

On Subclasses of Multivalent Functions Defined by a Multiplier Operator Involving the Komatu Integral Operator

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Communicated by Ayman Badawi

MSC 2010 Classification: 37E45,37E30,37E40

Keywords and phrases: Multivalent functions, Komatu integral operator.

The author would like to thank the referee for his valuable suggestions and comments.

Abstract. This paper is devoted to the study of some new subclasses strongly close-to-convex p -valent functions. It is defined by a multiplier operator using the Komatu integral operator and studies their inclusion relationships with the integral preserving properties.

1. INTRODUCTION

Let $A(p, n)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{N}) \tag{1.1}$$

which are analytic and p -valent in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$.

The generalized Komatu integral operator $K_{c,p}^{\delta} : A(p, n) \rightarrow A(p, n)$ is defined for $\delta > 0$ and $c > -p$ as

$$K_{c,p}^{\delta} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta)} z^c \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t) dt \tag{1.2}$$

Now, in terms of $K_{c,p}^{\delta}$, we introduce the linear multiplier operator

$J_{c,p,\lambda}^{m,\delta} : A(p, n) \rightarrow A(p, n)$ as follows:

$$J_{c,p,\lambda}^{0,\delta} f(z) = f(z) \tag{1.3}$$

$$J_{c,p,\lambda}^{1,\delta} f(z) = (1-\lambda)K_{c,p}^{\delta} f(z) + \frac{\lambda z}{p} (K_{c,p}^{\delta} f(z))' = J_{c,p,\lambda}^{\delta} f(z)$$

⋮

$$J_{c,p,\lambda}^{m,\delta} f(z) = J_{c,p,\lambda}^{\delta} (J_{c,p,\lambda}^{m-1,\delta} f(z))$$

for $\delta > 0, c > -p, \lambda \geq 0$ and $m \in \mathbb{N}$.

If $f \in A(p, n)$ is given by (1.1), then

$$J_{c,p,\lambda}^{m,\delta} f(z) = z^p + \sum_{k=p+n}^{\infty} B_{k,m}(c, p, \lambda, \delta) a_k z^k \tag{1.4}$$

where

$$B_{k,m}(c,p,\lambda,\delta) = \left[\left(\frac{c+p}{c+k} \right)^\delta \frac{\lambda}{p} (k-p) \right]^m \quad (1.5)$$

for $\delta > 0$, $c > -p$, $\lambda \geq 0$ and $m \in \mathbb{R}$.

If $f(z)$ and $g(z)$ are analytic in U , we say that $f(z)$ is subordinate to $g(z)$, and if a Schwarz function $w(z)$ in U such that $f(z) = g(w(z))$ exists then we write it as $f \prec g$ or $f(z) \prec g(z)$.

2. Results

Let $S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$ be the class of functions $f \in A(p, n)$ satisfying the condition

$$\frac{1}{p-\eta} \left(\frac{z (J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \prec \frac{1+Az}{1+Bz} \quad (2.0)$$

, then the following results appear:

Lemma 2.1. Let $h(z)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re}\{vh(z) + \mu\} > 0$ ($v, \mu \in \mathbb{R}$). If $p(z)$ is analytic in U with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{vp(z) + \mu} \prec h(z), \quad (z \in U) \text{ which implies } p(z) \prec h(z), \quad (z \in U)$$

Lemma 2.2. Let $h(z)$ be convex univalent in U and $w(z)$ be analytic in U with $\operatorname{Re} w(z) \geq 0$. If $p(z)$ is analytic in U with $p(0) = h(0)$, then $p(z) + w(z)zp'(z) \prec h(z)$, ($z \in U$) which implies $p(z) \prec h(z)$, ($z \in U$).

