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UNIQUENESS AND DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING A POLYNOMIAL

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Abstract. In this paper, we deal with the uniqueness problems of meromorphic functions when certain nonlinear differential polynomials generated by them share a nonconstant polynomial counting multiplicities by considering that the functions share infinity, ignoring multiplicities. The research findings also include IM-analogues of the theorem in which the nonconstant polynomial is allowed to be shared ignoring multiplicities. Though the main concern of the paper is to find out a possible answer of an open question posed by Zhang and Xu [Computer Math. with Appl., 61(2011), 722-730], as a consequence of the main results we also improves the concerning results of Zhang - Xu and the present author [Bull. Math. Anal. Appl., 2(2010), 106-118].

1 Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We assume that the reader is familiar with the classical value distribution theory of meromorphic functions as described in, say, the standard monograph [7, 19]. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside of a possible exceptional set E of finite linear measure. The meromorphic function $a(z)$ is called a small function of f if $T(r, a) = S(r, f)$.

Two nonconstant meromorphic functions f and g share a small function a CM (counting multiplicities) provided that $f-a$ and $g-a$ have the same set of zeros with the same multiplicities; f and g share a IM (ignoring multiplicities) if we do not consider the multiplicities. A finite value z_0 is called a fixed point of $f(z)$ if $f(z_0) = z_0$.

The following result is well known in the value distribution theory (see [3, 4]).

Theorem A. Let f be a transcendental meromorphic function, and let $n(\geq 1)$ be an integer. Then $f^n f' = 1$ has infinitely many solutions.

Corresponding to Theorem A the following result was obtained by Yang and Hua [16] in 1997.

Theorem B. Let f and g be two nonconstant meromorphic functions, and let $n(\geq 11)$ be an integer. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Regarding fixed point the following result was proved by Fang [5] in 2000.

Theorem C. Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.

In the same year Fang and Qiu [6] proved the following result which corresponded to Theorem C.

Theorem D. Let f and g be two nonconstant meromorphic functions, and let $n(\geq 11)$ be an integer. If $f^n f'$ and $g^n g'$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t satisfying $t^{n+1} = 1$.

A new trend in this direction is to consider the uniqueness of a meromorphic functions concerning the value sharing of the k-th derivative of a linear expression of a meromorphic function. For the last decade many research works regarding the value sharing of nonlinear differential polynomials which are mainly the k-th derivative of some linear expressions of f and g have been done. (See [2], [12], [13] and [14], for example). We recall the following results of Xu, Lu and Yi [14] where an additional condition namely the sharing of poles by the meromorphic functions are taken into account.

Theorem E. Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers satisfying $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1 c_2)^n c^2 = -1$ or $f = tg$ for a constant t such that $t^n = 1$.

Theorem F. Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let n, k be two positive integers such that $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f = g$.

The following question is inevitable.

Question 1. What happened if one do not consider the multiplicity into account for sharing fixed point in Theorems E and F ?

Keeping in mind the above question, the present author [11] obtained the following result in 2010 for some general nonlinear differential polynomial.

Theorem G. Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers with $n > 9k + 4m + 13$. Let $P(z) = a_m z^m + ... + a_1 z + a_0$, where $a_0(\neq 0)$, a_1, ... , $a_m(\neq 0)$ are complex constants. Suppose that $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share z IM, f and g share ∞ IM. Then either $f = tg$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, 2, ..., m\}$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(f, g) = f^n(a_m f^m + ... + a_1 f + a_0) - g^n(a_m g^m + ... + a_1 g + a_0). \tag{1.1}$$

It is now quite natural to ask the following question.

Question 2. What can be said if sharing fixed point in Theorems E - G is replaced with sharing a nonzero polynomial ?

In 2011 Zhang and Xu [21] obtained the following result which dealt with Question 2.

