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On Subclasses of Multivalent Functions Defined by a Multiplier Operator Involving the Komatu Integral Operator

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Abstract. This paper is devoted to the study of some new subclasses strongly close-to-convex p-valent functions. It is defined by a multiplier operator using the Komatu integral operator and studies their inclusion relationships with the integral preserving properties.

1. INTRODUCTION

Let $A(p,n)$ be the class of functions $f(z)$ of the form

$$f(z)=z^{p}+\sum_{k=p+n}^{\infty}a_{k}z^{k}\ ,\ (p,n\in N) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U=\{z : z\in C,|z|<1\}$.

The generalized Komatu integral operator $K_{c,p}^{\delta}:A(p,n)\to A(p,n)$ is defined for $\delta>0$ and $c>-p$ as

$$K_{c,p}^{\delta}f(z)=\frac{(c+p)^{\delta}}{\Gamma(\delta)z^{c}}\int_{0}^{z}t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}f(t)dt \quad . \tag{1.2}$$

Now, in terms of $K_{c,p}^{\delta}$, we introduce the linear multiplier operator $J_{c,p,\lambda}^{m,\delta}:A(p,n)\to A(p,n)$ as follows:

$$J_{c,p,\lambda}^{0,0}f(z)=f(z) \tag{1.3}$$

$$J_{c,p,\lambda}^{1,\delta}f(z)=(1-\lambda)K_{c,p}^{\delta}f(z)+\frac{\lambda z}{p}(K_{c,p}^{\delta}f(z))'=J_{c,p,\lambda}^{\delta}f(z)$$

$$\vdots$$

$$J_{c,p,\lambda}^{m,\delta}f(z)=J_{c,p,\lambda}^{\delta}(J_{c,p,\lambda}^{m-1,\delta}f(z))$$

for $\delta>0,\ c>-p,\ \lambda\geq 0$ and $m\in\mathbb{N}$.

If $f\in A(p,n)$ is given by (1.1), then

$$J_{c,p,\lambda}^{m,\delta}f(z)=z^{p}+\sum_{k=p+n}^{\infty}B_{k,m}(c,p,\lambda,\delta)a_{k}z^{k} \tag{1.4}$$

where

$$B_{k,m}(c,p,\lambda,\delta)=\left[\left(\frac{c+p}{c+k}\right)^{\delta}\frac{\lambda}{p}\left(k-p\right)\right]^{m} \tag{1.5}$$

for $\delta>0,\ c>-p,\ \lambda\geq 0$ and $m\in\mathbb{N}$.

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, and if a Schwarz function $w(z)$ in U such that $f(z)=g(w(z))$ exists then we write it as $f\prec g$ or $f(z)\prec g(z)$.

2. **Results**

Let $S_{c,p,\lambda}^{m,\delta}(\eta,A,B)$ be the class of functions $f\in A(p,n)$ satisfying the condition

$$\frac{1}{p-\eta}\left(\frac{z\,(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right)\prec\frac{1+Az}{1+Bz} \tag{2.0}$$

, then the following results appear:

Lemma 2.1. Let $h(z)$ be convex univalent in U with $h(0)=1$ and $\operatorname{Re}\{vh(z)+\mu\}>0$ $(v,\mu\in\mathbb{C})$. If $p(z)$ is analytic in U with $p(0)=1$, then

$p(z)+\dfrac{zp'(z)}{vp(z)+\mu}\prec h(z),(z\in U)$ which implies $p(z)\prec h(z),\ (z\in U)$

Lemma 2.2. Let $h(z)$ be convex univalent in U and $w(z)$ be analytic in U with $\operatorname{Re}w(z)\geq 0$. If $p(z)$ is analytic in U with $p(0)=h(0)$, then $p(z)+w(z)zp'(z)\prec h(z),\ (z\in U)$ which implies $p(z)\prec h(z),\ (z\in U)$.

