Palestine Journal of Mathematics

Vol. 5(2) (2016) , 35–44

Besov continuity for Multipliers defined on compact Lie groups

Duván Cardona

Communicated by A. Aldroubi

MSC 2010 Classifications: Primary 43A22, 43A77; Secondary 43A15, 22A30.

Keywords and phrases: Besov spaces, Multipliers, Compact Lie groups, Pseudo-differential operators.

I would like to thank the anonymous referee for his/her remarks which helped to improve the manuscript.

Abstract In this note we give estimates for Fourier multipliers on Besov spaces. We present a class of bounded non-invariant pseudo-differential operators on Compact Lie groups.

1 Introduction

Given a compact Lie group G, in this paper we establish continuity properties of Fourier multipliers defined on compact Lie groups when acting on Besov spaces. In our analysis on a Lie group G, the set $\widehat{G}$ denotes the unitary dual of G, i.e. the set of equivalence classes of all strongly continuous irreducible unitary representations of G. So, if $\xi : G \rightarrow U(H_\xi)$ is an irreducible unitary representation and $\sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$, the corresponding Fourier multiplier $\mathrm{Op}(\sigma)$ is the pseudo-differential operator formally defined by the formula

$$\mathrm{Op}(\sigma)f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma(\xi)(\mathscr{F}f)(\xi)] \tag{1.1}$$

where, the summations is understood so that from each class $[\xi]$ we pick just one representative $\xi \in [\xi]$, $d_\xi = \dim(H_\xi)$ and $(\mathscr{F}f)(\xi)$ is the Fourier transform at ξ :

$$(\mathscr{F}f)(\xi) := \widehat{f}(\xi) = \int_G f(x)\xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi}. \tag{1.2}$$

Besov spaces on compact Lie groups where introduced and analyzed in [23] and they form scales $B^r_{p,q}(G)$ carrying three indices $r \in \mathbb{R}$, $0 < p, q \leq \infty$. There are several possibilities concerning the conditions to impose on a symbol σ in the attempt to establish a Fourier multiplier theorem of boundedness on Besov spaces and Lebesgue spaces (see [5, 6, 9, 10, 11, 30]). However, our work is closely related with a recent result by Michael Ruzhansky and Jens Wirth [30]:

Theorem 1.1. *Let G be a compact Lie group. Denote by $\varkappa$ the smallest even integer larger than $\frac{1}{2}dim(G)$. Let $\mathrm{Op}(\sigma)$ be a Fourier multiplier and assume that its symbol $\sigma(\xi)$ satisfies*

$$\|\mathbb{D}^\alpha \sigma(\xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, \textit{ for all } \quad |\alpha| \leq \varkappa, \textit{ and } [\xi] \in \widehat{G}. \tag{1.3}$$

Then, $\mathrm{Op}(\sigma)$ is of weak type $(1,1)$ and bounded on $L^p(G)$ for all $1 < p < \infty$.

In this paper we prove the following theorems

Theorem 1.2. *Let G be a compact Lie group and $\mathrm{Op}(\sigma)$ be a Fourier multiplier. If $\mathrm{Op}(\sigma)$ is bounded from $L^{p_1}(G)$ into $L^{p_2}(G)$, then $\mathrm{Op}(\sigma)$ extends to a bounded operator from $B^r_{p_1,q}(G)$ into $B^r_{p_2,q}(G)$, for all $r \in \mathbb{R}$ and $0 < q \leq \infty$.*

Theorem 1.3. *Let G be a compact Lie group and $Op(\sigma)$ be a Fourier multiplier. If $\mathrm{Op}(\sigma)$ is of weak type $(1,1)$ then $\mathrm{Op}(\sigma)$ extends to a bounded operator from $B^r_{1,q}(G)$, into $B^r_{p,q}(G)$ for all $r \in \mathbb{R}$, $0 < q \leq \infty$ and $0 < p < 1$.*

As a consequence of our theorems and the estimate by Ruzhansky and Wirth we obtain:

Theorem 1.4. *Let G be a compact Lie group. Denote by $\varkappa$ the smallest even integer larger than $\frac{1}{2}dim(G)$. Let $\mathrm{Op}(\sigma)$ be a Fourier multiplier and assume that its symbol $\sigma(\xi)$ satisfies*

$$\|\mathbb{D}^\alpha\sigma(\xi)\|_{op} \leq C_\alpha\langle\xi\rangle^{-|\alpha|}, \text{ for all } \quad |\alpha| \leq \varkappa, \text{ and } [\xi] \in \widehat{G}. \tag{1.4}$$

Then, $\mathrm{Op}(\sigma)$ *is a bounded operator on $B^r_{p,q}(G)$ for all $1 < p < \infty$, $r \in \mathbb{R}$ and $0 < q \leq \infty$.*

Theorem 1.5. *Let G be a compact Lie group. Denote by $\varkappa$ the smallest even integer larger than $\frac{1}{2}dim(G)$. Let $\mathrm{Op}(\sigma)$ be a Fourier multiplier and assume that its symbol $\sigma(\xi)$ satisfies*

$$\|\mathbb{D}^\alpha\sigma(\xi)\|_{op} \leq C_\alpha\langle\xi\rangle^{-|\alpha|}, \text{ for all } \quad |\alpha| \leq \varkappa, \text{ and } [\xi] \in \widehat{G}. \tag{1.5}$$