Theorem H. Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $deg(p) = l \leq 5$, n, k and m be three positive integers with $n > 3k + m + 7$. Let $P(w) = a_m w^m + ... + a_1 w + a_0$ be a nonzero polynomial. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $p(z)$ CM, f and g share ∞ IM, then one of the following three conclusions hold:
(i) $f = tg$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, 2, ..., m\}$;
(ii) f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (1.1);
(iii) $P(w)$ is reduced to a nonzero monomial, namely, $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, ..., m\}$; if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)dz$, c_1, c_2 and c are three constants satisfying $a_i^2 (c_1 c_2)^{n+i}[(n+i)c]^2 = -1$, if $p(z)$ is a nonzero constant b, then $f(z) = c_3 e^{cz}$, $g(z) = c_4 e^{-cz}$, where c_3, c_4 and c are three constants such that $(-1)^k a_i^2 (c_3 c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

In the same paper the authors posed the following question, as far as we know, this remains open.

Question 3. Is it really possible in any way to remove the condition $deg(p) = l \leq 5$ in Theorem H ?

In the paper, taking the possible answer of the above question into background we will prove following two theorems first one of which improves Theorem H. Our second theorem will not only improves Theorem G at the same time provide an IM-analogues result of Theorem H. The following are the main results of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions, $p(z)$ be a nonconstant polynomial of degree l, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers with $n > \max\{3k+m+6, k+2l\}$. In addition we suppose that either k, l are co-prime or $k > l$ when $l \geq 2$. Let $P(w)$ be defined as in Theorem H. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $p(z)$ CM; f and g share ∞ IM, then the following conclusions hold:*
(i) If $P(w) = a_mw^m + ... + a_1w + a_0$ is not a monomial, then either $f = tg$ for a constant t that satisfies $t^d = 1$, where $d = \gcd(n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, 2, ..., m\}$; or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (1.1). In particular when $m = 1$ and $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$ then $f = g$.
(ii) When $P(w) = c_0$ or $P(w) = a_mw^m$, then either $f = tg$ for a constant t that satisfies $t^{n+m^*} = 1$, or $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = p(z)$, b_1, b_2 and b are three constants satisfying $c_0^2(nb)^2(b_1b_2)^n = -1$ or $a_m^2((n+m)b)^2(b_1b_2)^{n+m} = -1$, where

$$m^* = \begin{cases} m & \text{if } P(w) \neq c_0; \\ 0 & \text{if } P(w) = c_0. \end{cases}$$

Remark 1.2. Theorem 1.1 improves Theorem H by reducing the lower bound of n.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions, $p(z)$ be a nonconstant polynomial of degree l, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers with $n > \max\{9k+4m+11, k+2l\}$. In addition we suppose that either k, l are co-prime or $k > l$ when $l \geq 2$. Let $P(w)$ be defined as in Theorem H. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $p(z)$ IM; f and g share ∞ IM, then the conclusions of Theorem 1.1 hold.*

Remark 1.4. Theorem 1.3 improves Theorem G by reducing the lower bound of n as well as by generalizing the concept of fixed point sharing with sharing a nonzero polynomial. Thus Theorem 1.3 is a two fold improvement of Theorem G. Also it provide an IM-analogues result of Theorem H.

We now explain the following definitions and notations which are used in the paper.

Definition 1.5. [8] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer p we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function. Analoguesly we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 1.6. Let a be any value in the extended complex plane, and let $k(\geq 0)$ be an integer. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq k).$$

Definition 1.7. [8] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.8. [1] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ ($\overline{N}_L(r,a;g)$) the reduced counting function of those a-points of f and g, where $p > q \ge 1$ ($q > p \ge 1$).

Definition 1.9. [8] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Then

$$\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g).$$

2 Lemmas

Let F and G be two nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. *[20] Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \le T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2.1}$$

$$N_p\left(r,0;f^{(k)}\right) \le k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2.2}$$

By the similar arguments to the proof of Lemma 3 [16] we get the following lemma.

Lemma 2.2. *Let F, G be two nonconstant meromorphic functions sharing 1 CM and ∞ IM, and assume that $H \not\equiv 0$. Then*
(i) $T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + S(r,F) + S(r,G)$;
(ii) $T(r,G) \le N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;G) + S(r,F) + S(r,G)$.