Lemma 2.3. Let $p(z)$ be analytic in U with $p(0) = 1$ and $p(z) \neq 0$ in U . If two points exist $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2}\alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\alpha_2 \quad (2.1)$$

for some α_1, α_2 ($\alpha_1, \alpha_2 > 0$) and for all ($|z| < |z_1| = |z_2|$), then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i \frac{(\alpha_1 + \alpha_2)}{2} m \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i \frac{(\alpha_1 + \alpha_2)}{2} m \quad (2.2)$$

where

$$m \geq \frac{1-|c^*|}{1+|c^*|} \quad \text{and} \quad c^* = i \tan \frac{\pi}{4} \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right) \quad (2.3)$$

Proposition 2.1. Let $\delta > 0$, $c > -p$, $\lambda \geq 0$, $m \in \mathbb{R}$ and $h(z)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$; if a function $f(z) \in A(p, n)$ satisfies the condition

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U),$$

then

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U).$$

Proof:

Let

$$d(z) = \frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \quad (2.4)$$

where $d(z)$ is analytic function in U

$$\begin{aligned} d(z) &= \frac{1}{p-\eta} \left(\frac{(c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z) - c J_{c,p,\lambda}^{m,\delta} f(z)}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right) \\ &= \frac{1}{p-\eta} \left(\frac{(c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)}{J_{c,p,\lambda}^{m,\delta} f(z)} - c - \eta \right) \\ &= \frac{1}{p-\eta} \left[\frac{cz^p + pz^p + \sum_{k=p+n}^{\infty} a_k z^k \frac{(c+p)^\delta}{(c+k)^{\delta m-1}} \left(1 + (k-p) \frac{\lambda}{p} \right)^m}{z^p + \sum_{k=p+n}^{\infty} \left[\left(\frac{c+p}{c+k} \right)^\delta \left(\left(1 + (k-p) \frac{\lambda}{p} \right) \right)^m \right] a_k z^k} - c - \eta \right] \end{aligned}$$

then $d(0) = 1$

From (2.4), we get

$$(p-\eta)d(z) + c + \eta = \frac{(c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)}{J_{c,p,\lambda}^{m,\delta} f(z)} \quad (2.5)$$

Differentiating both sides logarithmically with respect to z and multiplying them by z yields

$$\frac{z(p-\eta)d'(z)}{(p-\eta)d(z) + c + \eta} = \frac{z \left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} f(z)} - \frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} \quad (2.6)$$

Dividing both sides by $p-\eta$, we get

$$\frac{zd'(z)}{(p-\eta)d(z)+c+\eta}+d(z)=\frac{1}{p-\eta}\left(\frac{z(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z)}-\eta\right) \quad (2.7)$$

By using Lemma (2.1), it follows that $d(z) \prec h(z), (z \in U)$, then

$$\frac{1}{p-\eta}\left(\frac{z(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right) \prec h(z).$$

Proposition 2.2. Let $h(z)$ be a convex univalent in U with $h(0)=1$ and $\operatorname{Re}(h(z))>0$; if a function $f(z) \in A(p, n)$ satisfies the condition

$$\frac{1}{p-\eta}\left(\frac{z(J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}f(z)}-\eta\right) \prec h(z), \quad (0 \leq \eta < 1; z \in U),$$

then

$$\frac{1}{p-\eta}\left(\frac{z(J_{c,p,\lambda}^{m,\delta}L_{\theta}f(z))'}{J_{c,p,\lambda}^{m,\delta}L_{\theta}f(z)}-\eta\right) \prec h(z), \quad (0 \leq \eta < 1; z \in U),$$

where $L_{\theta}(f)$ is the integral operator defined by

$$L_{\theta}(f) = L_{\theta}f(z) = \frac{(\theta+1)}{z^{\theta}} \int_0^z t^{\theta-1} f(t) dt, \quad (\theta \geq 0) \quad (2.8)$$

Proof:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N})$$

Then

$$\begin{aligned} L_{\theta}f(z) &= \frac{(\theta+1)}{z^{\theta}} \left[\int_0^z t^{\theta-1} t^p dt + \int_0^z t^{\theta-1} \sum_{k=p+n}^{\infty} a_k t^k dt \right] \\ L_{\theta}f(z) &= \left(\frac{\theta+1}{\theta+p} \right) z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{\theta+1}{\theta+k} \right) \end{aligned} \quad (2.9)$$