Lemma 2.3. Let $p(z)$ be analytic in U with $p(0)=1$ and $p(z)\neq 0$ in U. If two points exist $z_1,z_2\in U$ such that

$$-\frac{\pi}{2}\alpha_1=\arg p(z_1)<\arg p(z)<\arg p(z_2)=\frac{\pi}{2}\alpha_2 \tag{2.1}$$

for some α_1,α_2 $(\alpha_1,\alpha_2>0)$ and for all ($|z|<|z_1|=|z_2|$), then we have

$$\frac{z_1p'(z_1)}{p(z_1)}=-i\,\frac{(\alpha_1+\alpha_2)}{2}m \quad\text{and}\quad \frac{z_2p'(z_2)}{p(z_2)}=i\,\frac{(\alpha_1+\alpha_2)}{2}m \tag{2.2}$$

where

$$m\geq\frac{1-|c^*|}{1+|c^*|}\quad\text{and}\quad c^*=i\,\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1}\right) \tag{2.3}$$

Proposition 2.1. Let $\delta>0,\ c>-p,\ \lambda\geq 0$, $m\in\mathbb{N}$ and $h(z)$ be convex univalent in U with $h(0)=1$ and $\operatorname{Re}h(z)>0$; if a function $f(z)\in A(p,n)$ satisfies the condition

$$\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)\right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)}-\eta\right)\prec h(z),\quad (0\le\eta<1; z\in U),$$

then

$$\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}f(z)\right)'}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right)\prec h(z),\quad (0\le\eta<1; z\in U).$$

Proof:
Let

$$d(z)=\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}f(z)\right)'}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right) \tag{2.4}$$

where $d(z)$ is analytic function in U

$$d(z)=\frac{1}{p-\eta}\left(\frac{(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)-cJ_{c,p,\lambda}^{m,\delta}f(z)}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right)$$

$$=\frac{1}{p-\eta}\left(\frac{(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)}{J_{c,p,\lambda}^{m,\delta}f(z)}-c-\eta\right)$$

$$=\frac{1}{p-\eta}\left[\frac{cz^p+pz^p+\sum_{k=p+n}^{\infty}a_k z^k\,\frac{(c+p)^{\delta}}{(c+k)^{\delta m-1}}\left(1+(k-p)\frac{\lambda}{p}\right)^m}{z^p+\sum_{k=p+n}^{\infty}\left[\left(\frac{c+p}{c+k}\right)^{\delta}\left(\left(1+(k-p)\frac{\lambda}{p}\right)\right)\right]^m a_k z^k}-c-\eta\right]$$

then $d(0)=1$

From (2.4), we get

$$(p-\eta)d(z)+c+\eta=\frac{(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)}{J_{c,p,\lambda}^{m,\delta}f(z)} \tag{2.5}$$

Differentiating both sides logarithmically with respect to z and multiplying them by z yields

$$\frac{z(p-\eta)d'(z)}{(p-\eta)d(z)+c+\eta}=\frac{z\left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)\right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)}-\frac{z\left(J_{c,p,\lambda}^{m,\delta}f(z)\right)'}{J_{c,p,\lambda}^{m,\delta}f(z)} \tag{2.6}$$

Dividing both sides by $p-\eta$, we get

$$\frac{zd'(z)}{(p-\eta)d(z)+c+\eta}+d(z)=\frac{1}{p-\eta}\left(\frac{z(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)}-\eta\right) \tag{2.7}$$

By using Lemma (2.1), it follows that $d(z)\prec h(z),(z\in U)$, then

$$\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}f(z)\right)'}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right)\prec h(z).$$

Proposition 2.2. Let $h(z)$ be a convex univalent in U with $h(0)=1$ and $\operatorname{Re}(h(z))>0$; if a function $f(z)\in A(p,n)$ satisfies the condition

$$\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}f(z)\right)'}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right)\prec h(z),\quad (0\le\eta<1; z\in U),$$

then

$$\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}L_\theta f(z)\right)'}{J_{c,p,\lambda}^{m,\delta}L_\theta f(z)}-\eta\right)\prec h(z),\quad (0\le\eta<1; z\in U),$$

where $L_\theta(f)$ is the integral operator defined by

$$L_\theta(f)=L_\theta f(z)=\frac{(\theta+1)}{z^\theta}\int_0^z t^{\theta-1}f(t)dt,\ (\theta\ge 0) \tag{2.8}$$

Proof:

$$f(z)=z^p+\sum_{k=p+n}^{\infty}a_k z^k\quad (p,n\in N)$$

Then

$$L_\theta f(z)=\frac{(\theta+1)}{z^\theta}\left[\int_0^z t^{\theta-1}t^p dt+\int_0^z t^{\theta-1}\sum_{k=p+n}^{\infty}a_k t^k dt\right]$$

$$L_\theta f(z)=\left(\frac{\theta+1}{\theta+p}\right)z^p+\sum_{k=p+n}^{\infty}a_k z^k\left(\frac{\theta+1}{\theta+k}\right) \tag{2.9}$$