Lemma 2.3. *[11] Let F, G be two nonconstant meromorphic functions that share 1, ∞ IM and $H \not\equiv 0$. Then*
(i) $T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G)$;
(ii) $T(r,G) \le N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 3\overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + \overline{N}(r,0;F) + 2\overline{N}(r,0;G) + S(r,F) + S(r,G)$.

Lemma 2.4. *[15] Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0,1,2,...,n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.5. *[7] Suppose that f is a nonconstant meromorphic function, $k \ge 2$ is an integer. If*

$$N(r,\infty;f) + N(r,0;f) + N(r,0;f^{(k)}) = S\left(r,\frac{f'}{f}\right),$$

then $f = e^{az+b}$, where $a(\ne 0)$, b are constants.

The following lemma can be proved in the line of the proof of Lemma 9 [21].

Lemma 2.6. *Let f and g be two transcendental meromorphic functions, $p(z)$ be a nonconstant polynomial of degree l, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers with $n > k+2l$. If f, g share ∞ IM and*

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} = p^2(z),$$

where $P(w)$ is same as in Theorem H, then $P(w)$ is reduced to a nonzero monomial, namely $P(w) = a_iw^i \not\equiv 0$ for some $i \in \{0, 1, ..., m\}$.

Lemma 2.7. *Let f and g be two nonconstant meromorphic functions, $p(z)$ be a nonconstant polynomial of degree l, and let n, m and k be three positive integers with $n > k+2l$. Futher assume that either k, l are coprime or $k > l$ when $l \geq 2$. If f, g share ∞ IM and*

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} = p^2(z), \tag{2.3}$$

where $P(z) = a_mz^m$ or $P(z) = c_0$, then $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $a_m^2((n+m)b)^2(b_1b_2)^{n+m} = -1$ or $c_0^2(nb)^2(b_1b_2)^n = -1$ and $Q(z)$ is same as in Theorem 1.1.

Proof. Let $P(z) = a_mz^m$. The case $P(z) = c_0$ can be proved similarly. First we assume that $k = 1$. Then (2.3) becomes

$$(a_mf^{n+m})'(a_mg^{n+m})' = p^2(z). \tag{2.4}$$

Noting that f and g share ∞ IM and $n > k+2l$, we deduce from (2.4) that f and g have no zeros. We put

$$f = e^{\alpha}, \qquad g = e^{\beta}, \tag{2.5}$$

where α and β are two nonconstant entire functions. Therefore from (2.4) we get

$$a_m^2(n+m)^2\alpha'\beta'e^{(n+m)(\alpha+\beta)} = p^2(z).$$

From this it follows that α, β must be polynomials and $\alpha+\beta \equiv k_1$, where k_1 is a constant. Thus $deg(\alpha) = deg(\beta)$. Therefore $\alpha' + \beta' \equiv 0$ and

$$a_m^2(n+m)^2\alpha'\beta'e^{(n+m)k_1} = p^2(z).$$

Simplifying we obtain $\alpha' = bp(z)$ and $\beta' = -bp(z)$, where $b(\neq 0)$ is a constant. This gives $\alpha = bQ(z)+d_1$ and $\beta = -bQ(z)+d_2$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = p(z)$ and d_1, d_2 are constants. Therefore

$$f = b_1e^{bQ(z)}, \qquad g = b_2e^{-bQ(z)},$$

where b_1, b_2 and b are three constants satisfying

$$a_m^2((n+m)b)^2(b_1b_2)^{n+m} = -1.$$

If $k \geq 2$ then (2.3) becomes

$$(a_mf^{n+m})^{(k)}(a_mg^{n+m})^{(k)} = p^2(z). \tag{2.6}$$

Since f and g are transcendental meromorphic functions sharing ∞ IM , from (2.6) we obtain

$$N\left(r, 0; (a_mf^{n+m})^{(k)}\right) = O\{\log r\}.$$

From this and (2.5) we get

$$N(r, \infty; a_mf^{n+m}) + N(r, 0; a_mf^{n+m}) + N\left(r, 0; (a_mf^{n+m})^{(k)}\right) = O\{\log r\}.$$

