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Cyclic codes of arbitrary length over $F_q + uF_q + u^2F_q + \ldots + u^{k-1}F_q$

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Abstract. In this paper, we study the structure of cyclic codes of an arbitrary length n over the ring $F_q + uF_q + u^2F_q + \ldots + u^{k-1}F_q$, where $u^k = 0$ and q is a power of prime. Also we study the rank for these codes, and we find their minimal spanning sets. This study is a generalization and extension of the works in references [9] and [12], the dual codes over the ring $F_q + uF_q + u^2F_q$, where $u^3 = 0$ are studied as well

1 Introduction

Among the four rings of four elements, the Galois field F_4 and more recently the ring of integers modulo four Z_4 are the most used in coding theory. Z_4 -codes are renowned for producing good nonlinear codes by the Gray map, namely Kerdok, preparata or Goethals codes. The structure of cyclic codes over rings of odd length n has been discussed in Bonnecaze and Udaya [4], Calderbank [5], Dougherty and Shiromoto [8], and van Lint [13]. Calderbank and Sloane [6], and Pless [11] presented a complete structure of cyclic codes over Z_4 of odd length. In [3], Blackford studied cyclic codes of length $n = 2k$ when k is odd. The cyclic codes over Z_4 of length a power of 2 are studied in Abualrub and Oehmke [2]. They showed that the ring $Z_4[x]/\langle x^n - 1\rangle$ is not a principal ideal ring and hence ideals may have more than one generator. Ping Li and Shixin Zhu in [12], studied cyclic codes of arbitrary length over the ring $F_q + uF_q$, with $u^2 = 0$ and F_q is a finite field of order q where q is a power of prime.

Let R_k be the ring $F_q + uF_q + u^2F_q + \ldots + u^{k-1}F_q$ with $u^k = 0,$ where q is a power of prime p.

In [1], Abualrub and Siap studied cyclic codes of an arbitrary length n over $F_2 + uF_2 = \{0, 1, u, u + 1\}$ where $u^2 = 0$ and over $F_2 + uF_2 + u^2F_2 = \{0, 1, u, u + 1, u^2, 1 + u^2, 1 + u + u^2, u + u^2\}$ where $u^3 = 0$ and $F_2 = \{0, 1\}$. In [9], the authors Mohammed Al-Ashker and Mohammed Hamoudeh extend these resultsto rings of the form $F_2 + uF_2 + u^2F_2 + \ldots + u^{k-1}F_2$ where $u^k = 0$.

A. Singh and P. kewat in [14] extend some of the results in [9] to the ring $F_p + uF_p + u^2F_p + \ldots + u^{k-1}F_p$ where $u^k = 0,$ and $F_p = \{0, 1, 2, \cdots, p - 1\}$.

In this paper, we study cyclic codes of an arbitrary length over $F_q + uF_q + u^2F_q + \cdots + u^{k-1}F_q$, where q is a power of prime p and $u^k = 0$, we also study their dual codes and find their properties over these rings. We give a unique set of generators for these codes as ideals in the ring $R_{k,n} = R_k[x]/\langle x^n - 1\rangle$. For this purpose, it is useful to obtain the divisors of $x^n - 1$, but this becomes difficult when the characteristic of the ring is not relatively prime to the length of the code, because then $x^n - 1$ does not factor uniquely over the ring $F_q + uF_q + u^2F_q + \cdots + u^{k-1}F_q$. We show that the results of [12] concerning the codes over the ring $F_q + uF_q$ with $u^2 = 0$ and of [9] concerning the codes over the ring $F_2 + uF_2 + u^2F_2 + \cdots u^{k-1}F_2$ with $u^k = 0$ are valid for $R_k = F_q + uF_q + u^2F_q + \ldots + u^{k-1}F_q$ with $u^k = 0$. The proofs of lemmas and Theorems in this paper are some what similar to those discussed in [12], [9] and slightly different from those discussed in [14]

The remaining part of this paper is organized as follows: In section 2, we give some basic definitions and results that are used in the sequel of this paper. In section 3, we study cyclic codes of an arbitrary length n over $F_q + uF_q + u^2F_q + \cdots + u^{k-1}F_q$. We find a unique set of generators for these codes. In section 4, we study the rank and find minimal spanning sets for these codes. In section 5, we study the dual codes of the codes over the ring $F_q + uF_q + u^2Fq$. In section 6, we include some examples of cyclic codes over R_k.

2 Preliminaries

Let F_q^n denote the vector space of all $n-$tuples over the finite field F_q. An (n, M) code C over F_q is a subset of F_q^n of size M. If C is a $k-$dimentional subspace of F_q^n, then we will called an $[n, k]$ linear code over F_q.
A linear code C of length n over F_q is cyclic provided that for each vector $c = (c_0c_1 \dots c_{n-2}c_{n-1})$ in C, the vector $(c_{n-1}c_0 \dots c_{n-2})$ obtained from c by the cyclic shift of coordinates $i \mapsto i + 1($ mod $n)$, is also in C.
A code of length n over a commutative ring R is a nonempty subset of R^n, and a code is linear over R if it is an $R-$submodule of R^n.
A free module C is a module with a basis (a linearly independent spanning set for C).
A linear code of length n is cyclic if it is invariant under the automorphism σ which is given by $\sigma(c_0, c_1, \dots, c_{n-1}) = (c_{n-1}, c_0, \dots, c_{n-2})$.

Definition 2.1. [7] An ideal I of a ring R is called principal if it is generated by one element. A ring R is a principal ideal ring if its ideals are principal. R is called a local ring if R has a unique maximal right (left) ideal. Furthermore, a ring R is called a right (left) chain ring if the set of all right (left) ideals of R is a chain under set-theoretic inclusion. If R is both a right and a left chain ring, we simply call R a chain ring.

Definition 2.2. The ring $R_k = F_q[u]/\langle u^k\rangle = F_q + uF_q + u^2F_q + \dots + u^{k-1}F_q$ is a commutative chain ring of q^k elements with maximal ideal uR_k, where $u^k = 0$.
Since u is nilpotent with nilpotent index k, we have

$$R_k \supset\ uR_k \supset\ u^2R_k \supset \dots \supset\ u^kR_k = 0.$$

Moreover $R_k/uR_k \cong\ F_q$ is the residue field and $|u^iR_k| = q|(u^{i+1}R_k)| = q^{k-i}$, $0 \le i \le (k-1)$.
Denote $R_1 = F_q$, $R_2 = F_q + uF_q$, $R_3 = F_q + uF_q + u^2F_q, \dots$ etc.