Now

$$\begin{aligned} K_{c,p}^{\delta} L_{\theta}f(z) &= \frac{(c+p)^{\delta}}{\Gamma(\delta)z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t} \right)^{\delta-1} L_{\theta}f(t) dt \\ &= \frac{(c+p)^{\delta}}{\Gamma(\delta)z^c} \left[\int_0^z t^{c-1} \left(\log \frac{z}{t} \right)^{\delta-1} \left(\frac{\theta+1}{\theta+p} \right) t^p dt + \sum_{k=p+n}^{\infty} a_k \left(\frac{\theta+1}{\theta+k} \right) \int_0^z t^{c-1} \left(\log \frac{z}{t} \right)^{\delta-1} t^k dt \right] \end{aligned}$$

then

$$K_{c,p}^{\delta} L_{\theta}f(z) = \frac{\theta+1}{\theta+p} z^p + \sum_{k=p+n}^{\infty} \left(\frac{\theta+1}{\theta+k} \right) \left(\frac{c+p}{c+k} \right)^{\delta} a_k z^k \quad (2.10)$$

and

$$\left(K_{c,p}^{\delta} (L_{\theta}f(z)) \right)' = p \left(\frac{\theta+1}{\theta+p} \right) z^{p-1} + \sum_{k=p+n}^{\infty} \left(\frac{\theta+1}{\theta+k} \right) \left(\frac{c+p}{c+k} \right)^{\delta} a_k k z^{k-1}$$

Then

$$J_{c,p,\lambda}^{1,\delta}(L_\theta f(z)) = (1-\lambda)K_{c,p}^\delta(L_\theta f(z)) + \frac{\lambda z}{p}(K_{c,p}^\delta(L_\theta f(z)))'$$

$$J_{c,p,\lambda}^{1,\delta}(L_\theta f(z)) = \left(\frac{\theta+1}{\theta+p}\right)z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{\theta+1}{\theta+k}\right)\left(\frac{c+p}{c+k}\right)^\delta \left[1 + \frac{\lambda}{p}(k-p)\right]$$

By induction

$$J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) = \frac{\theta+1}{\theta+p}z^p + \sum_{k=p+n}^{\infty} \frac{\theta+1}{\theta+k} \left(\frac{c+p}{c+k}\right)^\delta \left[1 + \frac{\lambda}{p}(k-p)\right]^m a_k z^k \quad (2.11)$$

and

$$z \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)' = (\theta+1)J_{c,p,\lambda}^{m,\delta} f(z) - \theta J_{c,p,\lambda}^{m,\delta}(L_\theta f(z))$$

Let

$$d(z) = \frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta f(z))} - \eta \right), \quad (z \in U) \quad (2.12)$$

where $d(z)$ is analytic function in U , with $d(0) = 1$

Now

$$(p-\eta)d(z) + \eta = \frac{z \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta f(z))},$$

$$(p-\eta)d(z) + \eta + \theta = (\theta+1) \frac{J_{c,p,\lambda}^{m,\delta} f(z)}{J_{c,p,\lambda}^{m,\delta}(L_\theta f(z))}$$

Differentiating logarithmically with respect to z and multiplying by z

$$\frac{z(p-\eta)d'(z)}{\theta + \eta + (p-\eta)d(z)} + [(p-\eta)d + \eta] = \frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)}$$

Dividing both sides by $(p-\eta)$ and adding and subtracting $\frac{\eta}{p-\eta}$

$$\frac{zd'(z)}{\theta + \eta + (p-\eta)d(z)} + d(z) = \frac{1}{p-\eta} \left[\frac{z \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right], \quad z \in U$$

Therefore by Lemma (2.1), we obtain:

$$d(z) + \frac{zd'(z)}{c + \eta + (p-\eta)d(z)} \prec h(z)$$

Then

$$d(z) = \frac{1}{p-\eta} \left[\frac{z \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta f(z))} - \eta \right] \prec h(z)$$

Corollary 2.1. If $f(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then $L_\theta(f) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, where $L_\theta(f)$ is the integral operator defined by (2.8).

Proof:

In Proposition (2.2), take $h(z) = \frac{1+Az}{1+Bz}$, then

$$\frac{1}{p-\eta} \left[\frac{z (J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} f(z)} - \eta \right] \prec \frac{1+Az}{1+Bz}$$

$$\frac{1}{p-\eta} \left[\frac{z (J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z)))'}{J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z))} - \eta \right] \prec \frac{1+Az}{1+Bz}.$$

Proposition 2.3. Let $h(z)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re}(h(z)) > 0$; if a function $f(z) \in A(p, n)$ satisfies the condition

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_{\theta} f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_{\theta} f(z))} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U)$$

Then

$$\frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z))} - \eta \right) \prec h(z), \quad (0 \leq \eta < 1; z \in U) \quad (2.13)$$

Where $L_{\theta}(f)$ is the integral operator defined by (2.8).