Suppose that α is a transcendental entire function. Then by Lemma 2.5 we deduce that α is a polynomial, a contradiction. Next we assume that α, β are polynomials of degree p_1 and p_2 respectively. If $p_1 = p_2 = 1$, then

$$f = e^{Az+B}, \qquad g = e^{Cz+D},$$

where $A(\neq 0)$, B, $C(\neq 0)$ and D are constants. So from (2.6) we obtain

$$a_m^2 (AC)^k (n+m)^{2k} e^{(n+m)\{(A+C)z+(B+D)\}} = p^2(z),$$

which is impossible. Thus $\max\{p_1, p_2\} > 1$. We assume that $p_1 > 1$. Then $(\lambda f^{n+m})^{(k)} = Q_1 e^{(n+m)\alpha}$ and $(\lambda g^{n+m})^{(k)} = Q_2 e^{(n+m)\beta}$, where Q_1, Q_2 are polynomials of degree $k(p_1-1)$ and $k(p_2-1)$ respectively. Therefore from (2.6) we obtain $\alpha+\beta \equiv k_2$, a constant, and therefore $p_1 = p_2$ and $k(p_1-1) = l$. This shows that $l \geq k \geq 2$, contradicting with the assumption that k, l are prime to each other. This proves the lemma. □

Lemma 2.8. *[18] Let f and g be two nonconstant meromorphic functions that share 1 IM. Then*

$$\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).$$

The similar result holds for g also.

Lemma 2.9. *[17] Suppose that F and G be two nonconstant meromorphic functions, and*

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right). \tag{2.7}$$

If F, G share ∞ IM and $V \equiv 0$, then $F \equiv G$.

Lemma 2.10. *Let f and g be two nonconstant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers. Suppose that V is given as in (2.7), where $F = \frac{(f^nP(f))^{(k)}}{p(z)}$, $G = \frac{(g^nP(g))^{(k)}}{p(z)}$, $P(w)$, $p(z)$ are defined as in Theorem 1.1. If $V \not\equiv 0$, F and G share 1 CM, f and g share ∞ IM, then the poles of F and G are zeros of V and*

$$\begin{aligned}
\big(n+m^*-k-1\big)\overline{N}(r,\infty;f\mid\geq 1) &= \big(n+m^*-k-1\big)\overline{N}(r,\infty;g\mid\geq 1)\\
&\leq (k+m^*+1)\{T(r,f)+T(r,g)\}\\
&\quad +S(r,f)+S(r,g),
\end{aligned}$$

where m^* is defined as in Theorem 1.1.

Proof. We note that the order of the possible poles of F and G are at least $n+m^*+k$ as f and g share ∞ IM. Thus F and G share $(\infty, n+m^*+k-1)$. Now using the Milloux theorem (See [7], p. 55) and Lemma 2.4, we obtain from the definition of V that

$$m(r,V) = S(r,f) + S(r,g).$$

Thus using (2.2) we get

$$\begin{aligned}
\big(n+m^*+k-1\big)\overline{N}(r,\infty;f\mid\geq 1) &= \big(n+m^*+k-1\big)\overline{N}(r,\infty;g\mid\geq 1)\\
&= \big(n+m^*+k-1\big)\overline{N}\big(r,\infty;F\mid\geq n+m^*+k\big)\\
&\leq N(r,0;V)\\
&\leq T(r,V)+O(1)\\
&\leq N(r,\infty;V)+m(r,V)+O(1)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,f)+S(r,g)\\
&\leq N_{k+1}(r,0;f^nP(f))+k\overline{N}(r,\infty;f\mid\geq 1)\\
&\quad +N_{k+1}(r,0;g^nP(g))+k\overline{N}(r,\infty;g\mid\geq 1)\\
&\quad +S(r,f)+S(r,g).
\end{aligned}$$

This gives

$$\begin{aligned}(n+m^*-k-1)\overline{N}(r,\infty;f\mid\geq 1) &= (n+m^*-k-1)\overline{N}(r,\infty;g\mid\geq 1)\\ &\leq (k+m^*+1)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

This completes the proof of the lemma. □

Arguing similarly as in the proof of Lemma 2.10 above and using Lemma 2.8 we obtain the following lemma.