Then, $\mathrm{Op}(\sigma)$ *is a bounded operator from $B^r_{p,q}(G)$ into $B^r_{1,q}(G)$ for all $0 < p < 1$, $r \in \mathbb{R}$ and $0 < q \leq \infty$.*

Theorem 1.4 can be extended to the case of non-invariant pseudo-differential operators. For this we use a version of the Sobolev embedding theorem. This approach was used by Ruzhansky and Wirth [30], (see also Ruzhansky-Turunen [28] and [29]) in order to get L^p multiplier theorems for non-invariant pseudo-differential operators. In this work we consider the global quantization of operators on compact Lie groups proposed in [24, 25]. If

$$\sigma(x,\xi) : G \times \widehat{G} \to \cup_{[\xi]\in\widehat{G}}\mathbb{C}^{d_\xi\times d_\xi}$$

is a measurable function, the corresponding non-invariant operator is the pseudo-differential operator defined formally by

$$\mathrm{Op}(\sigma)f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma(x,\xi)(\mathscr{F}f)(\xi)]. \tag{1.6}$$

With this in mind, we present the following result.

Theorem 1.6. *Let G be a compact Lie group and $n = dim(G)$ its dimension. Denote by $\varkappa$ the smallest even integer larger than $\frac{n}{2}$, and $l = [n/p] + 1$. Let $\mathrm{Op}(\sigma)$ be a pseudo-differential operator on G and assume that its symbol $\sigma(x,\xi)$ satisfies*

$$\|\partial_x^\beta\mathbb{D}^\alpha\sigma(x,\xi)\|_{op} \leq C_\alpha\langle\xi\rangle^{-|\alpha|}, \text{ for all } \quad |\alpha| \leq \varkappa, \quad |\beta| \leq l \text{ and } [\xi] \in \widehat{G}. \tag{1.7}$$

Then, $\mathrm{Op}(\sigma)$ *is a bounded operator on $B^r_{p,q}(G)$ for all $1 < p < \infty$, $r \in \mathbb{R}$ and $0 < q \leq \infty$.*

We note that in the commutative case of the n-dimensional torus $G = \mathbb{T}^n$ conditions on the number of derivatives on the symbol in the ξ-variable, and assumptions on the size of these derivatives can be relaxed. This is possible by the following result which was proved by Julio Delgado in [18].

Theorem 1.7. *Let $0 < \varepsilon < 1$ and $k \in \mathbb{N}$ with $k > \frac{n}{2}$, let σ be a symbol such that $|\mathbb{D}^\alpha\sigma(x,\xi)| \leq C_\alpha\langle\xi\rangle^{-\frac{n}{2}\varepsilon-(1-\varepsilon)|\alpha|}$, $|\partial_x^\beta\sigma(x,\xi)| \leq C_\beta\langle\xi\rangle^{-\frac{n}{2}}$, for $|\alpha|, |\beta| \leq k$. Then,* $\mathrm{Op}(\sigma)$ *is a bounded operator from $L^p(\mathbb{T}^n)$ into $L^p(\mathbb{T}^n)$ for $2 \leq p < \infty$.*

Now, by using Theorem 1.2 and the estimate by Delgado we deduce the following Besov estimate.

Theorem 1.8. *Let $0 < \varepsilon < 1$ and $k, l \in \mathbb{N}$ with $k > \frac{n}{2}$ and $l > \frac{n}{p}$, let $\sigma(x,\xi)$ be a symbol such that $|\partial_x^\beta\mathbb{D}^\alpha\sigma(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-\frac{n}{2}\varepsilon-(1-\varepsilon)|\alpha|}$ for $|\alpha| \leq k, |\beta| \leq l$. Then,* $\mathrm{Op}(\sigma)$ *is a bounded operator from $B^r_{p,q}(\mathbb{T}^n)$ into $B^r_{p,q}(\mathbb{T}^n)$ for all $1 < p < \infty$, $r \in \mathbb{R}$ and $0 < q \leq \infty$.*

This paper is organized as follows. In Section 2, we summarizes basic properties on the harmonic analysis on compact Lie groups including the Ruzhansky-Turunen quantization of global pseudo-differential operators on compact-Lie groups and the definition of Besov spaces on such groups. Finally, in Section 3 we proof our results on the boundedness of invariant and non-invariant pseudo-differential operators on Besov spaces.

2 Preliminaries

In this section we will introduce some preliminaries on pseudo-differential operators on compact Lie groups. There are two notions of pseudo-differential operators on compact Lie groups. The first notion in the case of general manifolds (based on the idea of *local symbols*, as in Hörmander [21]) and, in a much more recent context, the one of global pseudo-differential operators on compact Lie groups as defined by [28] (from *full symbols*, for which the notations and terminologies are taken from [28]). We will always equip a compact Lie group with the Haar measure μ_G. For simplicity, we will write $\int_G f dx$ for $\int_G f d\mu_G$, $L^p(G)$ for $L^p(G, \mu_G)$, etc. The following assumptions are based on the group Fourier transform

$$\widehat{\varphi}(\xi) = \int_G \varphi(x)\xi(x)^* dx, \qquad \varphi(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\widehat{\varphi}(\xi)).$$

The Peter-Weyl Theorem on G implies the Plancherel identity on $L^2(G)$,

$$\|\varphi\|_{L^2(G)} = \left(\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\widehat{\varphi}(\xi)\widehat{\varphi}(\xi)^*)\right)^{\frac{1}{2}} = \|\widehat{\varphi}\|_{L^2(\widehat{G})}.$$