Definition 2.3. Let C_k be a code of length n over the ring $R_k = F_q + uF_q + u^2F_q + \dots + u^{k-1}F_q$ with $u^k = 0$, we mean an additive submodule of the R_k-module R_k^n.
A cyclic code of length n over R_k is an ideal in the ring $R_{k,n} = R_k[x]/\langle x^n - 1\rangle$.

Definition 2.4. [1] Let $c = (c_0, \dots, c_{n-1})$ and $u = (u_0, \dots, u_{n-1})$ be any two vectors over a ring. We define their inner product by

$$c.u = c_0u_0 + \dots + c_{n-1}u_{n-1}.$$

If $c.u = 0$, then c and u are said to be orthogonal. We define the dual of a cyclic code C to be the set

$$C^\perp = \{c \in R_k^n : c \cdot u = 0 \text{ for all } u \in C\}.$$

Notation: We write a for $a(x)$, g for $g(x), \dots$etc.

Proposition 2.1. [7] *Let R be a finite commutative ring, then the following conditions are equivalent:*
(i) R is a local ring and the maximal ideal M of R is principal.
(ii) R is a local principal ideal ring.
(iii) R is a chain ring.

Notation:all rings studied in this paper are commutative chain rings.

3 A generator Construction

The structure of cyclic codes over R_i depends on cyclic codes over R_{i-1} for $i = 2, 3, \dots, k$ and the structure of cyclic codes over R_2 depends on cyclic codes over $R_1 = F_q$.
By following results in [1] and [9], let C_1 be a cyclic code in $R_{k,n} = R_k[x]/\langle x^n - 1\rangle$.
Define $\psi_1 : R_k \to R_{k-1}$ by $\psi_1(a) = a$, where $u^k = 0\ mod\ q$. ψ_1 is a ring homomorphism that can be extended to a homomorphism $\phi_1 : C_1 \to\ R_{k-1,n} = R_{k-1}[x]/\langle x^n - 1\rangle$ defined by

$$\phi_1(c_0 + c_1x + \dots + c_{n-1}x^{n-1}) = \psi_1(c_0) + \psi_1(c_1)x + \dots + \psi_1(c_{n-1})x^{n-1}.$$

$$\ker \phi_1 = \{u^{k-1}r(x) : r(x) \in F_q[x]\}.$$

Let $J_1 = \{r(x) : u^{k-1}r(x) \in \ker \phi_1\}$, J_1 is an ideal in $R_{1,n} = R_1[x]/\langle x^n - 1\rangle = F_q[x]/\langle x^n - 1\rangle$ and and hence a cyclic code in $F_q[x]/\langle x^n - 1\rangle$. So $J_1 = \langle a_{k-1}(x)\rangle$ and $\ker \phi_1 = \langle u^{k-1}a_{k-1}(x)\rangle$

with $a_{k-1}(x)|(x^n-1)$.
Let C_2 be a cyclic code in $R_{k-1,n}=R_{k-1}[x]/\langle x^n-1\rangle$.
Define $\psi_2: R_{k-1}\to R_{k-2}$ by $\psi_2(a)=a$. ψ_2 is a ring homomorphism that can be extended to a homomorphism $\phi_2: C_2\to R_{k-2}[x]/\langle x^n-1\rangle$ defined by

$$\phi_2(c_0+c_1x+\ldots+c_{n-1}x^{n-1})=\psi_2(c_0)+\psi_2(c_1)x+\ldots+\psi_2(c_{n-1})x^{n-1}.$$

$$\ker\phi_2=\{u^{k-2}r(x): r(x)\in F_q[x]\}.$$

Let $J_2=\{r(x)=u^{k-2}r(x)\in\ker\phi_2\}$ is an ideal in $R_{1,n}=F_q[x]/\langle x^n-1\rangle$ and hence a cyclic code in $F_q[x]/\langle x^n-1\rangle$. So $J_2=\langle a_{k-2}(x)\rangle$ and hence $\ker(\phi_2)=\langle u^{k-2}a_{k-2}(x)\rangle$ with $a_{k-2}(x)|(x^n-1)$.
Let C_3 be a cyclic code in $R_{k-2,n}=R_{k-2}[x]/\langle x^n-1\rangle$.
Define $\psi_3: R_{k-2}\to R_{k-3}$ by $\psi_3(a)=a$. ψ_3 is a ring homomorphism that can be extended to a homomorphism $\phi_3: C_3\to R_{k-3}[x]/\langle x^n-1\rangle$. Continue in the same way as above until we define $\psi_{k-1}: R_2\to R_1=F_q$ by $\psi_{k-1}(a)=a^q$. ψ_{k-1} is a ring homomorphism because $(a+b)^q=a^q+b^q$ in R_2 and in F_q.
Extend ψ_{k-1} to a homomorphism $\phi_{k-1}: C_{k-1}\to F_q[x]/\langle x^n-1\rangle=R_{1,n}$ defined by

$$\begin{aligned}\phi_{k-1}(c_0+c_1x+\ldots+c_{n-1}x^{n-1})&=\psi_{k-1}(c_0)+\psi_{k-1}(c_1)x+\ldots+\psi_{k-1}(c_{n-1})x^{n-1}\\&=c_0^q+c_1^qx+\ldots+c_{n-1}^qx^{n-1}=c_0+c_1c+\cdots+c_nx^{n-1},\end{aligned}$$

where C_{k-1} be a cyclic code in $R_{2,n}=R_2[x]/\langle x^n-1\rangle$, where $R_2=F_q+uF_q$ with $u^2=0$ mod q.

$$\begin{aligned}\ker\phi_{k-1}&=\{ur(x): r(x)\text{ is a polynomial in }F_q[x]/\langle x^n-1\rangle\}\\&=\langle ua_1(x)\rangle\text{ with }a_1(x)\big|(x^n-1).\end{aligned}$$

The image of ϕ_{k-1} is also an ideal and hence a cyclic code over F_q generated by $g(x)$ with $g(x)\big|(x^n-1)$. The cyclic code over $R_2=F_q+uF_q$ have the form in the following lemma:

Lemma 3.1. [12] *Let C_{k-1} an arbitrary ideal of ring $R_{2,n}$ (i.e., it's an arbitrary cyclic code of arbitrary length n over ring R_2), then there only exits $a_1(x)|g(x)|x^n-1$, and the polynomials $g(x),a_1(x),p(x)$ in $F_q[x]$ with $deg a_1>deg p$, such that $C_{k-1}=<g(x)+up(x),ua_1(x)>$.*

Note that $a_1\big|(p\frac{x^n-1}{g})$ because

$$\phi_{k-1}\big(\frac{x^n-1}{g}[g+up]\big)=\phi_{k-1}\big(up\frac{x^n-1}{g}\big)=0$$

$\Rightarrow\big(up\frac{x^n-1}{g}\big)\in\ker\phi_{k-1}=\langle ua_1\rangle$. Also $ug\in\ker\phi_{k-1}$ implies $a_1(x)\big|g(x)$.