Lemma 2.11. *Suppose that f and g be two nonconstant meromorphic functions. Let V be given by (2.7), F, G are defined as in Lemma 2.10 and $V \not\equiv 0$. If f and g share ∞ IM, F and G share 1 IM, then the poles of F and G are zeros of V and*

$$\begin{aligned}(n+m^*-3k-3)\overline{N}(r,\infty;f\mid\geq 1) &= (n+m^*-3k-3)\overline{N}(r,\infty;g\mid\geq 1)\\ &\quad \leq 2(k+m^*+1)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g),\end{aligned}$$

where m^* is defined as in Theorem 1.1.

Lemma 2.12. *Let f and g be two transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ be three integers. Suppose that F and G are defined as in Lemma 2.10. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n\leq 3k+m^*+3$.*

Proof. We omit the proof since it can be carried out in the line of Lemma 2.14 [11]. □

Lemma 2.13. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b)=g^n(ag+b)$$

implies $f=g$, where a, b are two nonzero constants.

Proof. The proof of the lemma can be carried out in the line of Lemma 6 [10]. Here we omit the details. □

3 Proof of the Theorem

Proof of Theorem 1.3. We discuss the following three cases separately.
Case (i) Let $P(z)=a_mz^m+a_{m-1}z^{m-1}+...+a_1z+a_0$ is not a monomial. Suppose that F and G are defined as in Lemma 2.10. Then F, G are transcendental meromorphic functions that share $(1,0)$ and $(\infty,n+m+k-1)$. Therefore

$$\overline{N}_*(r,\infty;F,G)\leq\overline{N}(r,\infty;F\mid\geq n+m+k)=\overline{N}(r,\infty;f\mid\geq 1).$$

We assume that $H\not\equiv 0$. Then $F\not\equiv G$. So from Lemma 2.9 we have $V\not\equiv 0$. From Lemma 2.4 and (2.1) we obtain

$$\begin{aligned}N_2(r,0;F) &\leq N_2\left(r,0;(f^nP(f))^{(k)}\right)+S(r,f)\\ &\leq T\left(r,(f^nP(f))^{(k)}\right)-(n+m)T(r,f)+N_{k+2}(r,0;f^nP(f))\\ &\quad +S(r,f)\\ &\leq T(r,F)-(n+m)T(r,f)+N_{k+2}(r,0;f^nP(f))\\ &\quad +O\{\log r\}+S(r,f).\end{aligned}\tag{3.1}$$

Similarly

$$N_2(r,0;G) \le T(r,G) - (n+m)T(r,g) + N_{k+2}(r,0;g^nP(g)) + O\{\log r\} + S(r,g). \tag{3.2}$$

Again by (2.2) we have

$$N_2(r,0;F) \le N_{k+2}(r,0;f^nP(f)) + k\overline{N}(r,\infty;f) + S(r,f), \tag{3.3}$$

and

$$N_2(r,0;G) \le N_{k+2}(r,0;g^nP(g)) + k\overline{N}(r,\infty;g) + S(r,g). \tag{3.4}$$

From (3.1) and (3.2) we get

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} &\le T(r,F)+T(r,G)+N_{k+2}(r,0;f^nP(f)) \\ &\quad +N_{k+2}(r,0;g^nP(g)) - N_2(r,0;F) \\ &\quad -N_2(r,0;G) + O\{\log r\} + S(r,f) + S(r,g). \end{aligned} \tag{3.5}$$

Then using Lemma 2.3, Lemma 2.4, (3.3) and (3.4) we obtain from (3.5)