Now

$$K_{c,p}^{\delta}L_\theta f(z)=\frac{(c+p)^\delta}{\Gamma(\delta)z^c}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}L_\theta f(t)dt$$

$$=\frac{(c+p)^\delta}{\Gamma(\delta)z^c}\left[\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}\left(\frac{\theta+1}{\theta+p}\right)t^p dt+\sum_{k=p+n}^{\infty}a_k\left(\frac{\theta+1}{\theta+k}\right)\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}t^k dt\right]$$

then

$$K_{c,p}^{\delta}L_\theta f(z)=\frac{\theta+1}{\theta+p}z^p+\sum_{k=p+n}^{\infty}\left(\frac{\theta+1}{\theta+k}\right)\left(\frac{c+p}{c+k}\right)^\delta a_k z^k \tag{2.10}$$

and

$$\left(K_{c,p}^{\delta}\left(L_\theta f(z)\right)\right)'=p\left(\frac{\theta+1}{\theta+p}\right)z^{p-1}+\sum_{k=p+n}^{\infty}\left(\frac{\theta+1}{\theta+k}\right)\left(\frac{c+p}{c+k}\right)^\delta a_k kz^{k-1}$$

Then

$$J^{1,\delta}_{c,p,\lambda}(L_\theta f(z)) = (1-\lambda)K^{\delta}_{c,p}(L_\theta f(z)) + \frac{\lambda z}{p}\left(K^{\delta}_{c,p}\left(L_\theta f(z)\right)\right)'$$

$$J^{1,\delta}_{c,p,\lambda}(L_\theta f(z)) = \left(\frac{\theta+1}{\theta+p}\right) z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{\theta+1}{\theta+k}\right)\left(\frac{c+p}{c+k}\right)^{\delta} [1+\frac{\lambda}{p}(k-p)]$$

By induction

$$J^{m,\delta}_{c,p,\lambda}(L_\theta f(z)) = \frac{\theta+1}{\theta+p} z^p + \sum_{k=p+n}^{\infty} \frac{\theta+1}{\theta+k}\left(\left(\frac{c+p}{c+k}\right)^{\delta}\left[1+\frac{\lambda}{p}(k-p)\right]\right)^m a_k z^k \tag{2.11}$$

and

$$z\left(J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)\right)' = (\theta+1) J^{m,\delta}_{c,p,\lambda} f(z) - \theta J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)$$

Let

$$d(z) = \frac{1}{p-\eta}\left(\frac{z\left(J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)\right)'}{J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)} - \eta\right), \ (z \in U) \tag{2.12}$$

where $d(z)$ is analytic function in U, with $d(0)=1$

Now

$$(p-\eta)d(z)+\eta = \frac{z\left(J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)\right)'}{J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)},$$

$$(p-\eta)d(z)+\eta+\theta = (\theta+1)\frac{J^{m,\delta}_{c,p,\lambda} f(z)}{J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)}$$

Differentiating logarithmically with respect to z and multiplying by z

$$\frac{z(p-\eta)d'(z)}{\theta+\eta+(p-\eta)d(z)} + [(p-\eta)d+\eta] = \frac{z(J^{m,\delta}_{c,p,\lambda} f(z))'}{J^{m,\delta}_{c,p,\lambda} f(z)}$$

Dividing both sides by $(p-\eta)$ and adding and subtracting $\dfrac{\eta}{p-\eta}$

$$\frac{zd'(z)}{\theta+\eta+(p-\eta)d(z)} + d(z) = \frac{1}{p-\eta}\left[\frac{z(J^{m,\delta}_{c,p,\lambda}(f(z))'}{J^{m,\delta}_{c,p,\lambda} f(z)} - \eta\right], \ z \in U$$

Therefore by Lemma (2.1), we obtain:

$$d(z) + \frac{zd'(z)}{c+\eta+(p-\eta)d(z)} \prec h(z)$$

Then
$$d(z) = \frac{1}{p-\eta}[\frac{z(J^{m,\delta}_{c,p,\lambda}(L_\theta f(z)))'}{J^{m,\delta}_{c,p,\lambda}(L_\theta f(z))} - \eta] \prec h(z)$$

Corollary 2.1. If $f(z) \in S^{m,\delta}_{c,p,\lambda}(\eta, A, B)$, then $L_\theta(f) \in S^{m,\delta}_{c,p,\lambda}(\eta, A, B)$, where $L_\theta(f)$ is the integral operator defined by (2.8).