Lemma 3.2. *If $C_{k-1}=\langle g(x)+up(x),ua_1(x)\rangle$ over $R_2=F_q+uF_q$ with ($u^2=0\mod q$), and $g(x)=a_1(x)$ with $\deg g(x)=r$, then $C_{k-1}=\langle g(x)+up(x)\rangle$ and $(g+up)\big|(x^n-1)$ in $R_2[x]$.*

Proof. Since $u(g+up)=ug$ and $g=a$ with $\deg g(x)=r$, then $C_{k-1}=(g(x)+up)$ and $(g+up)|x^n-1$ in $R_2[x]$. □

Lemma 3.3. *(1) Let C_{k-2} be a cyclic code in $R_{3,n}$, then $C_{k-2}=\langle g+up_1+u^2p_2,ua_1+u^2q_1,u^2a_2\rangle$ with $a_2|a_1|g|(x^n-1)$, $a_1(x)\big|p_1(x)\big(\frac{x^n-1}{g(x)}\big)\mod q$, $a_2|q_1\big(\frac{x^n-1}{a_1}\big)$, $a_2|p_1\big(\frac{x^n-1}{g}\big)$ and $a_2\big|p_2\big(\frac{x^n-1}{g}\big)\big(\frac{x^n-1}{a_1}\big)$. We may assume that $\deg p_2<\deg a_2$, $\deg q_1<\deg a_2$, $\deg p_1<\deg a_1$.*
(2)A cyclic code over the ring $R_{3,n}$ can be written uniquely as $C_{k-2}=\langle g+up_1+u^2p_2,ua_1+u^2q_1,u^2a_2\rangle$.
(3)If $C_{k-2}=\langle g+up_1+u^2p_2,ua_1+u^2q_1,u^2a_2\rangle$ over $R_3=F_q+uF_q+u^2F_q$ with ($u^3=0$), and $a_2=g$, then $C_{k-2}=\langle g+up_1+u^2p_2\rangle$ and $\big(g+up_1+u^2p_2\big)\big|(x^n-1)$ in R_3.
(4)If n is relatively prime to q, then $C_{k-2}=\langle g,ua_1,u^2a_2\rangle=\langle g+ua_1+u^2a_2\rangle$ over R_3.

Proof. (1) Since the image of ϕ_{k-2} is an ideal in $R_{2,n}=R_2[x]/\langle x^n-1\rangle$ (where $R_2=F_q+uF_q$ with $u^2=0$), then $Im(\phi_{k-2})=\langle g(x)+up_1(x),ua_1(x)\rangle$ with $a_1(x)\big|g(x)|(x^n-1)$ and $a_1(x)\big|p_1(x)\big(\frac{x^n-1}{g(x)}\big)$. Also, $\ker(\phi_{k-2})=\langle u^2a_2(x)\rangle$ with $a_2(x)\big|(x^n-1)$. Since $u^2a_1\in\ker(\phi_{k-2})=\langle u^2a_2\rangle$, then the cyclic code C_{k-2} over $R_3=F_q+uF_q+u^2F_q$ with $u^3=0$ is $C_{k-2}=\langle g+up_1+u^2p_2,ua_1+u^2q_1,u^2a_2\rangle$ with $a_2|a_1|g|(x^n-1)$, $a_1(x)\big|p_1(x)\big(\frac{x^n-1}{g(x)}\big)\mod q$. Since $\phi_{k-2}\big(\frac{x^n-1}{a_1}(ua_1+u^2q_1)\big)=\phi_{k-2}\big(u^2q_1\frac{x^n-1}{a_1}\big)=0$. Hence $\big(u^2q_1\frac{x^n-1}{a_1}\big)\in\ker\phi_{k-2}=\langle u^2a_2\rangle$.

This implies that $a_2(x)\big|\big(q_1\frac{x^n-1}{a_1}\big)$. Similarly, we have $a_1(x)|p_1(x)\big(\frac{x^n-1}{g(x)}\big)$ mod q. further, $\phi_{k-2}\big(\big(\frac{x^n-1}{g}\big)\big(\frac{x^n-1}{a_1}\big)\big(g+up_1+u^2p_2\big)\big) = \phi_{k-2}\big(\big(\frac{x^n-1}{g}\big)\big(\frac{x^n-1}{a_1}\big)u^2p_2\big) = 0$. Thus, $a_2\big|p_2\big(\frac{x^n-1}{g}\big)\big(\frac{x^n-1}{a_1}\big)$. We may assume that $\deg p_2 < \deg a_2$, $\deg q_1 < \deg a_2$, $\deg p_1 < \deg a_1$ because if $e = (a,b)$, then $e = (a, b+de)$ for any d.
(2)The proof is similar to Lemma 6 in [1].
(3) Since $a_2 = g$, then $a_1 = a_2 = g$. From lemma 3.1. we get that $\big(g+up\big)\big|(x^n-1)$ in R_2 and $C_{k-2} = \big\langle g+up_1+u^2p_2, u^2a_2\big\rangle$. The rest of the proof is similar to lemma 3.1.
(4) The proof is similar to Lemma 8 in [1]. □