Proof:

$$z \left(J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z)) \right)' = (c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_{\theta} f(z)) - c J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z))$$

Let

$$d(z) = \frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z))} - \eta \right), \quad (z \in U) \quad (2.14)$$

Where $d(z)$ is analytic function in U , with $d(0) = 1$

$$\text{Now } (p-\eta)d(z) + \eta = \frac{z \left(J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z))}$$

Differentiating logarithmically with respect to z and multiplying by z

$$\frac{z(p-\eta)d'(z)}{c+\eta+(p-\eta)d(z)} = \frac{z \left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_{\theta} f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_{\theta} f(z))} - \frac{z \left(J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z))}$$

Dividing both sides by $(p-\eta)$ and adding and subtracting $\frac{\eta}{p-\eta}$

$$\frac{zd'(z)}{c+\eta+(p-\eta)d(z)} + d(z) = \frac{1}{p-\eta} \left[\frac{z \left(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_{\theta} f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} (L_{\theta} f(z))} - \eta \right].$$

Therefore by Lemma (2.1), we obtain $d(z) + \frac{zd'(z)}{c+\eta+(p-\eta)d(z)} \prec h(z)$

Then

$$d(z) = \frac{1}{p-\eta} \left[\frac{z (J_{c,p,\lambda}^{m,\delta}(L_{\theta}f(z)))'}{J_{c,p,\lambda}^{m,\delta}(L_{\theta}f(z))} - \eta \right] \prec h(z).$$

Theorem 2.1. Let $f(z) \in A(p, n)$ and $0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1$, if

$$-\frac{\pi}{2} \delta_1 < \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}g(z)} - \gamma \right) < \frac{\pi}{2} \delta_2$$

for some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$-\frac{\pi}{2} \alpha_1 < \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)} - \gamma \right) < \frac{\pi}{2} \alpha_2$$

, where α_1, α_2 ($0 < \alpha_1, \alpha_2 \leq 1$), are the solutions of the equations :

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi} \tan^{-1} \left(\frac{(\alpha_1 + \alpha_2)(1 - |c^*|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p-\eta)(1+A)}{1+B} + \eta + c \right) (1 + |c^*|) + (\alpha_1 + \alpha_2)(1 - |c^*|) \sin \frac{\pi}{2} t_1} \right) & , \text{for } B \neq -1 \\ \alpha_1 & , \text{for } B = -1 \end{cases} \quad (2.15)$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi} \tan^{-1} \left(\frac{(\alpha_1 + \alpha_2)(1 - |c^*|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p-\eta)(1+A)}{1+B} + \eta + c \right) (1 + |c^*|) + (\alpha_1 + \alpha_2)(1 - |c^*|) \sin \frac{\pi}{2} t_1} \right) & \text{for } B \neq -1 \\ \alpha_2 & \text{for } B = -1 \end{cases} \quad (2.16)$$

where C^* is given by (2.3) and

$$t_1 = \frac{2}{\pi} \sin^{-1} \left(\frac{(p-\eta)(1-B)}{(p-\eta)(1-AB) + (\eta+c)(1-B^2)} \right) \quad (2.17)$$

Proof:

Let
$$d(z) = \frac{1}{p-\gamma} \left(\frac{z (J_{c,p,\lambda}^{m,\delta}f(z))'}{J_{c,p,\lambda}^{m,\delta}g(z)} - \gamma \right), (z \in U)$$

Now

$$z (J_{c,p,\lambda}^{m,\delta}f(z))' = (c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z) - c J_{c,p,\lambda}^{m,\delta}f(z)$$

and

$$((p-\gamma)d(z) + \gamma) J_{c,p,\lambda}^{m,\delta}g(z) = z (J_{c,p,\lambda}^{m,\delta}f(z))' = (c+p) J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z) - c J_{c,p,\lambda}^{m,\delta}f(z)$$