Here

$$\|A\|_{HS} = \mathrm{Tr}(AA^*),$$

denotes the Hilbert-Schmidt norm of matrices. Any linear operator A on G mapping $C^\infty(G)$ into $\mathcal{D}'(G)$ gives rise to a *matrix-valued global (or full) symbol* $\sigma_A(x,\xi) \in \mathbb{C}^{d_\xi\times d_\xi}$ given by

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x), \tag{2.1}$$

which can be understood from the distributional viewpoint. Then it can be shown that the operator A can be expressed in terms of such a symbol as [28]

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)]. \tag{2.2}$$

In this paper we use the notation $\mathrm{Op}(\sigma_A) = A$. $L^p(\widehat{G})$ spaces on the unitary dual can be well defined. If $p = 2$, $L^2(\widehat{G})$ is defined by the norm

$$\|\Gamma\|^2_{L^2(\widehat{G})} = \sum_{[\xi]\in\widehat{G}} d_\xi \|\Gamma(\xi)\|^2_{HS}.$$

Now, we want to introduce Sobolev spaces and, for this, we give some basic tools. Let $\xi \in Rep(G) := \cup\widehat{G}$, if $x \in G$ is fixed, $\xi(x) : H_\xi \to H_\xi$ is an unitary operator and $d_\xi := \dim H_\xi < \infty$. There exists a non-negative real number $\lambda_{[\xi]}$ depending only on the equivalence class $[\xi] \in \hat{G}$, but not on the representation ξ, such that $-\mathcal{L}_G\xi(x) = \lambda_{[\xi]}\xi(x)$; here $\mathcal{L}_G$ is the Laplacian on the group G (in this case, defined as the Casimir element on G). Let $\langle\xi\rangle$ denote the function $\langle\xi\rangle = (1+\lambda_{[\xi]})^{\frac{1}{2}}$.

Definition 2.1. For every $s \in \mathbb{R}$, the *Sobolev space* $H^s(G)$ on the Lie group G is defined by the condition: $f \in H^s(G)$ if only if $\langle\xi\rangle^s\widehat{f} \in L^2(\widehat{G})$.

The Sobolev space $H^s(G)$ is a Hilbert space endowed with the inner product $\langle f, g\rangle_s = \langle \Lambda_s f, \Lambda_s g\rangle_{L^2(G)}$, where, for every $r \in \mathbb{R}$, $\Lambda_s : H^r \to H^{r-s}$ is the bounded pseudo-differential operator with symbol $\langle\xi\rangle^s I_\xi$.

Definition 2.2. Let $(Y_j)_{j=1}^{\dim(G)}$ be a basis for the Lie algebra $\mathfrak{g}$ of G, and let ∂_j be the left-invariant vector fields corresponding to Y_j. We define the differential operator associated to such a basis by $D_{Y_j} = \partial_j$ and, for every $\alpha \in \mathbb{N}^n$, the *differential operator* ∂_x^α is the one given by $\partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$. Now, if ξ_0 is a fixed irreducible representation, the matrix-valued *difference operator* is the given by $\mathbb{D}_{\xi_0} = (\mathbb{D}_{\xi_0,i,j})_{i,j=1}^{d_{\xi_0}} = \xi_0(\cdot) - I_{d_{\xi_0}}$. If the representation is fixed we omit the index ξ_0 so that, from a sequence $\mathbb{D}_1 = \mathbb{D}_{\xi_0,j_1,i_1}, \cdots, \mathbb{D}_n = \mathbb{D}_{\xi_0,j_n,i_n}$ of operators of this type we define $\mathbb{D}^\alpha = \mathbb{D}_1^{\alpha_1}\cdots\mathbb{D}_n^{\alpha_n}$, where $\alpha \in \mathbb{N}^n$.

Definition 2.3. We introduce the Besov spaces on compact Lie groups using the Fourier transform on the group G as follow. Let $r \in \mathbb{R}$, $0 \leq q < \infty$ and $0 < p \leq \infty$. If f is a measurable function on G, we say that $f \in B^r_{p,q}(G)$ if f satisfies

$$\|f\|_{B^r_{p,q}} := \left(\sum_{m=0}^{\infty} 2^{mrq} \| \sum_{2^m \leq \langle \xi \rangle < 2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\widehat{f}(\xi)] \|^q_{L^p(G)} \right)^{\frac{1}{q}} < \infty. \tag{2.3}$$

If $q = \infty$, $B^r_{p,\infty}(G)$ consists of those functions f satisfying

$$\|f\|_{B^r_{p,\infty}} := \sup_{m\in\mathbb{N}} 2^{mr} \| \sum_{2^m \leq \langle \xi \rangle < 2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\widehat{f}(\xi)] \|_{L^p(G)} < \infty. \tag{2.4}$$

In the case of $p = q = \infty$ and $0 < r < 1$ and $G = \mathbb{T}$ we obtain $B^r_{\infty,\infty}(\mathbb{T}) = \Lambda^r(\mathbb{T})$, that is the familiar space of Hölder continuous functions of order r. According to the usual definition, a function f belongs to Λ^r if there exists a constant A such that: $|f(x)| \leq A$ and

$$|f|_{\Lambda^r} := \sup_{x,y} \frac{|f(x-y) - f(x)|}{|y|^r} \leq A.$$

These are Banach spaces together with the norm

$$\|f\|_{\Lambda^r} = |f|_{\Lambda^r} + \sup_{x\in\mathbb{T}} |f(x)|.$$

3 Proofs

In this section we prove our results which were presented above. We treat the theory with basic methods. In our tools we consider the Sobolev embedding theorem and basics on Fourier analysis. We star with the proof of the Theorem 1.2.