Following the same process we find the cyclic code C_{k-3} over $R_4 = F_q + uF_q + u^2F_q + u^3F_q$ with $(u^4 = 0)$. So, since the image of ϕ_{k-3} is an ideal in $R_{3,n} = R_3[x]/\big\langle x^n-1\big\rangle$ (where $R_3 = F_q + uF_q + u^2F_q$ with $u^3 = 0$), then $Im(\phi_{k-3}) = \big\langle g(x)+up_1(x)+u^2p_2(x), ua_1(x)+u^2q_1(x), u^2a_2(x)\big\rangle$ with $a_2|a_1|g|(x^n-1)$, $a_1(x)|p_1(x)\big(\frac{x^n-1}{g(x)}\big)$, $a_2|q_1(x)\Big(\frac{(x^n-1)}{a_1(x)}\Big)$ and $a_2\big|p_2(x)\big(\frac{x^n-1}{g(x)}\big)\big(\frac{x^n-1}{a_1(x)}\big)$. Also $\ker(\phi_{k-3}) = \big\langle u^3a_3(x)\big\rangle$ with $a_3(x)\big|(x^n-1)$. Since $u^3a_2 \in \ker(\phi_{k-3}) = \big\langle u^3a_3(x)\big\rangle$, then the cyclic code C_{k-3} over $R_4 = Z_2 + uZ_2 + u^2Z_2 + u^3Z_2$ with $(u^4 = 0)$ is $C_{k-3} = \big\langle g+up_1+u^2p_2+u^3p_3, ua_1+u^2q_1+u^3q_2, u^2a_2+u^3l_1, u^3a_3\big\rangle$ with

$$a_3\big|a_2\big|a_1\big|g\big|(x^n-1) \mod q,\ a_1(x)\big|p_1(x)\big(\frac{x^n-1}{g(x)}\big),$$

$$a_2\big|q_1(x)\Big(\frac{(x^n-1)}{a_1(x)}\Big),\ a_2\big|p_2(x)\big(\frac{x^n-1}{g(x)}\big)\big(\frac{x^n-1}{a_1(x)}\big),$$

$$a_3\big|l_1(x)\Big(\frac{(x^n-1)}{a_2(x)}\Big),\ a_3\big|q_2(x)\big(\frac{x^n-1}{q_1(x)}\big)\big(\frac{x^n-1}{a_1(x)}\big)$$

and $a_3(x)\big|p_3(x)\big(\frac{x^n-1}{g(x)}\big)\big(\frac{x^n-1}{a_2(x)}\big)\big(\frac{x^n-1}{a_1(x)}\big)$. Moreover $\deg p_3 < dega_3$, $\deg q_2 < dega_3$, $\deg l_1 < dega_3$, $\deg p_2 < dega_2$, $\deg q_1 < dega_2$, $\deg p_1 < dega_1$.

Lemma 3.4. *If $C_{k-3} = \big\langle g+up_1+u^2p_2+u^3p_3, ua_1+u^2q_1+u^3q_2, u^2a_2+u^3l_1, u^3a_3\big\rangle$ over $R_4 = F_q + uF2q + u^2F_q + u^3F_q$ with $(u^4 = 0)$, and $a_3 = g$, then $C_{k-3} = \big\langle g+up_1+u^2p_2+u^3p_3\big\rangle$and $\big(g+up_1+u^2p_2+u^3p_3\big)\big|(x^n-1)$ in R_4.*

Proof. Since $a_3 = g$, then $a_1 = a_2 = a_3 = g$. From lemma 3.3 we get that $(g+up_1+u^2p_2)\big|(x^n-1)$ in R_3 and $C_{k-3} = \big\langle g+up_1+u^2p_2+u^3p_3, ua_1+u^2q_1+u^3q_2, u^3a_3\big\rangle$. The rest of the proof is similar to lemma 3.3. □

Lemma 3.5. *If n is relatively prime to q, then the cyclic code C_{k-3} over R_4 can be written as*

$$C_{k-3} = \big\langle g, ua_1, u^2a_2, u^3a_3\big\rangle = \big\langle g+ua_1+u^2a_2+u^3a_3\big\rangle.$$

Proof. The proof is similar to Lemma 3.5 in [9]. □

From all the above discussion, we can construct any cyclic code C_1 over R_k, $k \geq 4$ by using the same process and induction on k to get the following theorem:

Theorem 3.6. *Let C_1 be a cyclic code in $R_{k,n} = R_k[x]/\big\langle x^n-1\big\rangle$, $R_k = F_q + uF_q + u^2F_q + \ldots + u^{k-1}F_q$ with $u^k = 0$.*
(1) If n is relatively prime to q, then $R_{k,n}$ is a principal ideal ring and
$C_1 = \big\langle g, ua_1, u^2a_2, \ldots, u^{k-1}a_{k-1}\big\rangle = \big\langle g+ua_1+u^2a_2+\ldots+u^{k-1}a_{k-1}\big\rangle$
where $g(x), a_1(x), a_2(x), \ldots, a_{k-1}(x)$ are polynomials over F_q with
$a_{k-1}(x)\big|a_{k-2}(x)\big|\ldots\big|a_2(x)\big|a_1(x)\big|g(x)$.
(2) If n is not relatively prime to q, then
(a) $C_1 = \big\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\big\rangle$ where $g(x)$, $p_i(x)$ are polynomials over F_q $\forall i = 1, 2, \ldots, k-1$ with $g(x)\big|(x^n-1)$, $(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1})\big|(x^n-1)$ in R_k and deg *$p_i < deg\ p_{i-1}$ for all $2 \leq i \leq k-1$.*
OR
(b) $C_1 = \big\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}, u^{k-1}a_{k-1}\big\rangle$ where $a_{k-1}|g|(x^n-1)$, $(g+up_1)|(x^n-1)$ in R_2, $g(x)|p_1\big(\frac{x^n-1}{g(x)}\big)$ and $a_{k-1}|p_1\big(\frac{x^n-1}{g(x)}\big)$, $a_{k-1}|p_2\big(\frac{x^n-1}{g(x)}\big)\big(\frac{x^n-1}{g(x)}\big), \ldots$ and $a_{k-1}|p_{k-1}\big(\frac{x^n-1}{g(x)}\big)\ldots\big(\frac{x^n-1}{g(x)}\big)(k-1, times)$ and $deg\ p_{k-1} < deg\ a_{k-1}$.
OR
(c) $C_1 = \big\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}, ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2}, u^2a_2+u^3l_1+\ldots+$

$u^{k-1}l_{k-3},\ldots,u^{k-2}a_{k-2}+u^{k-1}t_1,u^{k-1}a_{k-1}\rangle$ *with* $a_{k-1}|a_{k-2}|\ldots|a_2|a_1|g|(x^n-1)$, $a_{k-2}|p_1(\frac{x^n-1}{g}),\ldots,a_{k-1}|t_1(\frac{x^n-1}{a_{k-2}}),\ldots,a_{k-1}|p_{k-1}(\frac{x^n-1}{g})\ldots(\frac{x^n-1}{a_{k-2}})$.
Moreover $deg\ p_{k-1} < deg\ a_{k-1},\ldots,deg\ t_1 < deg\ a_{k-1},\ldots$ *and* $deg\ p_1 < deg\ a_{k-2}$.