Proof:

In Proposition (2.2), take $h(z)=\dfrac{1+Az}{1+Bz}$, then

$$\frac{1}{p-\eta}\left[\frac{z\,(J_{c,p,\lambda}^{m,\delta}f\,(z))'}{J_{c,p,\lambda}^{m,\delta}f\,(z)}-\eta\right]\prec\frac{1+Az}{1+Bz}$$

$$\frac{1}{p-\eta}\left[\frac{z\,(J_{c,p,\lambda}^{m,\delta}(L_\theta f\,(z)))'}{J_{c,p,\lambda}^{m,\delta}(L_\theta f\,(z))}-\eta\right]\prec\frac{1+Az}{1+Bz}.$$

Proposition 2.3. Let $h(z)$ be convex univalent in U with $h(0)=1$ and $\operatorname{Re}(h(z))>0$; if a function $f\,(z)\in A(p,n)$ satisfies the condition

$$\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}\left(L_\theta f\,(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}\left(L_\theta f\,(z)\right)}-\eta\right)\prec h(z)\;,\quad(0\le\eta<1;z\in U)$$

Then

$$\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)}-\eta\right)\prec h(z)\quad,\ (0\le\eta<1;z\in U)\tag{2.13}$$

Where $L_\theta(f)$ is the integral operator defined by (2.8).

Proof:

$$z\left(J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)\right)'=(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}\left(L_\theta f\,(z)\right)-c\,J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)$$

Let

$$d(z)=\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)}-\eta\right)\quad,(z\in U)\tag{2.14}$$

Where $d(z)$ is analytic function in U, with $d(0)=1$

Now $\quad(p-\eta)d(z)+\eta=\dfrac{z\left(J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_\theta f\,(z)\right)}$

Differentiating logarithmically with respect to z and multiplying by z

$$\frac{z\,(p-\eta)d'(z)}{c+\eta+(p-\eta)d(z)}=\frac{z\,(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}(L_\theta f\,(z)))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}(L_\theta f\,(z))}-\frac{z\,(J_{c,p,\lambda}^{m,\delta}(L_\theta f\,(z)))'}{J_{c,p,\lambda}^{m,\delta}(L_\theta f\,(z))}$$

Dividing both sides by $(p-\eta)$ and adding and subtracting $\dfrac{\eta}{p-\eta}$

$$\frac{zd'(z)}{c+\eta+(p-\eta)d(z)}+d(z)=\frac{1}{p-\eta}\left[\frac{z\,(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}(L_\theta f\,(z)))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}(L_\theta f\,(z))}-\eta\right].$$

Therefore by Lemma (2.1), we obtain $d(z)+\dfrac{zd'(z)}{c+\eta+(p-\eta)d(z)}\prec h(z)$

Then

$$d(z)=\frac{1}{p-\eta}\left[\frac{z(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)))'}{J_{c,p,\lambda}^{m,\delta}(L_\theta f(z))}-\eta\right]\prec h(z)\,.$$

Theorem 2.1. Let $f(z)\in A(p,n)$ and $0<\delta_1,\delta_2\le 1,\ 0\le\gamma<1$, if

$$-\frac{\pi}{2}\delta_1<\arg\left(\frac{z(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}g(z)}-\gamma\right)<\frac{\pi}{2}\delta_2$$

for some $g(z)\in S_{c,p,\lambda}^{m,\delta}(\eta,A,B)$, then

$$-\frac{\pi}{2}\alpha_1<\arg\left(\frac{z(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)}-\gamma\right)<\frac{\pi}{2}\alpha_2$$

, where α_1,α_2 $(0<\alpha_1,\alpha_2\le 1)$, are the solutions of the equations :

$$\delta_1=\begin{cases}\alpha_1+\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_1}{2(\frac{(p-\eta)(1+\mathrm{A})}{1+\mathrm{B}}+\eta+c)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_1}\right. & ,\text{for } B\neq -1\\ \alpha_1 & ,\text{for } B=-1\end{cases}\qquad(2.15)$$

and

$$\delta_2=\begin{cases}\alpha_2+\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_1}{2(\frac{(p-\eta)(1+A)}{1+B}+\eta+c)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_1}\right. & \text{for } B\neq -1\\ \alpha_2 & \text{for } B=-1\end{cases}\qquad(2.16)$$

where C^* is given by (2.3) and

$$t_1=\frac{2}{\pi}\sin^{-1}\left(\frac{(p-\eta)(1-B)}{(p-\eta)(1-AB)+(\eta+c)(1-B^2)}\right)\qquad(2.17)$$