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} &\le N_2(r,0;F)+N_2(r,0;G)+5\overline{N}(r,\infty;F)+5\overline{N}(r,\infty;G) \\ &\quad +2\overline{N}_*(r,\infty;F,G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G) \\ &\quad +N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g)) \\ &\quad +O\{\log r\}+S(r,f)+S(r,g) \\ &\le 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g)) \\ &\quad +3N_{k+1}(r,0;f^nP(f))+3N_{k+1}(r,0;g^nP(g)) \\ &\quad +(4k+5)\overline{N}(r,\infty;f)+(4k+5)\overline{N}(r,\infty;g) \\ &\quad +2\overline{N}_*(r,\infty;F,G)+O\{\log r\}+S(r,f)+S(r,g) \\ &\le (5k+5m+7)\{T(r,f)+T(r,g)\}+(4k+6)(\overline{N}(r,\infty;f) \\ &\quad +\overline{N}(r,\infty;g))+O\{\log r\}+S(r,f)+S(r,g). \end{aligned} \tag{3.6}$$

We note that as f and g are transcendental meromorphic functions

$$\log r = o\{T(r,f)\}.$$

Therefore using Lemma 2.4 and Lemma 2.11 we deduce from (3.6)

$$\begin{aligned}(n-9k-4m-7)\{T(r,f)+T(r,g)\} &\le 12\overline{N}(r,\infty;f\mid\ge 1) \\ &\le \frac{24(k+m+1)}{n+m-3k-3}\{T(r,f)+T(r,g)\} \\ &\quad +S(r,f)+S(r,g). \end{aligned}$$

This gives

$$\begin{aligned}&[(n-9k-4m-7)(n+m-3k-3)-24(k+m+1)]\{T(r,f)+T(r,g)\} \\ &\qquad\qquad \le S(r,f)+S(r,g), \end{aligned}$$

a contradiction as $n > \max\{9k+4m+11, k+2l\}$.
Next we assume that $H=0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{3.7}$$

where $A(\neq 0)$, B are constants. We now discuss the following three subcases.
Subcase (i) Let $B \neq 0$ and $A = B$. Then from (3.7) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.8}$$

If $B = -1$, then from (3.8) we obtain

$$FG = 1,$$

i.e.,

$$(f^n P(f))^{(k)}(g^n P(g))^{(k)} = p^2(z),$$

a contradiction by Lemma 2.6.
If $B \neq -1$, from (3.8), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r, \infty; G) + S(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned}$$

Using (2.1) and (2.2) we obtain from above inequality

$$\begin{aligned} T(r, G) \leq\ & N_{k+1}(r, 0; f^n P(f)) + k\overline{N}(r, \infty; f) + T(r, G) + N_{k+1}(r, 0; g^n P(g)) \\ & -(n+m)T(r, g) + \overline{N}(r, \infty; g) + O\{\log r\} + S(r, g). \end{aligned}$$

Hence

$$(n+m)T(r, g) \leq (2k+m+1)T(r, f) + (k+m+2)T(r, g) + S(r, g).$$

Similarly

$$(n+m)T(r, f) \leq (k+m+2)T(r, f) + (2k+m+1)T(r, g) + S(r, g).$$

Combining the above two inequality we obtain

$$(n-3k-m-3)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

a contradiction as $n > \max\{9k+4m+11, k+2l\}$.
Subcase (ii) Let $B \neq 0$ and $A \neq B$. Then from (3.7) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding as in Subcase (i) we arrive at a contradiction.
Subcase (iii) Let $B = 0$ and $A \neq 0$. Then (3.7) gives $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. Using Lemma 2.12 we have $n \leq 3k+m+3$, a contradiction. Thus $A = 1$ and hence $F = G$. Then

$$[f^n P(f)]^{(k)} = [g^n P(g)]^{(k)}.$$

Integrating we get

$$[f^n P(f)]^{(k-1)} = [g^n P(g)]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using Lemma 2.12 we deduce that $n \leq 3k+m$, a contradiction. Thus $c_{k-1} = 0$. Repeating the process k-times, we obtain

$$f^n P(f) = g^n P(g).$$

That is

$$f^n(a_m f^m + ... + a_1 f + a_0) = g^n(a_m g^m + ... + a_1 g + a_0). \tag{3.9}$$

