**B—TANGENT DEVELOPABLE OF BIHARMONIC B—SLANT HELICES ACCORDING TO BISHOP FRAME IN**

**THE $\widetilde{SL}_2(\mathbb{R})$**

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**Abstract.** In this paper, we study $B$—tangent developable of biharmonic $B$—slant helices in the $\widetilde{SL}_2(\mathbb{R})$. We characterize $B$—tangent developable of biharmonic $B$—slant helices in terms of their curvature and torsion. Finally, we find out their explicit equations.

1 **Introduction**

Modeling developable surfaces through approximation is attractive as designers do not have to concern themselves with developability constraints during the modeling process. Ideally, they can freely utilize all sorts of modeling tools (e.g., blends, fillets) and then rely on an approximation algorithm to yield a developable result. In practice though, the approximation approach is highly restricted since the methods can only succeed if the original input surfaces already have fairly small Gaussian curvature. Moreover, in most cases the final result is not analytically developable. While this is not a problem for applications such as texture-mapping, it can be problematic for manufacturing, where the surfaces need to be realised from planar patterns (e.g., sewing).

This study is organised as follows: Firstly, we study tangent developable of biharmonic $B$—slant helices in the $\widetilde{SL}_2(\mathbb{R})$. Secondly, we characterize tangent developable of biharmonic $B$—slant helices in terms of their curvature and torsion. Finally, we find out their explicit equations.

2 **$\widetilde{SL}_2(\mathbb{R})$**

We identify $\widetilde{SL}_2(\mathbb{R})$ with $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ endowed with the metric

$$g = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$  

The following set of left-invariant vector fields forms an orthonormal basis for $\widetilde{SL}_2(\mathbb{R})$

$$e_1 = \frac{\partial}{\partial x}, e_2 = z \frac{\partial}{\partial y} - \frac{\partial}{\partial x}, e_3 = z \frac{\partial}{\partial z}. \quad (2.1)$$

The characterizing properties of $g$ defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0.$$

The Riemannian connection $\nabla$ of the metric $g$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$
which is known as Koszul’s formula.

Using the Koszul’s formula, we obtain

$$\nabla e_i e_1 = 0, \quad \nabla e_i e_2 = \frac{1}{2} e_1, \quad \nabla e_i e_3 = -\frac{1}{2} e_2, \quad \nabla e_i e_4 = \frac{1}{2} e_3, \quad \nabla e_i e_5 = -\frac{1}{2} e_1 - e_2,$$

(2.2)

Moreover we put

$$R_{ijkl} = R(i, j) e_k, \quad R_{ijkl} = R(i, e_j) e_k,$$

where the indices $i, j, k$ and $l$ take the values 1, 2 and 3

$$R_{1212} = R_{1313} = \frac{1}{4}, \quad R_{2323} = -\frac{7}{4}.$$

(2.3)

3 Biharmonic $B-$Slant Helices in $\hat{\mathcal{SL}}_2(\mathbb{R})$

Assume that $\{T, N, B\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B,$$

(3.1)

$$\nabla_T B = -\tau N,$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(T, T) = 1, \quad g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(N, N) = 1, \quad g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(B, B) = 1,$$

(3.2)

$$g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(T, N) = g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(T, B) = g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(N, B) = 0.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\nabla_T T = k_1 M_1 + k_2 M_2,$$

$$\nabla_T M_1 = -k_1 T,$$

$$\nabla_T M_2 = -k_2 T,$$

(3.3)

where

$$g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(T, T) = 1, \quad g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(M_1, M_1) = 1, \quad g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(M_2, M_2) = 1,$$

(3.4)

$$g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(T, M_1) = g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(T, M_2) = g_{\hat{\mathcal{SL}}_2(\mathbb{R})}(M_1, M_2) = 0.$$

Here, we shall call the set $\{T, M_1, M_1\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures and $\varphi(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \varphi'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$k_1 = \kappa(s) \cos \varphi(s),$$

$$k_2 = \kappa(s) \sin \varphi(s).$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$T = T^1 e_1 + T^2 e_2 + T^3 e_3, M_1 = M^1_1 e_1 + M^2_1 e_2 + M^3_1 e_3,$$

$$M_2 = M^1_2 e_1 + M^2_2 e_2 + M^3_2 e_3.$$

(3.5)