Theorem 3.1. *Let G be a compact Lie group and $Op(\sigma)$ be a Fourier multiplier. If* $\mathrm{Op}(\sigma)$ *is bounded from $L^{p_1}(G)$ into $L^{p_2}(G)$, then* $\mathrm{Op}(\sigma)$ *extends to a bounded operator from $B^r_{p_1,q}(G)$ into $B^r_{p_2,q}(G)$, for all $r \in \mathbb{R}$ and $0 < q \leq \infty$.*

Proof. First, let us consider a multiplier operator $\mathrm{Op}(\sigma)$ bounded from $L^{p_1}(G)$ into $L^{p_2}(G)$, and $f \in C^\infty(G)$. Then, we have

$$\|Tf\|_{L^{p_2}(G)} \leq C\|f\|_{L^{p_1}(G)},$$

where $C = \|T\|_{B(L^{p_1},L^{p_2})}$ is the usual operator norm. We denote by $\chi_m(\xi)$ the characteristic function of

$$D_m := \{\xi : 2^m \leq \langle \xi \rangle < 2^{m+1}\}$$

and $\mathrm{Op}(\chi_m)$ the corresponding Fourier multiplier of the symbol $\chi_m(\xi)I_\xi$. Here, I_ξ is the identity operator in $\mathbb{C}^{d_\xi \times d_\xi}$. By the definition of Besov norm, if $0 < q < \infty$ we have

$$\begin{aligned}
\|\mathrm{Op}(\sigma)f\|^q_{B^r{}_{p_2,q}} &= \sum_{m=0}^{\infty} 2^{mrq} \| \sum_{2^m \leq \langle \xi \rangle < 2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)] \|^q_{L^{p_2}(G)} \\
&= \sum_{m=0}^{\infty} 2^{mrq} \| \sum_{[\xi]\in\widehat{G}} d_\xi \cdot \chi_m(\xi)\mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)] \|^q_{L^{p_2}(G)} \\
&= \sum_{m=0}^{\infty} 2^{mrq} \| \sum_{[\xi]\in\widehat{G}} d_\xi \cdot \mathrm{Tr}[\xi(x)\chi_m(\xi)\sigma(\xi)(\mathscr{F}f)(\xi)] \|^q_{L^{p_2}(G)} \\
&= \sum_{m=0}^{\infty} 2^{mrq} \| \sum_{[\xi]\in\widehat{G}} d_\xi \cdot \mathrm{Tr}[\xi(x)\sigma(\xi)\mathscr{F}(\mathrm{Op}(\chi_m)f)(\xi)] \|^q_{L^{p_2}(G)} \\
&= \sum_{m=0}^{\infty} 2^{mrq} \|\mathrm{Op}(\sigma)[(\mathrm{Op}(\chi_m)f)]\|^q_{L^{p_2}(G)}.
\end{aligned}$$

By the boundedness of $\mathrm{Op}(\sigma)$ from $L^{p_1}(G)$ into $L^{p_2}(G)$ we get,

$$\begin{aligned}
\|\mathrm{Op}(\sigma)f\|^q_{B^r{}_{p_2,q}} &\leq \sum_{m=0}^{\infty} 2^{mrq}C^q\|\mathrm{Op}(\chi_m)f\|^q_{L^{p_1}(G)} \\
&= \sum_{m=0}^{\infty} 2^{mrq}C^q\| \sum_{[\xi]\in\widehat{G}} d_\xi\cdot \mathrm{Tr}[\xi(x)\chi_m(\xi)I_\xi\mathscr{F}(f)(\xi)]\|^q_{L^{p_1}(G)} \\
&= \sum_{m=0}^{\infty} 2^{mrq}C^q\| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi\cdot \mathrm{Tr}[\xi(x)I_\xi\mathscr{F}(f)(\xi)]\|^q_{L^{p_1}(G)} \\
&= \sum_{m=0}^{\infty} 2^{mrq}C^q\| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi\cdot \mathrm{Tr}[\xi(x)I_\xi\mathscr{F}(f)(\xi)]\|^q_{L^{p_1}(G)} \\
&= C^q\|f\|^q_{B^r{}_{p_1,q}}
\end{aligned}$$

Hence,

$$\|\mathrm{Op}(\sigma)f\|_{B^r{}_{p_2,q}} \leq C\|f\|_{B^r{}_{p_1,q}}.$$

If $q=\infty$ we have

$$\begin{aligned}
\|\mathrm{Op}(\sigma)f\|_{B^r{}_{p_2,\infty}} &= \sup_{m\in\mathbb{N}} 2^{mr}\| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)]\|_{L^{p_2}(G)} \\
&= \sup_{m\in\mathbb{N}} 2^{mr}\| \sum_{[\xi]\in\widehat{G}} d_\xi\cdot\chi_m(\xi)\mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)]\|_{L^{p_2}(G)} \\
&= \sup_{m\in\mathbb{N}} 2^{mr}\| \sum_{[\xi]\in\widehat{G}} d_\xi\cdot \mathrm{Tr}[\xi(x)\chi_m(\xi)\sigma(\xi)(\mathscr{F}f)(\xi)]\|_{L^{p_2}(G)} \\
&= \sup_{m\in\mathbb{N}} 2^{mr}\| \sum_{[\xi]\in\widehat{G}} d_\xi\cdot \mathrm{Tr}[\xi(x)\sigma(\xi)\mathscr{F}(\mathrm{Op}(\chi_m)f)(\xi)]\|_{L^{p_2}(G)} \\
&= \sup_{m\in\mathbb{N}} 2^{mr}\|\mathrm{Op}(\sigma)[(\mathrm{Op}(\chi_m)f)]\|_{L^{p_2}(G)}.
\end{aligned}$$

