem 2.10. *If A and B are the sets of second category at least one of which have the property of Baire, then $P_c(f, A \times B)$ is a non-empty open set.*

Proof. Assume first that A has the property of Baire but B is only second category. Then $A = G \Delta F$, where G is a non-empty open set and F is a set of first category and Δ stands for symmetric difference. Then

$$P_c(f, A \times B) = P_c(f, G \times B) \text{ ....(2.10.1)}$$

Let $S = \{x \in \mathbf{R} : B \text{ is of first category at } x\}$ and $T = \text{Int}(\mathbf{R} \setminus S)$. Then B is of second category at each point of the open set T. Let $B_1 = B \cap T$. Then B_1 is of second category at each point of T and $B \setminus B_1$ is a set of first category. Hence

$$P_c(f, G \times B) = P_c(f, G \times B_1) \text{ ....(2.10.2)}$$

In view of (2.10.1), (2.10.2) and Theorem 2.6 it is sufficient to prove that $P_c(f, G \times B_1) = P(f, G \times T)$. Since $B_1 \subset T$, it is evident that $P_c(f, G \times B_1) \subseteq P(f, G \times B_1) \subseteq P(f, G \times T)$. Let $c \in P(f, G \times T)$ then $(f(G) + c) \cap T \neq \emptyset$ implies that $(f(G) + c) \cap B_1$ is a set of second category because B_1 is of the second category at each point of T. Therefore, $c \in P_c(f, G \times B_1)$. Thus $P(f, G \times T) \subseteq P_c(f, G \times B_1)$. As $P(f, G \times T)$ is an open set by Theorem 2.6, hence $P_c(f, A \times B)$ is an open set.
The proof is similar when A is only second category but B is of second category having the property of Baire. □

Remark 2.11. The above theorem is no longer valid when A and B either have to property of Baire or are of second category.

The following example [1] supports the Remark 2.11.

Example 2.12. Let us take A as the set $\mathbf{Q}$ of rationals and B be that of set I of irrationals. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = mx$ for all $x \in \mathbf{R}$, where m is a real number. Then from the Proposition 2.1 we get $P(f, A \times B) = I - m\mathbf{Q}$ and hence

$$P(f, A \times B) = \begin{cases} I, & \text{if } m = 0; \\ I - \mathbf{Q}, & \text{if } m \in \mathbf{Q}, m \neq 0 \\ \mathbf{R}, & \text{if } m \notin \mathbf{Q}. \end{cases}$$

Hence $P(f, A \times B) \subset I$ if $m \in \mathbf{Q}$ and this implies that $P(f, A \times B)$ does not contain an interval although A being a set of rationals is an F_σ-set and B is a G_δ-set (the complement of F_σ set) and so both of these sets have the property of Baire.

Answering a question raised by J.Ceder and D.K.Ganguly in [2] “It is unknown whether or not a second category set A can be found such that projection (category) of $A \times B$ fails to have a non-empty interior in each direction", Tomas Katkaniec [7] constructed with the help of Martin axiom a linear set A of second category such that category projection of $A \times A$ onto each line has empty interior.

We now constructed with the help of Martin axiom a planar second category set which does not have Lebesgue measure zero such that projection of this set has empty interior in each line. For Martin’s axiom the reader may consult with the paper [9].

3 Main Result

Theorem 3.1. *Assuming Martin’s axiom, there exists a planar second category set which does not have Lebesgue measure zero and whose projection on each line has non empty interior.*

Proof. Let $\mathcal{G}$ be the family of all Borel sets in the plane which can not be covered by less than c lines where c denotes the power of the continuum and let $\{G_\alpha\}_{\alpha<c}$ be a well-ordering of $\mathcal{G}$. Let $\{m_\alpha\}_{\alpha<c}$ be a well-ordering of real number in $[0, \pi)$ with $m_0 = 0$. Also let L_α denotes the line through $(0, 0)$ and e^{im_α}. Further let $\{P_\alpha\}_{\alpha<c}$ be a well-ordering of $\aleph_0 \times c$ where $\aleph_0$ represents the power of the set of positive integers. Let us take $P_\alpha = (P_\alpha^{(1)}, P_\alpha^{(2)})$ and let $\{B_n\}_{n=0}^\infty$ be a countable open base for $\mathbf{R}$. Pick E_0 to be the line perpendicular to $L_{P_0^{(2)}}$ such that $E_0 \cap L_{P_0^{(2)}} \subseteq B_{P_0^{(1)}} e^{im_{P_0^{(2)}}}$ and choose $a_0 \in G_0 \setminus E_0$. By transfinite induction, assume that for each $\xi < \beta$ we have picked a line E_ξ and a point a_ξ such that $a_\xi \in G_\xi \setminus \bigcup_{\gamma\leq\xi} E_\gamma$ and E_ξ is perpendicular to $L_{P_\xi^{(2)}}$ and $E_\xi \cap L_{P_\xi^{(2)}} \subseteq B_{P_\xi^{(1)}} e^{im_{P_\xi^{(2)}}}$. Now we proceed to define E_β and a_β as follows:
Since the projection of $\{a_\alpha : \alpha < \beta\}$ on $L_{P_\beta^{(2)}}$ has cardinality less than c, we can find a line E_β perpendicular to $L_{P_\beta^{(2)}}$ such that $L_{P_\beta^{(2)}} \cap E_\beta \subseteq B_{P_\beta^{(1)}} e^{im_{P_\beta^{(2)}}}$ and $E_\beta \cap \{a_\alpha : \alpha < \beta\} = \emptyset$. Pick $a_\beta \in G_\beta \setminus \bigcup\{E_\alpha : \alpha \leq \beta\}$ and put $A = \{a_\alpha : \alpha < c\}$. For each α, the projection of A upon L_α misses the set $\bigcup\{L_\alpha \cap E_\beta : \alpha = P_\beta^{(2)}\}$ which is countable dense in L_α. By Martin’s axiom, it is clear that the set A is of second category.
If A is of positive outer measure, then there exists a unit square I^2 such that $I^2 \cap A$ has positive outer measure. If not, then $I^2 \setminus A$ has positive outer measure and contains a F_σ set, B, of positive measure. Then $B \in \mathcal{G}$, contradict the fact $A \cap B = \emptyset$. Hence A is not Lebesgue measure zero. □

4 Some open questions

We now arise some open questions in this regard.
Question 1: If A and B are of second category, does the projection of $A \times B$ on each line con-

tains an interval?
Question 2: Let $\{L_x\}_{x\in\mathbf{R}}$ be a family of lines such that slope $L_x = x$. Is $\bigcup\{L_x : x \in \mathbf{R}\}$ second category?
Question 3: If $\{L_x\}_{x\in\mathbf{R}}$ be a family of lines with slope $L_x = x$, does $\bigcup\{L_x : x \in \mathbf{R}\}$ have measure zero?
Question 4: Does planar set of full measure (or even positive Lebesgue measure) have a projection with non-empty interior?

Note: If the answer of **Question 2** is yes, then we have the following theorem.

Theorem 4.1. *Every residual set has a projection containing an interval.*

Proof. Suppose there exists a residual set A having no projection with non-empty interiors. Then there exist a family of lines $\{L_x\}_{x\in\mathbf{R}}$ with slope $l_x = x$ each misses A. By the **Question 2**, $\mathbf{R}^2$ $\setminus A$ must be second category and hence contradiction. Hence the theorem. □

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