Motivated by the work in [7], [10], the structure of cyclic codes over R_k of length n relatively prime to q can be given in another way as follows: Let R_k be a finite chain ring with the maximal ideal $< u >$ and k be the nilpotent index of u. Assume that n is not divisible by the characteristic of the residue field F_q, so that x^n-1 has a unique decomposition as a product of basic irreducible pairwise coprime polynomials in $R_k[x]$ (cf.proposition 2.7 in [7]).

Theorem 3.7. *Let C be a cyclic code of length n relatively prime to q over R_k, which has maximal ideal $< u >$ and k is the nilpotent index of u. Then there exist polynomials $g_0,g_1,\ldots,g_{k-1}$ in $R_k[x]$ such that $C=\langle g_0,ug_1,\ldots,u^{k-1}g_{k-1}\rangle$ and $g_{k-1}|g_{k-2}|\ldots|g_1|g_0|(x^n-1)$.*

Theorem 3.8. *Let C be a cyclic code of length n relatively prime to q over R_k, which has maximal ideal $< u >$ and k is the nilpotent index of u, $F=\hat{F}_1+u\hat{F}_2+\ldots+u^{k-1}\hat{F}_k$, where $F_i(x)$ is a factor of x^n-1, $\hat{F}_i(x)=\frac{x^n-1}{F_i(x)}$. Then $C=\langle F\rangle$.*

Corollary 3.9. *The ring $R_k[x]/\langle x^n-1\rangle$ with n relatively prime to q is a principal ideal ring.*

4 Ranks and minimal spanning sets for cyclic codes over R_k

In this section we will discuss the ranks and minimal spanning sets for cyclic codes over R_k. In [9], the authors have shown the following Theorem:

Lemma 4.1. [9] *Let C_1 be a cyclic code of even length n over $R_k=Z_2+uZ_2+u^2Z_2+\ldots+u^{k-1}Z_2$ with $u^k=0$. The constraints on the generator polynomials as in theorem 3.6.*
(1) *If* $C_1=\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\rangle$, deg $g(x)=r$, *then C_1 is a free module with* rank$(C_1)=n-r$ *and basis*
$\beta=\Big\{(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ x(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ \ldots,\ x^{n-r-1}(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1})\Big\}$.
(2) *If* $C_1=\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1},ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2},u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3},\ldots,u^{k-2}a_{k-2}+u^{k-1}t_1,u^{k-1}a_{k-1}\rangle$ *with* deg $g(x)=r_1$, deg $a_1(x)=r_2$, deg $a_2(x)=r_3,\ldots,$ deg $a_{k-1}=r_k$, *then C_1 has* rank$(C_1)=n-r_k$ *and a minimal spanning set given by*
$\chi=\Big\{\big(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\big),\ x\big(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\big),\ \ldots,\ x^{n-r_1-1}\big(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\big),\ \big(ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2}\big),\ x\big(ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2}\big),\ \ldots,\ x^{r_1-r_2-1}\big(ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2}\big),\ (u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3}),\ x(u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3}),\ \ldots,\ x^{r_2-r_3-1}(u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3}),\ \ldots,\ u^{k-1}a_{k-1}(x),\ xu^{k-1}a_{k-1}(x),\ \ldots,\ x^{r_{k-1}-r_k-1}u^{k-1}a_{k-1}(x)\Big\}$.
(3) *If* $C_1=\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1},u^{k-1}a_{k-1}\rangle$ *with deg $g(x)=r$,deg $a_{k-1}=t$ then C_1 has* rank$(C_1)=n-t$ *and a minimal spanning set given by*
$\Gamma=\Big\{(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ x(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ \ldots,\ x^{n-r-1}(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ u^{k-1}a_{k-1},\ xu^{k-1}a_{k-1},\ldots,x^{r-t-1}u^{k-1}a_{k-1}\Big\}$.

Now we use the technology to obtain the similar results:

Theorem 4.2. *Let C_1 be a cyclic code of length n not relatively prime to q over $R_k=F_q+uF_q+u^2F_q+\ldots+u^{k-1}F_q$ with $u^k=0$. The constraints on the generator polynomials as in theorem 3.6.*
(1) *If* $C_1=\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\rangle$, deg $g(x)=r$, *then C_1 is a free module with* rank$(C_1)=n-r$ *and basis*
$\beta=\Big\{(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ x(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ \ldots,\ x^{n-r-1}(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1})\Big\}$.
(2) *If* $C_1=\langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1},ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2},u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3},\ldots,u^{k-2}a_{k-2}+u^{k-1}t_1,u^{k-1}a_{k-1}\rangle$ *with* deg $g(x)=r_1$, deg $a_1(x)=r_2$, deg $a_2(x)=r_3,\ldots,$ deg $a_{k-1}=r_k$, *then C_1 has* rank$(C_1)=n-r_k$ *and a minimal spanning set given by*

$\chi = \Big\{ \big(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\big),\ x\big(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\big),\ \ldots,\ x^{n-r_1-1}\big(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}\big),\ \big(ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2}\big),\ x\big(ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2}\big),\ \ldots,\ x^{r_1-r_2-1}\big(ua_1+u^2q_1+\ldots+u^{k-1}q_{k-2}\big),\ (u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3}),\ x(u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3}),\ \ldots,\ x^{r_2-r_3-1}(u^2a_2+u^3l_1+\ldots+u^{k-1}l_{k-3}),\ \ldots,\ u^{k-1}a_{k-1}(x),\ xu^{k-1}a_{k-1}(x),\ \ldots,\ x^{r_{k-1}-r_k-1}u^{k-1}a_{k-1}(x)\Big\}.$