Newly, by using the fact that $\mathrm{Op}(\sigma)$ is a bounded operator from $L^{p_1}(G)$ into $L^{p_2}(G)$ we have,

$$\begin{aligned}
\|\mathrm{Op}(\sigma)f\|_{B^r{}_{p_2,\infty}} &\leq \sup_{m\in\mathbb{N}} 2^{mr}C\|\mathrm{Op}(\chi_m)f\|_{L^{p_1}(G)} \\
&= \sup_{m\in\mathbb{N}} 2^{mr}C\| \sum_{[\xi]\in\widehat{G}} d_\xi\cdot \mathrm{Tr}[\xi(x)\chi_m(\xi)I_\xi\mathscr{F}(f)(\xi)]\|_{L^{p_1}(G)} \\
&= \sup_{m\in\mathbb{N}} 2^{mr}C\| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi\cdot \mathrm{Tr}[\xi(x)I_\xi\mathscr{F}(f)(\xi)]\|_{L^{p_1}(G)} \\
&= \sup_{m\in\mathbb{N}} 2^{mr}C\| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi\cdot \mathrm{Tr}[\xi(x)I_\xi\mathscr{F}(f)(\xi)]\|_{L^{p_1}(G)} \\
&= C\|f\|_{B^r{}_{p_1,\infty}}.
\end{aligned}$$

This implies that,

$$\|\mathrm{Op}(\sigma)f\|_{B^r{}_{p_2,\infty}} \leq C\|f\|_{B^r{}_{p_1,\infty}}.$$

With the last inequality we end the proof. □

The proof of Theorem 1.3 depends on the following result due to Kolmogorov (see Lemma 5.16 in [19]).

Lemma 3.2. *Given an operator $S:(X,\mu)\to(Y,\nu)$ which is of weak type (1,1), $0<v<1$, and a set $E\subset Y$ of finite measure, there exists a $C>0$ such that*

$$\int_E |Sf(x)|^v\,dx \leq C\mu(E)^{1-v}\|f\|^v_{L^1(X)}. \tag{3.1}$$

Theorem 3.3. *Let G be a compact Lie group and $\mathrm{Op}(\sigma)$ be a Fourier multiplier. If $\mathrm{Op}(\sigma)$ is of weak type $(1,1)$ then $\mathrm{Op}(\sigma)$ extends to a bounded operator from from $B^r_{1,q}(G)$, into $B^r_{p,q}(G)$ for all $r\in\mathbb{R}$, $0<q\leq\infty$ and $0<p<1$.*

Proof. Let us consider a Fourier multiplier $\mathrm{Op}(\sigma)$ on G of weak type (1,1). By Lemma 3.2 we deduce the boundedness of $\mathrm{Op}(\sigma)$ from $L^1(G)$ into $L^p(G)$ for all $0<p<1$. Finally, by Theorem 1.2 we obtain that $\mathrm{Op}(\sigma)$ is a continuous multiplier from $B^r_{1,q}(G)$, into $B^r_{p,q}(G)$ for all $0<p<1$, $r\in\mathbb{R}$ and $0<q\leq\infty$. □

Remark 3.4. We observe that Theorems 1.4 and 1.5 are an immediate consequence of the Theorems 1.1, 1.2 and 1.3.

Now we extend the boundedness of Fourier multipliers on Besov spaces, to the case of non-invariant pseudo-differential operators.

Theorem 3.5. *Let G be a compact Lie group and $n=dim(G)$ its dimension. Denote by $\varkappa$ the smallest even integer larger than $\frac{n}{2}$, and $l=[n/p]+1$. Let $\mathrm{Op}(\sigma)$ be a pseudo-differential operator and assume that its symbol $\sigma(x,\xi)$ satisfies*

$$\|\partial_x^\beta\mathbb{D}^\alpha\sigma(x,\xi)\|_{op}\leq C_\alpha\langle\xi\rangle^{-|\alpha|},\ \textit{for all}\quad |\alpha|\leq\varkappa,\quad |\beta|\leq l \textit{ and } [\xi]\in\widehat{G}. \tag{3.2}$$

Then, $\mathrm{Op}(\sigma)$ is a bounded operator on $B^r_{p,q}(G)$ for all $1<p<\infty$, $r\in\mathbb{R}$ and $0<q\leq\infty$.