**Theorem 3.1.** ([9]) $\gamma : I \rightarrow \hat{\mathcal{SL}}_2(\mathbb{R})$ is a biharmonic curve according to Bishop frame if and only if

$$k_1^2 + k_2^2 = \text{constant} \neq 0, k_1'' - [k_1^2 + k_2^2] k_1 = -k_1 \left[ \frac{15}{4} M_2^1 - \frac{1}{4} \right] - 2k_2 M_1^1 M_2^1,$$

(3.6)
\[ k'' - [k_1^2 + k_2^2] k_2 = 2k_1M_1^1M_2^1 - k_3 \left[ \frac{15}{4} M_1^1 - \frac{1}{4} \right]. \]

**Definition 3.2.** A regular curve \( \gamma : I \rightarrow \widetilde{SL}_2(\mathbb{R}) \) is called a slant helix provided the unit vector \( M_1 \) of the curve \( \gamma \) has constant angle \( \mathfrak{W} \) with some fixed unit vector \( u \), that is
\[ g_{\widetilde{SL}_2(\mathbb{R})} (\gamma(s), u) = \cos \mathfrak{W} \text{ for all } s \in I. \] (3.7)

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as \( B \)-slant helix.

We shall also use the following lemma.

**Lemma 3.3.** Let \( \gamma : I \rightarrow \widetilde{SL}_2(\mathbb{R}) \) be a unit speed curve. Then \( \gamma \) is a \( B \)-slant helix if and only if
\[ k_1 = -k_2 \tan \mathfrak{W}. \] (3.8)

**Theorem 3.4.** ([9]) Let \( \gamma : I \rightarrow \widetilde{SL}_2(\mathbb{R}) \) be a unit speed non-geodesic biharmonic \( B \)-slant helix. Then, the parametric equations of \( \gamma \) are
\[
\begin{align*}
x(s) &= \frac{1}{\Omega_1} \cos \mathfrak{W} \sin [\Omega_1 s + \Omega_2] + \frac{1}{\Omega_1} \cos \mathfrak{W} \cos [\Omega_1 s + \Omega_2] + \Omega_4, \\
y(s) &= -\frac{\Omega_3}{\Omega_1^2 + \sin^2 \mathfrak{W}} \cos \mathfrak{W} e^{-\sin \mathfrak{W} s} (\cos [\Omega_1 s + \Omega_2] + \sin \mathfrak{W} \sin [\Omega_1 s + \Omega_2]) + \Omega_5, \\
z(s) &= \Omega_3 e^{-\sin \mathfrak{W} s},
\end{align*}
\]
where \( \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5 \) are constants of integration.

We can use Mathematica in above theorem, yields
4 \( B \)-Tangent Developable Surfaces of \( B \)-Slant Helices in \( \mathcal{S}L_2(\mathbb{R}) \)

To separate a tangent developable according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for this surface as \( B \)-tangent developable.

The purpose of this section is to study \( B \)-tangent developable surfaces of \( B \)-slant helices in \( \mathcal{S}L_2(\mathbb{R}) \).

The \( B \)-tangent developable of \( \gamma \) is a ruled surface

\[
\mathcal{R}(s, u) = \gamma(s) + u\gamma'(s)
\]

Theorem 4.1. Let \( \gamma : I \rightarrow \mathcal{S}L_2(\mathbb{R}) \) is a unit speed non-geodesic biharmonic \( B \)-slant helix in \( \mathcal{S}L_2(\mathbb{R}) \). Then, the parametric equations of \( B \)-tangent developable of \( \gamma \) are

\[
\begin{align*}
\mathbf{x}_\mathcal{R}(s, u) &= \frac{1}{\Omega_1} \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2] + \frac{1}{\Omega_1} \cos \mathcal{W} \cos[\Omega_1 s + \Omega_2] + u[\cos \mathcal{W} \cos[\Omega_1 s + \Omega_2] - \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2]] + \Omega_4, \\
\mathbf{y}_\mathcal{R}(s, u) &= -\frac{\Omega_1}{\Omega_1^2 + \sin^2 \mathcal{W}} \cos \mathcal{W} e^{-\sin \mathcal{W}}(\Omega_1 \cos[\Omega_1 s + \Omega_2] + \sin \mathcal{W} \sin[\Omega_1 s + \Omega_2]) + u \Omega_3 e^{-\sin \mathcal{W}} \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2] + \Omega_5, \\
\mathbf{z}_\mathcal{R}(s, u) &= \Omega_3 e^{-\sin \mathcal{W}} - \Omega_3 e^{-\sin \mathcal{W}} \sin \mathcal{W},
\end{align*}
\]

where \( \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5 \) are constants of integration.