Proof:

Let $$d(z)=\frac{1}{p-\gamma}\left(\frac{z(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)}-\gamma\right),\ (z\in U)$$

Now

$$z(J_{c,p,\lambda}^{m,\delta}f(z))'=(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)-cJ_{c,p,\lambda}^{m,\delta}f(z)$$

and

$$((p-\gamma)d(z)+\gamma)J_{c,p,\lambda}^{m,\delta}g(z)=z(J_{c,p,\lambda}^{m,\delta}f(z))'=(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)-cJ_{c,p,\lambda}^{m,\delta}f(z)$$

Differentiating both sides with respect to z and multiplying by z

$$\begin{aligned}&z(p-\gamma)d'(z)J_{c,p,\lambda}^{m,\delta}g(z)+z[(p-\gamma)d(z)+\gamma](J_{c,p,\lambda}^{m,\delta}g(z))'\\&\qquad=z(c+p)(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'-zc(J_{c,p,\lambda}^{m,\delta}f(z))'\end{aligned}\qquad(2.18)$$

Let

$$q(z)=\frac{1}{p-\eta}\left(\frac{z\,(J^{m,\delta}_{c,p,\lambda}g(z))'}{J^{m,\delta}_{c,p,\lambda}g(z)}-\eta\right),\ (z\in U)$$

Then

$$(p-\eta)q+\eta=\frac{z\,(J^{m,\delta}_{c,p,\lambda}g(z))'}{J^{m,\delta}_{c,p,\lambda}g(z)}=\frac{(c+p)J^{m,\delta-\frac{1}{m}}_{c,p,\lambda}g(z)}{J^{m,\delta}_{c,p,\lambda}g(z)}-c$$

$$(p-\eta)q+\eta+c=\frac{(c+p)J^{m,\delta-\frac{1}{m}}_{c,p,\lambda}g(z)}{J^{m,\delta}_{c,p,\lambda}g(z)} \tag{2.19}$$

From (2.18) and (2.19), we have

$$z\,(p-\gamma)d\,'(z)J^{m,\delta}_{c,p,\lambda}g(z)+z\,[(p-\gamma)d(z)+\gamma](J^{m,\delta}_{c,p,\lambda}g(z))'$$
$$=z\,(c+p)(J^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z))'-zc(J^{m,\delta}_{c,p,\lambda}f(z))'$$

Dividing both sides by $(p-\gamma)J^{m,\delta}_{c,p,\lambda}g(z)$ implies

$$zd\,'(z)+\frac{[(p-\gamma)d+\gamma]}{(p-\gamma)}((p-\eta)q+\eta)=\frac{z\,(c+p)}{(p-\gamma)}\frac{(J^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z))'}{J^{m,\delta}_{c,p,\lambda}g(z)}-\frac{[(p-\gamma)d(z)+\gamma]c}{(p-\gamma)}$$

$$zd\,'(z)+\frac{[(p-\gamma)d+\gamma]}{(p-\gamma)}((p-\eta)q+\eta+c)=\frac{z\,(c+p)}{(p-\gamma)}\frac{(J^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z))'}{J^{m,\delta}_{c,p,\lambda}g(z)}$$

Dividing both sides by $[(p-\eta)q+\eta+c]$, then we get

$$\frac{zd\,'(z)}{(p-\eta)q+\eta+c}+d(z)=\frac{1}{(p-\gamma)}\left[\frac{z\,(J^{m,\delta-\frac{1}{m}}_{c,p,\lambda}f(z))'}{J^{m,\delta-\frac{1}{m}}_{c,p,\lambda}g(z)}-\gamma\right]$$

While, by using the result of Silverman and Silvia [3], we have

$$\left|q(z)-\frac{1-AB}{1-B^2}\right|<\frac{(A-B)}{1-B^2},\ (z\in U;B\neq-1) \tag{2.20}$$

and

$$\mathrm{Re}\{q(z)\}>\frac{1-A}{2},\quad (z\in U;B=-1) \tag{2.21}$$

Then from (2.20) and (2.21), we obtain

$$(p-\eta)q+\eta+c=\rho e^{\frac{i\pi\varphi}{2}}$$

Where

$$\left\{\begin{array}{ll}\frac{(p-\eta)(1-A)}{(1-B)}+\eta+c<\rho<\frac{(p-\eta)(1+A)}{(1+B)}+\eta+c & \\ -t_1<\phi<t_1 & for\ \ B\neq-1\end{array}\right\}$$

where t_1 is given by (2.17), and

$$\left\{\begin{array}{ll}\frac{(p-\eta)(1-A)}{2}+\eta+c<\rho<\infty & \\ -1<\phi<1 & for\ \ B=-1\end{array}\right\}$$