Proof. Let $f\in C^\infty(G)$. To prove this theorem we write

$$\begin{aligned}\mathrm{Op}(\sigma)f(x) &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma(x,\xi)\widehat{f}(\xi)]\\ &= \int_G\left(\sum_{[\xi]\in\widehat{G}} d_\xi\mathrm{Tr}[\xi(y^{-1}x)\sigma(x,\xi)]\right)f(y)dy\\ &= \int_G\left(\sum_{[\xi]\in\widehat{G}} d_\xi\mathrm{Tr}[\xi(y)\sigma(x,\xi)]\right)f(xy^{-1})dy.\end{aligned}$$

Hence $\mathrm{Op}(\sigma)f(x)=(\kappa(x,\cdot)*f)(x)$, where

$$\kappa(z,y)=\sum_{[\xi]\in\widehat{G}} d_\xi\mathrm{Tr}[\xi(y)\sigma(z,\xi)] \tag{3.3}$$

and $*$ is the right convolution operator. Moreover, if we define $A_zf(x)=(\kappa(z,\cdot)*f)(x)$ for every element $z\in G$, we have

$$A_xf(x)=\mathrm{Op}(\sigma)f(x),\ \ x\in G.$$

For all $0\leq|\beta|\leq[n/p]+1$ we have $\partial_z^\beta A_zf(x)=\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f(x)$. So, by the condition 3.2 and Theorem 1.2, for every $z\in G$, $\partial_z^\beta A_zf=\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f$ is a bounded operator on $B^r_{p,q}(G)$ for all $1<p<\infty$, $r\in\mathbb{R}$ and $0<q\leq\infty$. Now, we want to estimate the Besov norm of

$\mathrm{Op}(\sigma(\cdot,\cdot))$. First, we observe that

$$\begin{aligned}
&\left\| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)] \right\|_{L^p}^p \\
&:= \int_G \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\int_G \mathrm{Op}(\sigma)f(y)\xi(y)^*dy] \right|^p dx \\
&= \int_G \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\int_G A_y f(y)\xi(y)^*dy] \right|^p dx \\
&\leq \int_G \sup_{z\in G} \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(A_z f)(\xi)] \right|^p dx
\end{aligned}$$

By the Sobolev embedding theorem (Theorem 3.1.3 of [1]), we have

$$\begin{aligned}
&\sup_{z\in G} \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(A_z f)(\xi)] \right|^p \\
&\lesssim \sum_{|\beta|\leq l} \int_G \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)] \right|^p dz \\
&\lesssim \sup_{|\beta|\leq l} \int_G \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)] \right|^p dz
\end{aligned}$$

From this, and the Sobolev embedding theorem we have

$$\begin{aligned}
&\int_G \sup_{z\in G} \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)A_z f(y)\xi(y)^*] \right|^p dy \\
&\lesssim \sum_{|\beta|\leq l} \int_G\int_G \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)] \right|^p dzdx \\
&\lesssim \sup_{|\beta|\leq l} \int_G\int_G \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)] \right|^p dxdz \\
&\leq \sup_{|\beta|\leq l, z\in G} \int_G \left| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)] \right|^p dx \\
&= \sup_{|\beta|\leq l, z\in G} \| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)] \|_{L^p}^p
\end{aligned}$$

Hence,

$$\begin{aligned}
&\| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)] \|_{L^p} \\
&\lesssim \sup_{|\beta|\leq l, z\in G} \| \sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)] \|_{L^p}
\end{aligned}$$

Thus, considering $0 < q < \infty$ we obtain

$$\begin{aligned}\|\mathrm{Op}(\sigma)f\|_{B^r_{p,q}(\mathbb{T})} &:= \left(\sum_{m=0}^{\infty} 2^{mrq}\left\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)]\right\|^q_{L^p}\right)^{\frac{1}{q}}\\ &\lesssim \left(\sum_{m=0}^{\infty} 2^{mrq}\sup_{|\beta|\leq l, z\in G}\left\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)]\right\|^q_{L^p}\right)^{\frac{1}{q}}\end{aligned}$$

We define for every $z\in\mathbb{T}$ the non-negative function $z\mapsto g(z)$ by

$$g(z) = \sup_{|\beta|\leq l}\left\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)]\right\|^q_{L^p}.$$

We can choose a sequence $(z_n)_{n\in\mathbb{N}}\subset\mathbb{T}$ such that $(g(z_n))_n$ is increasing and

$$\lim_{n\to\infty} g(z_n) = \sup_{z\in\mathbb{T}} g(z).$$

So, by an application of the monotone convergence theorem we have

$$\begin{aligned}\left(\sum_{m=0}^{\infty} 2^{mrq}\sup_{z\in\mathbb{T}} g(z)\right)^{\frac{1}{q}} &= \left(\sum_{m=0}^{\infty} 2^{mrq}\lim_{n\to\infty} g(z_n)\right)^{\frac{1}{q}}\\ &= \left(\lim_{n\to\infty}\sum_{m=0}^{\infty} 2^{mrq} g(z_n)\right)^{\frac{1}{q}}\\ &= \lim_{n\to\infty}\left(\sum_{m=0}^{\infty} 2^{mrq} g(z_n)\right)^{\frac{1}{q}}\\ &\leq \sup_{z\in\mathbb{T}}\left(\sum_{m=0}^{\infty} 2^{mrq} g(z)\right)^{\frac{1}{q}}\end{aligned}$$

Hence, we can write

$$\begin{aligned}\|\mathrm{Op}(\sigma)f\|_{B^r_{p,q}(\mathbb{T})} &\lesssim \left(\sum_{m=0}^{\infty} 2^{mrq}\sup_{|\beta|\leq l, z\in G}\left\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)]\right\|^q_{L^p}\right)^{\frac{1}{q}}\\ &\leq \sup_{|\beta|\leq l, z\in\mathbb{T}}\|\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f\|_{B^r_{p,q}(\mathbb{T})}\\ &\leq \left[\sup_{|\beta|\leq l, z\in G}\|\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))\|_{B(B^r_{p,q})}\right]\|f\|_{B^r_{p,q}}.\end{aligned}$$