Proof. By the Bishop formula, we have the following equation

\[
\mathbf{T} = \cos \mathcal{W} \cos[\Omega_1 s + \Omega_2] \mathbf{e}_1 + \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2] \mathbf{e}_2 - \sin \mathcal{W} \mathbf{e}_3.
\]

Using (2.1) in (4.3), we obtain

\[
\mathbf{T} = (\cos \mathcal{W} \cos[\Omega_1 s + \Omega_2] - \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2], \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2], -z \sin \mathcal{W}).
\]

In terms of (2.4) and (4.4), we may give:

\[
\mathbf{T} = (\cos \mathcal{W} \cos[\Omega_1 s + \Omega_2] - \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2], \Omega_3 e^{-\sin \mathcal{W}} \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2], -\Omega_3 e^{-\sin \mathcal{W}} \sin \mathcal{W}).
\]

Consequently, the parametric equations of \( \mathcal{R} \) can be found from (4.1), (4.5). This concludes the proof of Theorem.

We can prove the following interesting main result.

Theorem 4.2. Let \( \gamma : I \rightarrow \mathcal{S}L_2(\mathbb{R}) \) be a unit speed non-geodesic biharmonic \( B \)-slant helix and \( \mathcal{R} \) its \( B \)-tangent developable surface in \( \mathcal{S}L_2(\mathbb{R}) \). Then the equation of \( B \)-tangent developable is

\[
\begin{align*}
\mathcal{R}(s, u) &= \left[\frac{1}{\Omega_1} \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2] + \frac{1}{\Omega_1} \cos \mathcal{W} \cos[\Omega_1 s + \Omega_2] + \Omega_4\right] + u \left[\frac{\Omega_3 \sin \mathcal{W} s}{\Omega_3} + \Omega_5 \sin \mathcal{W} s + \sin \mathcal{W} \cos[\Omega_1 s + \Omega_2]\right] \mathbf{e}_1 \\
&\quad + \sin \mathcal{W} \sin[\Omega_1 s + \Omega_2] + \frac{\Omega_3 \sin \mathcal{W} s}{\Omega_3} + u \cos \mathcal{W} \cos[\Omega_1 s + \Omega_2]\right] \mathbf{e}_2 \\
&\quad - \frac{1}{\Omega_3^2 + \sin^2 \mathcal{W}} \cos \mathcal{W} \sin[\Omega_1 s + \Omega_2] + \sin \mathcal{W} \sin[\Omega_1 s + \Omega_2] + \frac{\Omega_3 \sin \mathcal{W} s}{\Omega_3} \mathbf{e}_1,
\end{align*}
\]

where \( \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5 \) are constants of integration.
where $Q_1, Q_2, Q_3, Q_4, Q_5$ are constants of integration.

**Proof.** We assume that $\gamma$ is a unit speed $B$–slant helix. Substituting (2.4) to (4.2), we have (4.6). Thus, the proof is completed.

Thus, we proved the following:

**Corollary 4.3.** Let $\gamma : I \rightarrow S\mathcal{L}_2(\mathbb{R})$ be a unit speed non-geodesic biharmonic $B$–slant helix and $R$ its $B$–tangent developable surface in $S\mathcal{L}_2(\mathbb{R})$. Then, normal of $B$–tangent developable of $\gamma$ is

$$n_R = \left[uk_1 \sin [Q_1 s + Q_2] - u k_2 \sin M \cos [Q_1 s + Q_2]e_1 + [-u k_1 \cos [Q_1 s + Q_2] - u k_2 \sin M \sin [Q_1 s + Q_2]]e_2ight.$$

$$\left.- u k_2 \cos M e_3,\right.$$  

where $Q_1, Q_2$ are constants of integration.

**Proof.** Assume that $n_R$ be the normal vector field on $B$–tangent developable defined by

$$n_R = R_s \wedge R_u. \tag{4.8}$$

From Definition 3.2, we have the following equation

$$M_1 = \sin M \cos [Q_1 s + Q_2] e_1 + \sin M \sin [Q_1 s + Q_2] e_2 + \cos M e_3, \tag{4.9}$$

where $Q_1, Q_2$ are constants of integration.

On the other hand, using Bishop formulas (3.3) and (2.1), we have

$$M_2 = \sin [Q_1 s + Q_2] e_1 - \cos [Q_1 s + Q_2] e_2. \tag{4.10}$$

Substituting (4.9) and (4.10) to (4.8), we have (4.7). This concludes the proof of corollary.

Finally, the obtained parametric equations are illustrated in Figures 1 and 2:
References


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