When $m=1$ the theorem follows from Lemma 2.13. When $m\geq 2$, let $h=\frac{f}{g}$. If h is a constant, putting $f=gh$ in (3.9) we get

$$a_m g^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+...+a_0g^n(h^n-1)=0,$$

which implies $h^d=1$, where $d=\gcd(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,...,m\}$. Thus $f=tg$ for a constant t such that $t^d=1$, $d=\gcd(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i\in\{0,1,...,m\}$.

If h is not a constant, then from (3.9) we see that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by (1.1).

Case (ii) Now we assume that $P(z)=a_mz^m$, where a_m $(\neq 0)$ is a complex constant. Let $F=\frac{(a_mf^{n+m})^{(k)}}{p(z)}$ and $G=\frac{(a_mg^{n+m})^{(k)}}{p(z)}$. Then F and G are transcendental meromorphic functions that share $(1,0)$ and $(\infty,n+m+k-1)$. Proceeding in the like manner as Case (i) above we obtain either $FG=1$ or $F=G$.

If $FG=1$, then

$$(a_mf^{n+m})^{(k)}(a_mg^{n+m})^{(k)}=p^2(z).$$

So by Lemma 2.8 we obtain $f(z)=b_1e^{bQ(z)}$, $g(z)=b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $a_m^2((n+m)b)^2(b_1b_2)^{n+m}=-1$ and $Q(z)$ is same as in Theorem 1.1. If $F=G$, then using Lemma 2.12 and proceeding similarly as in Case (i) we obtain $f=tg$ for a constant t such that $t^{n+m}=1$.

Case (iii) Let $P(z)=c_0$ where c_0 is a complex constant. Taking $F=\frac{(c_0f^n)^{(k)}}{p(z)}$, $G=\frac{(c_0g^n)^{(k)}}{p(z)}$ and arguing similarly as in Case (ii) we obtain either $f(z)=b_1e^{bQ(z)}$, $g(z)=b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $c_0^2(nb)^2(b_1b_2)^n=-1$, $Q(z)$ is same as in Theorem 1.1 or $f=tg$ for a constant t satisfying $t^n=1$. This completes the proof of the theorem. □

Proof of Theorem 1.1. Let $P(z)=a_mz^m+a_{m-1}z^{m-1}+...+a_1z+a_0$ is not a monomial. Suppose that F and G are defined as in Lemma 2.10. Then F, G are transcendental meromorphic functions that share $(1,\infty)$ and $(\infty,n+m+k-1)$. Therefore

$$\overline{N}_*(r,\infty;F,G)\leq\overline{N}(r,\infty;F\mid\geq n+m+k)=\overline{N}(r,\infty;f\mid\geq 1).$$

If possible, we assume that $H\not\equiv 0$. Then $F\not\equiv G$. So from Lemma 2.9 we have $V\not\equiv 0$. Then using Lemma 2.2, Lemma 2.4, (3.3) and (3.4) we deduce from (3.5)

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} &\leq N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)+3\overline{N}(r,\infty;G)\\ &\quad +N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ &\quad +O\{\log r\}+S(r,f)+S(r,g)\\ &\leq 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ &\quad +(k+3)\overline{N}(r,\infty;f)+(k+3)\overline{N}(r,\infty;g)\\ &\quad +O\{\log r\}+S(r,f)+S(r,g)\\ &\leq (2k+2m+4)\{T(r,f)+T(r,g)\}+(k+3)(\overline{N}(r,\infty;f)\\ &\quad +\overline{N}(r,\infty;g))+O\{\log r\}+S(r,f)+S(r,g). \end{aligned} \tag{3.10}$$