So, we deduce the boundedness of $\mathrm{Op}(\sigma)$. Now, we treat of a similar way the boundedness of $\mathrm{Op}(\sigma)$ if $q=\infty$. In fact, from the inequality

$$\begin{aligned}&\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)]\|_{L^p}\\ &\lesssim \sup_{|\beta|\leq l, z\in G}\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)]\|_{L^p}\end{aligned}$$

we have

$$\begin{aligned}&2^{mr}\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\sigma)f)(\xi)]\|_{L^p}\\ &\lesssim 2^{mr}\sup_{|\beta|\leq l, z\in G}\|\sum_{2^m\leq\langle\xi\rangle<2^{m+1}} d_\xi \mathrm{Tr}[\xi(x)\mathscr{F}(\mathrm{Op}(\partial_z^\beta\sigma(z,\cdot))f)(\xi)]\|_{L^p}.\end{aligned}$$

So we get

$$\|\mathrm{Op}(\sigma)f\|_{B^r_{p,\infty}(\mathbb{T})} \lesssim \left[\sup_{|\beta|\leq l, z\in G} \|\mathrm{Op}(\partial_z^\beta \sigma(z,\cdot))\|_{B(B^r_{p,\infty})}\right] \|f\|_{B^r_{p,\infty}}.$$

With the last inequality we end the proof. □

We note that in the $n-$dimensional case of the torus some conditions on the derivatives of the symbol (number of derivatives and size of them) can be relaxed:

Theorem 3.6. *Let* $0 < \varepsilon < 1$ *and* $k, l \in \mathbb{N}$ *with* $k > \frac{n}{2}$ *and* $l > \frac{n}{p}$, *let* $\sigma(x,\xi)$ *be a symbol such that* $|\partial_x^\beta \mathbb{D}^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-\frac{n}{2}\varepsilon-(1-\varepsilon)|\alpha|}$ *for* $|\alpha| \leq k, |\beta| \leq l$. *Then,* $\mathrm{Op}(\sigma)$ *is a bounded operator from* $B^r_{p,q}(\mathbb{T}^n)$ *into* $B^r_{p,q}(\mathbb{T}^n)$ *for all* $1 < p < \infty$, $r \in \mathbb{R}$ *and* $0 < q \leq \infty$.

Proof. First, we prove this theorem in the case of Fourier multipliers. If we consider a toroidal symbol σ satisfying

$$|\mathbb{D}^\alpha \sigma(\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-\frac{n}{2}\varepsilon-(1-\varepsilon)|\alpha|},$$

by the Theorem 1.2 we obtain the boundedness of $\mathrm{Op}(\sigma)$ on $L^p(\mathbb{T}^n)$ for $2 \leq p < \infty$. If $1 < p \leq 2$ we apply the boundedness of $\mathrm{Op}(\sigma)$ for $p' \geq 2$, where $1/p + 1/p' = 1$:

$$\begin{aligned}\|\mathrm{Op}(\sigma)f\|_{L^p(\mathbb{T}^n)} &= \sup\{|\langle \mathrm{Op}(\sigma)f, g\rangle| : \|g\|_{L^{p'}(\mathbb{T}^n)} \leq 1\} \\ &= \sup\{|\langle f, \mathrm{Op}(\sigma)^* g\rangle| : \|g\|_{L^{p'}(\mathbb{T}^n)} \leq 1\} \\ &= \sup\{|\langle f, \mathrm{Op}(\overline{\sigma}(\cdot))g\rangle| : \|g\|_{L^{p'}(\mathbb{T}^n)} \leq 1\} \\ &\leq \|\mathrm{Op}(\overline{\sigma}(\cdot))\|_{B(L^{p'})}\|f\|_{L^p(\mathbb{T}^n)}.\end{aligned}$$

Now we have the global continuity of $\mathrm{Op}(\sigma)$ on $L^p(\mathbb{T}^n)$ for all $1 < p < \infty$. By Theorem 1.2 we deduce the boundedness of the multiplier $\mathrm{Op}(\sigma)$ on $B^r_{p,q}(\mathbb{T}^n)$ for all $1 < p < \infty$, $r \in \mathbb{R}$ and $0 < q \leq \infty$. Finally, if we reproduce the proof of theorem 1.4 in the particular case of the torus $\mathbb{T}^n$, we extend this estimate to the case of pseudo-differential operators $\mathrm{Op}(\sigma(\cdot,\cdot))$ with symbol $\sigma(x,\xi)$ satisfying the following toroidal inequalities

$$|\partial_x^\beta \mathbb{D}^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-\frac{n}{2}\varepsilon-(1-\varepsilon)|\alpha|}, \quad |\alpha| \leq k, |\beta| \leq l.$$

□

Remark 3.7. We end this section with some references on the topic. The boundedness of Fourier multipliers in L^p-spaces, Hölder spaces and Besov spaces has been considered by many authors for a long time. In the general case of Compact Lie groups we refer the reader to the works of Alexopoulos, Anker, Coifman, Ruzhansky and Wirth [2, 3, 17, 26, 27, 28, 29, 30]. The particular case of Fourier multipliers on the torus has been investigated in [4, 5, 6, 9, 10, 11]. L^p and Hölder estimates of periodic pseudo-differential operators can be found in [12, 13, 14, 18] and [22].

References

[1] D. Applebaum, Probability on compact Lie groups. With a foreword by Herbert Heyer. *Probability Theory and Stochastic Modelling* **70**, Springer, Cham, (2014).