(3) If $C_1 = \langle g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}, u^{k-1}a_{k-1}\rangle$ with deg $g(x)=r$,deg $a_{k-1}=t$ then C_1 has rank$(C_1)=n-t$ *and a minimal spanning set given by*
$\Gamma = \Big\{ (g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ x(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ \ldots,\ x^{n-r-1}(g+up_1+u^2p_2+\ldots+u^{k-1}p_{k-1}),\ u^{k-1}a_{k-1},\ xu^{k-1}a_{k-1},\ldots,x^{r-t-1}u^{k-1}a_{k-1}\Big\}.$

Proof. The proof is similar to the prove of lemma (4.1) in [9]. □

5 Dual codes over rings $F_q + uF_q + u^2F_q$

This section study the dual codes of cyclic codes over $R_3 = F_q + uF_q + u^2F_q$. Let I be the ideal of $R_{i,n} = R_i[x]/\langle x^n-1\rangle$, where $2 \le i \le k$, then the set $A(I) = \{g(x) : f(x)g(x) = 0, \forall f(x) \in I\}$ is called the annihilator of I in $R_{i,n}$; reciprocal polynomial of degree r of the polynomial $f(x) = c_0 + c_1x + \cdots + c_rx^r$ is defined as $f^*(x) = c_r + c_{r-1}x + \cdots + c_0x^r$; It's obvious that if C is a cyclic code with associated ideal I then the associate ideal of $C^\perp$ is $A(I)^* = \{g^*(x) : \forall g(x) \in I\}$.

Lemma 5.1. [12] *If $(n,p) \neq 1$, let C_{k-1} be an arbitrary ideal of the ring $R_{2,n} = R_2[x]/\langle x^n-1\rangle$ (i.e., it's an arbitrary cyclic code of arbitrary length n over ring R_2), then there only exits $a_1(x)|g(x)|x^n-1$, and the polynomials $g(x), a_1(x), p_1(x)$ in $F_q[x]$ with $dega_1 > degp_1$, such that $C_{k-1} = \langle g(x)+up_1(x), ua_1(x)\rangle$:*
(I) if $a(x)=g(x)$, then $C_{k-1} = \langle g+up\rangle$, and $(g+up)|x^n-1$ in $R_2[x]$, thus $A(C_{k-1}) = \langle \frac{x^n-1}{g+up}\rangle$, also we have $C_{k-1}^\perp = \langle (\frac{x^n-1}{g+up})^*\rangle$
(II)Otherwise, $C_{k-1} =< g+up, ua >$, then $A(C_{k-1}) = \langle \frac{x^n-1}{a} - u\frac{p\frac{x^n-1}{g}}{a}, u\frac{x^n-1}{g}\rangle$, also we have $C_{k-1}^\perp = \langle (\frac{x^n-1}{a} - u\frac{p\frac{x^n-1}{g}}{a})^*, u(\frac{x^n-1}{g})^*\rangle$

Lemma 5.2. *Let C_{k-2} be a cyclic code in $R_{3,n}$,then*
(I) if $(n,p)\neq 1$ and $C_{k-2} = \langle g+up_1+u^2p_2\rangle$ with $a_2|a_1|g|(x^n-1) mod\ q$, $(g+up)|x^n-1$, and $(g+up_1+u^2p_2) \mid (x^n-1)$ and $\deg p_2 < \deg p_1$, *then*
$A(C_{k-2}) = \langle \frac{x^n-1}{g+up_1+u^2p_2}\rangle$, also have $C_{k-2}^\perp = \langle (\frac{x^n-1}{g+up_1+u^2p_2})^*\rangle$
(II) if $(n,p)\neq 1$ and $C_{k-2} = \langle g+up_1+u^2p_2, u^2a_2\rangle$ with $a_2|g|(x^n-1) mod\ q,\ (g+up)|x^n-1$, $g(x)\big|p_1(x)\big(\frac{x^n-1}{g(x)}\big)$, $a_2|p_1\big(\frac{x^n-1}{g}\big)$ and $a_2\big|p_2\big(\frac{x^n-1}{g}\big)\big(\frac{x^n-1}{a_1}\big)$, and $degp_2 < dega_2$, then

$$A(C_{k-2}) = \Big\langle \frac{(x^n-1)^2}{ga_2} - u\frac{p_1(\frac{x^n-1}{g})^2}{a_2} + u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2} - u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2}, u\frac{x^n-1}{g}\Big\rangle.$$

Also have $C_{k-2}^\perp = \Big\langle \Big(\frac{(x^n-1)^2}{ga_2} - u\frac{p_1(\frac{x^n-1}{g})^2}{a_2} + u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2} - u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2}\Big)^*, \big(u\frac{x^n-1}{g}\big)^*\Big\rangle.$
(III) $C_{k-2} = \langle g+up_1+u^2p_2, ua_1+u^2q_1, u^2a_2\rangle$ with $a_2|a_1|g|(x^n-1)$, $a_1(x)\big|p_1(x)\big(\frac{x^n-1}{g(x)}\big)$ mod *q, $a_2|q_1\big(\frac{x^n-1}{a_1}\big)$, $a_2|p_1\big(\frac{x^n-1}{g}\big)$ and $a_2\big|p_2\big(\frac{x^n-1}{g}\big)\big(\frac{x^n-1}{a_1}\big)$, moreover,* $\deg p_2 < \deg a_2, \deg q_1 < \deg a_2, \deg p_1 < \deg a_1$, *then*

$$A(C_{k-2}) = \Big\langle \frac{(x^n-1)^3}{ga_1a_2} - u\frac{p_1(x^n-1)^3}{g^2a_1a_2} + u^2\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2} - u^2\frac{p_2(x^n-1)^3}{g^2a_1a_2}, u\frac{(x^n-1)^2}{ga_1} - u^2\frac{p_1(\frac{x^n-1}{g})^2}{a_1}, u^3\frac{x^n-1}{g}\Big\rangle,$$

Also have $C_{k-2}^\perp = \Big\langle \Big(\frac{(x^n-1)^3}{ga_1a_2} - u\frac{p_1(x^n-1)^3}{g^2a_1a_2} + u^2\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2} - u^2\frac{p_2(x^n-1)^3}{g^2a_1a_2}\Big)^*, \Big(u\frac{(x^n-1)^2}{ga_1} - u^2\frac{p_1(\frac{x^n-1}{g})^2}{a_1}\Big)^*, \big(u^3\frac{x^n-1}{g}\big)^*\Big\rangle.$