Noting that f and g are transcendental meromorphic functions, using Lemma 2.4 and Lemma 2.10 we deduce from (3.10)

$$\begin{aligned}(n-2k-m-4)\{T(r,f)+T(r,g)\} &\leq 2(k+3)\overline{N}(r,\infty;f\mid\geq 1)\\ &\leq \frac{2(k+3)(k+m+1)}{n+m-k-1}\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

This gives

$$\begin{aligned}&[(n-2k-m-4)(n+m-k-1)-2(k+3)(k+m+1)]\{T(r,f)+T(r,g)\}\\ &\qquad\leq S(r,f)+S(r,g),\end{aligned}$$

which leads to a contradiction as $n > \max\{3k+m+6, k+2l\}$. If $H = 0$, arguing similarly as in the proof of Theorem 1.3 above we obtain the conclusions of Theorem 1.1. The case when $P(z) = a_m z^m$, where a_m $(\neq 0)$ is a complex constant or $P(z) = c_0$ the theorem follows from Case (ii) and Case (iii) of Theorem 1.3. This proves the theorem. □

References

[1] T.C. Alzahary and H.X. Yi, Weighted value sharing and a question of I. Lahiri, *Complex Var. Theory Appl.,* **49**, 1063–1078 (2004).

[2] A. Banerjee, Uniqueness of certain nonlinear differential polynomials sharing 1-points, *Kyungpook Math. J.,* **51**, 43–58 (2011).

[3] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoam,* **11**, 355–373 (1995).

[4] H.H. Chen and M.L. Fang, On the value distribution of $f^n f'$, *Sci. China Ser. A* **38**, 789–798 (1995).

[5] M.L. Fang, A note on a problem of Hayman, *Analysis(Munich),* **20**, 45–49 (2000).

[6] M.L. Fang and H.L. Qiu, Meromorphic functions that share fixed points, *J. Math. Anal. Appl.,* **268**, 426–439 (2002).

[7] W.K. Hayman, Meromorphic Functions, *The Clarendon Press,* Oxford (1964).

[8] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, *Complex Var. Theory Appl.,* **46**, 241–253 (2001).

[9] I. Lahiri, Value distribution of certain differential polynomials, *Int. J. Math. Math. Sc.,* **28**, 83–91 (2001).

[10] I. Lahiri, On a question of Hong Xun Yi, *Arch Math. (Brno),* **38**, 119–128 (2002).

[11] P. Sahoo, Meromorphic functions that share fixed points with finite weights, *Bull. Math. Anal. Appl.,* **2**, 106–118 (2010).

[12] P. Sahoo, Uniqueness and weighted sharing of meromorphic functions, *Ann. Polon. Math.,* **100**, 127–145 (2011).

[13] J. Wang, W. Lu and Y. Chen, Value sharing of meromorphic functions and their derivatives, *Applied Math. E-Notes,* **11**, 91–100 (2011).

[14] J.F. Xu, F. Lu and H.X. Yi, Fixed points and uniqueness of meromorphic functions, *Comput. Math. Appl.,* **59**, 9–17 (2010).

[15] C.C. Yang, On deficiencies of differential polynomials II, *Math. Z.,* **125**, 107–112 (1972).

[16] C.C. Yang and X.H. Hua, Uniqueness and value sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.* **22**, 395–406 (1997).

[17] H.X. Yi, Meromorphic functions that share three sets, *Kodai Math. J.,* **20**, 22–32 (1997).

[18] H.X. Yi, Meromorphic functions that shares one or two values II, *Kodai Math. J.,* **22**, 264–272 (1999).

[19] H.X. Yi and C.C. Yang, Uniqueness Theory of Meromorphic Functions, *Science Press,* Beijing, 1995.

[20] J.L. Zhang and L.Z. Yang, Some results related to a conjecture of R. Bruck, *J. Inequal. Pure Appl. Math.,* **8**, Art. 18 (2007).

[21] X.B. Zhang and J.F. Xu, Uniqueness of meromorphic functions sharing a small function and its applications, *Comput. Math. Appl.,* **61**, 722–730 (2011).

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