Here we note that $d(z)$ is analytic in U with $d(0)=1$ and $\operatorname{Re}(d(z))>0$ in U by applying the assumption and Lemma (2.2) with $w(z)=\dfrac{1}{(p-\eta)q+\eta+c}$.

Here $d(z)\neq 0$ in U. If the following two points exist: $z_1, z_2 \in U$ such that the condition (2.1) is satisfied; then (by Lemma 2.3) we obtain (2.2) under the restriction (2.3). At first, for the case $B\neq -1$, we obtain:

$$\arg\left(d(z_1)+\frac{z_1 d'(z_1)}{(p-\eta)q(z_1)+\eta+c}\right)$$
$$\leq -\frac{\pi}{2}\alpha_1-\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_1}{2(\frac{(p-\eta)(1+A)}{(1+B)}+\eta+c)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_1}\right\}=-\frac{\pi}{2}\delta_1$$

and

$$\arg\left(d(z_2)+\frac{z_2 d'(z_2)}{(p-\eta)q(z_2)+\eta+c}\right)$$
$$\geq \frac{\pi}{2}\alpha_2+\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_1}{2(\frac{(p-\eta)(1+A)}{(1+B)}+\eta+c)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_1}\right\}=\frac{\pi}{2}\delta_2$$

where we have used inequality (2.3), and δ_1,δ_2, t_1 are given by (2.15) , (2.16) and (2.17) respectively .

Similarly, for the case $B=-1$, we obtain

$$\arg\left(d(z_1)+\frac{z_1 d'(z_1)}{(p-\eta)q(z_1)+\eta+c}\right)\leq -\frac{\pi}{2}\alpha_1$$

and

$$\arg\left(d(z_2)+\frac{z_2 d'(z_2)}{(p-\eta)q(z_2)+\eta+c}\right)\geq \frac{\pi}{2}\alpha_2$$

Which contradicts the assumption of the Theorem, and hence the proof is completed.

Corollary 2. 2. Let $f(z)\in A(p,n)$, if

$$\left|\arg\left(\frac{z(I_p^{\delta-1}f(z))'}{I_p^{\delta-1}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta$$

Then

$$\left|\arg\left(\frac{z(I_p^{\delta}f(z))'}{I_p^{\delta}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha$$

where

$$I_p^{\delta-1}f(z)=J_{1,p,0}^{1,\delta-1}f(z)=z^p+\sum_{k=p+n}^{\infty}a_k z^k\left(\frac{1+p}{1+k}\right)^{\delta-1} \tag{2.22}$$

Proof:

In Theorem (2.1), if we put $m=1, c=1, \lambda=0$, $\delta_1=\delta_2=\delta$ and $\alpha_1=\alpha_2=\alpha$, we get

$$I_p^{\delta-1} f(z)=J_{1,p,0}^{1,\delta-1} f(z)=z^p+\sum_{k=p+n}^{\infty} a_k z^k\left(\frac{1+p}{1+k}\right)^{\delta-1}$$

then

$$\left|\arg\left(\frac{z\,(I_p^{\delta-1} f(z))'}{I_p^{\delta-1} g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta$$

and

$$\left|\arg\left(\frac{z\,(I_p^{\delta} f(z))'}{I_p^{\delta} g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha$$