Proof. (I)Since $(g+up_1+u^2p_2) \mid (x^n-1)$, the proof of the conclusion is similar to the generator of dual codes in the ring R_1.
(II)Let $D = \Big\langle \frac{(x^n-1)^2}{ga_2} - u\frac{p_1(\frac{x^n-1}{g})^2}{a_2} + u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2} - u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2}, u\frac{x^n-1}{g}\Big\rangle$, it is easy to prove that $\frac{(x^n-1)^2}{ga_2} - u\frac{p_1(\frac{x^n-1}{g})^2}{a_2} + u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2} - u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2} \in A(C_{k-2})$, $u\frac{x^n-1}{g} \in A(C_{k-2})$.
Since $A(C_{k-2})$ is an ideal of the ring of $R_{3,n}$, we assume that $A(C_{k-2}) = \langle h+uv_1+u^2v_2, u^2d_2\rangle$.
Since $(h+uv_1+u^2v_2)(g+up_1+u^2p_2) = 0,$ *and* $(h+uv_1+u^2v_2)(u^2a_2) = 0,$
then

$$a_2h = 0, gh = 0, gv_1 + p_1h = 0, gv_2 + p_1v_1 + p_2h = 0.$$

From the above equalities, we assume that
$h=\varphi(x)\frac{(x^n-1)^2}{ga_2}$, and we can obtain that $v_1=-p_1\varphi(x)\frac{(x^n-1)^2}{g^2a_2}$, $v_2=\varphi\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}-\varphi\frac{p_2(\frac{x^n-1}{g})^2}{a_2}$.
we also can get $v_2-\varphi\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}+\varphi\frac{p_2(\frac{x^n-1}{g})^2}{a_2}=\xi\frac{x^n-1}{g}$, i.e. $v_2=\varphi\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}-\varphi\frac{p_2(\frac{x^n-1}{g})^2}{a_2}+\xi\frac{x^n-1}{g}$.
Then

$$h+uv_1+u^2v_2=\varphi(x)\frac{(x^n-1)^2}{ga_2}-up_1\varphi(x)\frac{(x^n-1)^2}{g^2a_2}+u^2\varphi\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}-u^2\varphi\frac{p_2(\frac{x^n-1}{g})^2}{a_2}+u^2\xi\frac{x^n-1}{g}$$

$$=\varphi(x)[\frac{(x^n-1)^2}{ga_2}-up_1\frac{(x^n-1)^2}{g^2a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}-u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2}]+\xi u^2\frac{x^n-1}{g},$$

which implies that $h+uv_1+u^2v_2\in D$, it is easy to prove that $u^2d_2\in D$, then $A(C_{k-2})\in D$.
Hence $A(C_{k-2})=\langle\frac{(x^n-1)^2}{ga_2}-u\frac{p_1(\frac{x^n-1}{g})^2}{a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}-u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2},u\frac{x^n-1}{g}\rangle$,
and $C_{k-2}^{\perp}=\langle(\frac{(x^n-1)^2}{ga_2}-u\frac{p_1(\frac{x^n-1}{g})^2}{a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}-u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2})^*,(u\frac{x^n-1}{g})^*\rangle$.
(III)Let$D_1=\langle\frac{(x^n-1)^3}{ga_1a_2}-u\frac{p_1(x^n-1)^3}{g^2a_1a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}-u^2\frac{p_2(x^n-1)^3}{g^2a_1a_2},u\frac{(x^n-1)^2}{ga_1}-u^2\frac{p_1(\frac{x^n-1}{g})^2}{a_1},u^3\frac{x^n-1}{g}\rangle$,
it is easy to prove that $\frac{(x^n-1)^3}{ga_1a_2}-u\frac{p_1(x^n-1)^3}{g^2a_1a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}-u^2\frac{p_2(x^n-1)^3}{g^2a_1a_2}\in A(C_{k-2})$, $u\frac{(x^n-1)^2}{ga_1}-u^2\frac{p_1(\frac{x^n-1}{g})^2}{a_1}\in A(C_{k-2})$, $u^3\frac{x^n-1}{g}\in A(C_{k-2})$.
Since $A(C_{k-2})$ is an ideal of the ring of $R_{3,n}$,
we assume that $A(C_{k-2})=\langle h+uv_1+u^2v_2,ud_1+u^2l_1,u^2d_2\rangle$.
Since $(h+uv_1+u^2v_2)(g+up_1+u^2p_2)=0$, $(h+uv_1+u^2v_2)(ua_1+u^2q_1)=0$, $(h+uv_1+u^2v_2)(u^2a_2)=0$, $(g+up_1+u^2p_2)(ud_1+u^2l_1)=0$, $(ua_1+u^2q_1)(ud_1+u^2l_1)=0$, and $(h+uv_1+u^2v_2)(u^2a_2)=0$,
then

$$a_2h=0,ha_1=0,hq_1+a_1v_1=0,gh=0,gv_1+p_1h=0,gv_2+p_1v_1+p_2h=0.$$

From the above equalities, we assume that $h=\eta(x)\frac{(x^n-1)^3}{ga_1a_2}$,
and we can obtain that $v_1=-p_1\eta(x)\frac{(x^n-1)^3}{g^2a_1a_2}$, $v_2=\eta(x)\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}-\eta(x)\frac{p_2((x^n-1)^3}{g^2a_1a_2}$.
We also can get $v_2-\eta(x)\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}+\eta(x)\frac{p_2((x^n-1)^3}{g^2a_1a_2}=\delta\frac{x^n-1}{g}$, i.e. $v_2=\eta(x)\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}-\eta(x)\frac{p_2((x^n-1)^3}{g^2a_1a_2}+\delta\frac{x^n-1}{g}$.
Then

$$h+uv_1+u^2v_2=\eta(x)\frac{(x^n-1)^3}{ga_1a_2}-up_1\eta(x)\frac{(x^n-1)^3}{g^2a_1a_2}$$

$$+u^2\eta(x)\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}-u^2\eta(x)\frac{p_2((x^n-1)^3}{g^2a_1a_2}+u^2\delta\frac{x^n-1}{g}$$

$$=\eta(x)[\frac{(x^n-1)^2}{ga_2}-up_1\frac{(x^n-1)^2}{g^2a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^2}{ga_2}-u^2\frac{p_2(\frac{x^n-1}{g})^2}{a_2}]+\delta u^2\frac{x^n-1}{g},$$