[2] G. Alexopoulos, Spectral multipliers on Lie groups of polynomial growth., *Proc. Amer. Math. Soc.* **120**, 973–979 (1994).

[3] J. Anker, L_p Fourier multipliers on Riemannian symmetric spaces of the noncompact type, *Ann. of Math.* **132**, 597–628 (1990).

[4] W. Arendt, Operator-valued Fourier multipliers on periodic Besov spaces and applications, *Proc. Edinb. Math. Soc.* **47(1)**, 15–33 (2004).

[5] B. Barraza, I. González, and J. Hernández, Operator-valued Fourier multipliers on periodic Besov spaces, arXiv:1504.04408

[6] B. Barraza, R. Denk, J. Hernandez and T. Nau, Generation of semigroups for vector-valued pseudodifferential operators on the torus $\mathbb{T}^n$, to appear in *J. Fourier Anal. Appl.*

[7] S. Bernstein. Sur la convergence absolue des séries trigonométriques, *Comptes Rendum Hebdomadaires des Séances de l'Academie des Sciences, Paris* **158**, 1661–1663 (1914).

[8] W. R. Bloom, Bernstein's inequality for locally compact Abelian groups, *Ann. of Math.* **132**, 597–628 (1990).

[9] S. Bu, and J. Kim, Operator-valued Fourier multiplier theorems on Lp -spaces *Archiv. der Math.* **82** 404–414 (2004).

[10] S. Bu, and J. Kim, Operator-valued Fourier Multipliers on Periodic Triebel Spaces, *Acta Math. Sin. (Engl. Ser.)* **21(5)** 1049–1056 (2005).

[11] S. Bu and J. Kim, A note on operator-valued Fourier multipliers on Besov spaces, *Math. Nachr.* **278(14)** 1659-1664 (2005).

[12] D. Cardona, Estimativos L^2 para una clase de operadores pseudodiferenciales definidos en el toro, *Rev. Integr. Temas Mat.* **31(2)**, 147–152 (2013).

[13] D. Cardona, Weak type $(1, 1)$ bounds for a class of periodic pseudo-differential operators, *J. Pseudo-Differ. Oper. Appl.* **5(4)**, 507–515 (2014).

[14] D. Cardona, Hölder estimates for pseudo-differential operators on $\mathbb{T}^1$. *J. Pseudo-Differ. Oper. Appl.* **5(4)** 517-525 (2014).

[15] R. Coifman, and M. de Guzmán, Singular integrals and multipliers on homogeneous spaces, *Rev. Un. Mat. Argentina,* **25.** 137–143 (1970).

[16] R. Coifman, and G. Weiss, Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes, *Lecture Notes in Mathematics.* **242**, Springer-Verlag, Berlin Heidelberg New York, 1971.

[17] R. Coifman, and G. Weiss, Central multiplier theorems for compact Lie groups, *Bull. Am. Math. Soc.* **80**, 124–126 (1973).

[18] J. Delgado, L^p bounds for pseudo-differential operators on the torus, *Oper. Theory Adv. Appl.* **231**, 103–116 (2012).

[19] J. Duoandikoetxea, Fourier Analysis, *American Mathematical Society, Providence.* (2000).

[20] C. Fefferman, L^p bounds for pseudo-differential operators, *Israel J. Math.* **132**, 413–417 (1973).

[21] L. Hörmander, Estimates for translation invariant operators in L^p spaces, *Acta Math.* **104**, 93–140 (1960).

[22] S. Molahajloo, and M. W. Wong, Pseudo-differential Operators on $\mathbb{S}^1$, *New developments in pseudo-differential operators.* Eds. L. Rodino and M.W. Wong, 297–306 (2008).

[23] E. Nursultanov, M. Ruzhansky, and S. Tikhonov, Nikolskii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds. arXiv:1403.3430

[24] M. Ruzhansky, and V. Turunen, On the Fourier analysis of operators on the torus, Modern trends in pseudo-differential operator. *Oper. Theory Adv. Appl.* **172**, 87–105 (2007).

[25] M. Ruzhansky and V. Turunen, On the toroidal quantization of periodic pseudo-differential operators *Numer. Funct. Anal. Optim.* **104** 1098–1124 (2009)

[26] M. Ruzhansky, and V. Turunen, Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics *Birkhaüser-Verlag Basel*, (2010).

[27] M. Ruzhansky, and V. Turunen, Quantization of Pseudo-Differential Operators on the Torus, *J Fourier Annal Appl.* **16** 943–982 (2010).

[28] M. Ruzhansky and V. Turunen Global quantization of pseudo-differential operators on compact Lie groups, SU(2), 3-sphere, and homogeneous spaces, *Int. Math. Res. Not.* 2439–2496 (2013).

[29] M. Ruzhansky, V. Turunen and J. Wirth, Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity, *J. Fourier Anal. Appl.* **20** 476–499 (2014).

[30] M. Ruzhansky, and J. Wirth, L^p Fourier multipliers on compact Lie groups *Math. Z.* 621–642 (2015).

[31] E. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press, 1993.

[32] M. W. Wong, Discrete Fourier Analysis, Birkhäuser, 2011.

Author information

Duván Cardona, Department of Mathematics, Universidad de los Andes, Bogotá , Colombia.
E-mail: `duvanc306@gmail.com - d.cardona@uniandes.edu.co`

Received: June 29, 2015.

Accepted: December 4, 2015.