Theorem 2. 2. Let $f(z)\in A(p,n)$ and $0<\delta_1,\delta_2\le 1, 0\le\gamma<1$, if

$$-\frac{\pi}{2}\delta_1<\arg\left(\frac{z\,(J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)}-\gamma\right)<\frac{\pi}{2}\delta_2$$

for some $g(z)\in S_{c,p,\lambda}^{m,\delta}(\eta,A,B)$, then

$$-\frac{\pi}{2}\alpha_1<\arg\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}\left(L_\theta f(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_\theta g(z)\right)}-\gamma\right)<\frac{\pi}{2}\alpha_2\,,$$

where $L_\theta(f)$ is defined by (2.8), α_1, α_2 $(0<\alpha_1,\alpha_2\le 1)$ is the solutions of the equations

$$\delta_1=\begin{cases}\alpha_1+\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\theta\right)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_2}\right) & \text{for } B\neq -1\\ \alpha_1 & \text{for } B=-1\end{cases} \tag{2.23}$$

and

$$\delta_2=\begin{cases}\alpha_2+\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\theta\right)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_2}\right) & \text{for } B\neq -1\\ \alpha_2 & \text{for } B=-1\end{cases} \tag{2.24}$$

Where c^* is given by (2.3) and

$$t_2=\frac{2}{\pi}\sin^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+(\eta+\theta)(1-B^2)}\right) \tag{2.25}$$

Proof:

Let

$$d(z)=\frac{1}{p-\gamma}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}-\gamma\right),$$

Since $g(z)\in S_{c,p,\lambda}^{m,\delta}(\eta,A,B)$, then by Corollary (2.1), $L_{\theta}(g)\in S_{c,p,\lambda}^{m,\delta}(\eta,A,B)$, then

$$[(p-\gamma)d(z)+\gamma]J_{c,p,\lambda}^{m,\delta}(L_{\theta}g(z))=z(J_{c,p,\lambda}^{m,\delta}(L_{\theta}f(z)))'$$

From Proposition (2.2)

$$z\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)\right)'=(\theta+1)J_{c,p,\lambda}^{m,\delta}f(z)-\theta J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)$$

$$((p-\gamma)d+\gamma)J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)=(\theta+1)J_{c,p,\lambda}^{m,\delta}f(z)-\theta J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)$$

Differentiating both sides with respect to z

$$((p-\gamma)d+\gamma)\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)\right)'+(p-\gamma)d'(z)J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)=(\theta+1)\left(J_{c,p,\lambda}^{m,\delta}f(z)\right)'-\theta\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)\right)'$$

Dividing both sides by $(J_{c,p,\lambda}^{m,\delta}L_{\theta}g(z))$ and multiplying both sides by z, then

$$z((p-\gamma)d+\gamma)\frac{(J_{c,p,\lambda}^{m,\delta}L_{\theta}g(z))'}{J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}+(p-\gamma)zd'(z)=z(\theta+1)\frac{\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}-z\theta\frac{\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}$$

Let

$$q(z)=\frac{1}{p-\eta}\left(\frac{z\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}-\eta\right)$$

Then

$$((p-\gamma)d+\gamma)(q(p-\eta)+\eta+\theta)+(p-\gamma)zd'(z)=z(\theta+1)\frac{\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}$$

Now dividing both sides by $(p-\gamma)$

$$(d+\frac{\gamma}{(p-\gamma)})(q(p-\eta)+\eta+\theta)+zd'(z)=\frac{z(\theta+1)}{(p-\gamma)}\frac{\left(J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}f(z)\right)\right)'}{J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}$$

Dividing both sides by $(q(p-\eta)+\eta+\theta)$

$$d(z)+\frac{zd'(z)}{(q(p-\eta)+\eta+\theta)}=\frac{1}{(p-\gamma)}\left[\frac{z(\theta+1)(J_{c,p,\lambda}^{m,\delta}(f(z))'}{(q(p-\eta)+\eta+\theta)J_{c,p,\lambda}^{m,\delta}\left(L_{\theta}g(z)\right)}-\gamma\right]$$

$$d(z)+\frac{zd'(z)}{(q(p-\eta)+\eta+\theta)}=\frac{1}{(p-\gamma)}\left[\frac{z(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)}-\gamma\right]$$

Then from (2.20) and (2.21), we obtain

$$(p-\eta)q+\eta+\theta=\rho e^{\frac{i\pi\varphi}{2}}$$

where

$$\begin{cases}\frac{(p-\eta)(1-\mathrm{A})}{(1-\mathrm{B})}+\eta+\theta<\rho<\frac{(p-\eta)(1+\mathrm{A})}{(1+\mathrm{B})}+\eta+\theta \\ -\mathrm{t}_2<\phi<\mathrm{t}_2 \qquad\qquad for\ \mathrm{B}\neq-1\end{cases}$$

where t_2 is given by (2.25), and

$$\begin{cases}\frac{(p-\eta)(1-\mathrm{A})}{2}+\eta+\theta<\rho<\infty \\ -1<\phi<1 \qquad\qquad for\ \mathrm{B}=-1\end{cases}$$

Here, we note that $d(z)$ is analytic in U with $\mathrm{d}(0)=1$ in U by applying the assumption and Lemma (2.2) with $w(z)=\frac{1}{(p-\eta)q+\eta+\theta}$

Hence, $d(z)\neq 0$ in U if the following two points exist $z_1,z_2\in U$, such that the condition (2.1) is satisfied then (by Lemma 2.3), we obtain (2.2) under the restriction (2.3).