which implies that $h+uv_1+u^2v_2\in D$, it is easy to prove that $ud_1+u^2l_1\in D$, and $u^2d_2\in D$, then $A(C_{k-2})\in D$.
Hence $A(C_{k-2})=\langle\frac{(x^n-1)^3}{ga_1a_2}-u\frac{p_1(x^n-1)^3}{g^2a_1a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}-u^2\frac{p_2(x^n-1)^3}{g^2a_1a_2},u\frac{(x^n-1)^2}{ga_1}-u^2\frac{p_1(\frac{x^n-1}{g})^2}{a_1},u^3\frac{x^n-1}{g}\rangle$,
and
$C_{k-2}^{\perp}=\langle((\langle\frac{(x^n-1)^3}{ga_1a_2}-u\frac{p_1(x^n-1)^3}{g^2a_1a_2}+u^2\frac{p_1^2(\frac{x^n-1}{g})^3}{a_1a_2}-u^2\frac{p_2(x^n-1)^3}{g^2a_1a_2})^*,(u\frac{(x^n-1)^2}{ga_1}-u^2\frac{p_1(\frac{x^n-1}{g})^2}{a_1})^*,(u^3\frac{x^n-1}{g})^*\rangle$.
□

6 Examples

Example 6.1. *Cyclic codes of length 3 over $F_3+uF_3+u^2F_3+u^3F_3$ with $u^4=0$.*
Now, $x^3-1=(x+2)^3=g(x)^3$
The Nonzero cyclic codes of length 3 over $F_3+uF_3+u^2F_3+u^3F_3$ with generator polynomials are on the following table 1:

Non zero generator polynomials
$\langle 1\rangle,\ \langle g\rangle,\ \langle g^2\rangle$
$\langle u\rangle,\ \langle ug\rangle,\ \langle ug^2\rangle$
$\langle u^2\rangle,\ \langle u^2g\rangle,\ \langle u^2g^2\rangle$
$\langle u^3\rangle,\ \langle u^3g\rangle,\ \langle u^3g^2\rangle$
$\langle g,u\rangle,\ \langle g^2,u\rangle,\ \langle g,u^2\rangle,\ \langle g^2,u^2\rangle,\ \langle g^2,u^2g\rangle$
$\langle g,u^3\rangle,\ \langle g^2,u^3\rangle,\ \langle g^2,u^3g\rangle$
$\langle ug,u^2\rangle,\ \langle ug^2,u^2\rangle,\ \langle ug^2,u^2g\rangle$
$\langle u^2g,u^3\rangle,\ \langle u^2g^2,u^3\rangle,\ \langle u^2g^2,u^3g\rangle$

Table 1 : Cyclic codes of length 3 over $F_3+uF_3+u^2F_3+u^3F_3$.

Example 6.2. *If $n=4$ over $F_3+uF_3+u^2F_3$ with $u^3=0$.*
$x^4-1=(x+1)(x+2)(x^2+1)=f_1(x)f_2(x)f_3(x)$.
The nonzero free/non free module cyclic codes over $F_3+uF_3+u^2F_3$ are on the following tables 2,3:

Non zero generator polynomial(s)
$\langle 1\rangle,\ \langle f_1\rangle,\ \langle f_2\rangle,\ \langle f_3\rangle,\ \langle f_1+u\rangle,\ \langle f_2+u\rangle,\ \langle f_3+u\rangle,\ \langle f_1+u^2\rangle,\ \langle f_2+u^2\rangle,\ \langle f_3+u^2\rangle$
$\langle f_1f_2+u(c_0+c_1x)\rangle,\ \langle f_1f_2+u^2(c_0+c_1x)\rangle$
$\langle f_1f_3+u(c_0+c_1x+c_2x^2)\rangle,\ \langle f_1f_3+u^2(c_0+c_1x+c_2x^2)\rangle$
$\langle f_2f_3+u(c_0+c_1x+c_2x^2)\rangle,\ \langle f_2f_3+u^2(c_0+c_1x+c_2x^2)\rangle$

Table 2 : Non zero Free module cyclic codes of length 4 over $F_3+uF_3+u^2F_3$.

Non zero generator polynomial(s): g=x+1
$\langle u\rangle,\ \langle u^2\rangle$
$\langle uf_i\rangle,\ i=1,\dots,3,\ \langle u^2f_i\rangle,\ i=1,\dots,3.$
$\langle uf_1f_2\rangle, \langle uf_1f_3\rangle, \langle uf_2f_3\rangle$
$\langle u^2f_1f_2\rangle, \langle u^2f_1f_3\rangle, \langle u^2f_2f_3\rangle$
$\langle f_1,u\rangle, \langle f_2,u\rangle, \langle f_3,u\rangle, \langle f_1f_2,u\rangle, \langle f_1f_3,u\rangle, \langle f_2f_3,u\rangle,$
$\langle f_1,u^2\rangle, \langle f_2,u^2\rangle, \langle f_3,u^2\rangle, \langle f_1f_2,u^2\rangle, \langle f_1f_3,u^2\rangle, \langle f_2f_3,u^2\rangle,$
$\langle f_1f_2+uc_0,uf_1\rangle,\ \langle f_1f_2+u^2c_0,u^2f_1\rangle,\ \langle f_1f_2+uc_0,uf_2\rangle,\ \langle f_1f_2+u^2c_0,u^2f_2\rangle$
$\langle f_1f_3+uc_0,uf_1\rangle,\ \langle f_1f_3+u^2c_0,u^2f_1\rangle,\ \langle f_1f_3+u(c_0+c_1x),uf_2\rangle,\ \langle f_1f_3+u^2(c_0+c_1x),u^2f_3\rangle$
$\langle f_2f_3+uc_0,uf_2\rangle,\ \langle f_2f_3+u^2c_0,u^2f_2\rangle,\ \langle f_2f_3+u(c_0+c_1x),uf_2\rangle,\ \langle f_2f_3+u^2(c_0+c_1x),u^2f_3\rangle$

Table 3 : Non Free module cyclic codes of length 4 over $F_3+uF_3+u^2F_3$

7 Conclusion

In this paper, we studied cyclic codes of an arbitrary length over the ring $F_q+uF_q+u^2F_q+\ldots+u^{k-1}F_q$, with $u^k=0$. The rank and minimum spanning of this family of codes are studied as well. We also study dual codes and find their properties over the ring $F_q+uF_q+u^2F_q$.

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