At first, for the case $B\neq-1$

$$\arg(d(z_1)+\frac{z_1d'(z_1)}{(p-\eta)q(z_1)+\eta+\theta})\leq-\frac{\pi}{2}\alpha_1-\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_2}{2(\frac{(p-\eta)(1+A)}{(1+B)}+\eta+\theta)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_2}\right\}$$

$$=-\frac{\pi}{2}\delta_1$$

and

$$\arg(d(z_2)+\frac{z_2d'(z_2)}{(p-\eta)q(z_2)+\eta+\theta})\geq\frac{\pi}{2}\alpha_2+\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|c^*|)\cos\frac{\pi}{2}t_2}{2(\frac{(p-\eta)(1+A)}{(1+B)}+\eta+\theta)(1+|c^*|)+(\alpha_1+\alpha_2)(1-|c^*|)\sin\frac{\pi}{2}t_2}\right\}$$

$$=\frac{\pi}{2}\delta_2$$

where we have used the inequality (2.3), and δ_1,δ_2, t_2 are given by (2.15), (2.16) and (2.25) respectively.

Similarly, for the case $B=-1$, we obtain

$$\arg(d(z_1)+\frac{z_1d'(z_1)}{(p-\eta)q(z_1)+\eta+\theta})\leq-\frac{\pi}{2}\alpha_1$$

and

$$\arg(d(z_2)+\frac{z_2 d'(z_2)}{(p-\eta)q(z_2)+\eta+\theta})\geq\frac{\pi}{2}\alpha_2$$

These are contradiction to the assumption of Theorem (2.2).

This completes the proof of the Theorem (2.2).

Corollary 2.3. Let $f(z)\in \mathrm{A}(p,n)$ and $0<\delta\leq 1,\ 0\leq\gamma<1$ if

$$\left|\arg(\frac{z(J^{m,\delta}_{c,p,\lambda}f(z))'}{J^{m,\delta}_{c,p,\lambda}g(z)}-\gamma)\right|<\frac{\pi}{2}\delta$$

for some $g(z)\in S^{m,\delta}_{c,p,\lambda}(\eta,A,B)$

, then

$$\left|\arg(\frac{z(J^{m,\delta}_{c,p,\lambda}L_\theta f(z))'}{J^{m,\delta}_{c,p,\lambda}L_\theta g(z)}-\gamma)\right|<\frac{\pi}{2}\alpha$$

where $L_\theta(f)$ is defined by (2.8), and $(0<\alpha\leq 1)$ is the solution of the equation

$$\delta=\begin{cases}\alpha+\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{\alpha\cos\frac{\pi}{2}t_2}{\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\theta\right)+\alpha\sin\frac{\pi}{2}t_2}\right) & for\ \ B\neq -1\\ \alpha & for\ \ B=-1\end{cases}$$

where t_2 is given by (2.25)

Proof:

Take $\delta_1=\delta_2=\delta$ and $\alpha_1=\alpha_2=\alpha$ in Theorem (2.2)

$$\left|\arg\left(\frac{z(J^{m,\delta}_{c,p,\lambda}f(z))'}{J^{m,\delta}_{c,p,\lambda}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta$$

, for some $g(z)\in S^{m,\delta}_{c,p,\lambda}(\eta,A,B)$, then

$$\left|\arg\left(\frac{z\left(J^{m,\delta}_{c,p,\lambda}\left(L_\theta f(z)\right)\right)'}{J^{m,\delta}_{c,p,\lambda}\left(L_\theta g(z)\right)}-\gamma\right)\right|<\frac{\pi}{2}\alpha$$

where

$$\delta=\begin{cases}\alpha+\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{\alpha\cos\frac{\pi}{2}t_2}{\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\theta\right)+\alpha\sin\frac{\pi}{2}t_2}\right) & for\ \ B\neq-1\\ \alpha & for\ \ B=-1\end{cases}$$

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