Differentiating both sides with respect to z and multiplying by z

$$\begin{aligned} z(p-\gamma)d'(z) J_{c,p,\lambda}^{m,\delta}g(z) + z[(p-\gamma)d(z) + \gamma](J_{c,p,\lambda}^{m,\delta}g(z))' \\ = z(c+p)(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}}f(z))' - zc(J_{c,p,\lambda}^{m,\delta}f(z))' \end{aligned} \quad (2.18)$$

Let

$$q(z) = \frac{1}{p-\eta} \left(\frac{z (J_{c,p,\lambda}^{m,\delta} g(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \eta \right), \quad (z \in U)$$

Then

$$\begin{aligned} (p-\eta)q + \eta &= \frac{z (J_{c,p,\lambda}^{m,\delta} g(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} = \frac{(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z)}{J_{c,p,\lambda}^{m,\delta} g(z)} - c \\ (p-\eta)q + \eta + c &= \frac{(c+p)J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z)}{J_{c,p,\lambda}^{m,\delta} g(z)} \end{aligned} \quad (2.19)$$

From (2.18) and (2.19), we have

$$\begin{aligned} z(p-\gamma)d'(z)J_{c,p,\lambda}^{m,\delta} g(z) + z[(p-\gamma)d(z) + \gamma](J_{c,p,\lambda}^{m,\delta} g(z))' \\ = z(c+p)(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z))' - zc(J_{c,p,\lambda}^{m,\delta} g(z))' \end{aligned}$$

Dividing both sides by $(p-\gamma)J_{c,p,\lambda}^{m,\delta} g(z)$ implies

$$\begin{aligned} zd'(z) + \frac{[(p-\gamma)d + \gamma]}{(p-\gamma)}((p-\eta)q + \eta) &= \frac{z(c+p)}{(p-\gamma)} \frac{(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \frac{[(p-\gamma)d(z) + \gamma]c}{(p-\gamma)} \\ zd'(z) + \frac{[(p-\gamma)d + \gamma]}{(p-\gamma)}((p-\eta)q + \eta + c) &= \frac{z(c+p)}{(p-\gamma)} \frac{(J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} \end{aligned}$$

Dividing both sides by $[(p-\eta)q + \eta + c]$, then we get

$$\frac{zd'(z)}{(p-\eta)q + \eta + c} + d(z) = \frac{1}{(p-\gamma)} \left[\frac{z (J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z))'}{J_{c,p,\lambda}^{m,\delta-\frac{1}{m}} g(z)} - \gamma \right]$$

While, by using the result of Silverman and Silvia [3], we have

$$\left| q(z) - \frac{1-AB}{1-B^2} \right| < \frac{(A-B)}{1-B^2}, \quad (z \in U; B \neq -1) \quad (2.20)$$

and

$$\operatorname{Re}\{q(z)\} > \frac{1-A}{2}, \quad (z \in U; B = -1) \quad (2.21)$$

Then from (2.20) and (2.21), we obtain

$$(p-\eta)q + \eta + c = \rho e^{\frac{i\pi\phi}{2}}$$

Where

$$\left\{ \begin{array}{l} \frac{(p-\eta)(1-A)}{(1-B)} + \eta + c < \rho < \frac{(p-\eta)(1+A)}{(1+B)} + \eta + c \\ -t_1 < \phi < t_1 \end{array} \right\} \quad \text{for } B \neq -1$$

where t_1 is given by (2.17), and

$$\left\{ \begin{array}{l} \frac{(p-\eta)(1-A)}{2} + \eta + c < \rho < \infty \\ -1 < \phi < 1 \end{array} \right\} \quad \text{for } B = -1$$

Here we note that $d(z)$ is analytic in U with $d(0) = 1$ and $\operatorname{Re}(d(z)) > 0$ in U by applying the assumption and Lemma (2.2) with $w(z) = \frac{1}{(p-\eta)q + \eta + c}$.

Here $d(z) \neq 0$ in U . If the following two points exist: $z_1, z_2 \in U$ such that the condition (2.1) is satisfied; then (by Lemma 2.3) we obtain (2.2) under the restriction (2.3). At first, for the case $B \neq -1$, we obtain:

$$\begin{aligned} & \arg(d(z_1) + \frac{z_1 d'(z_1)}{(p-\eta)q(z_1) + \eta + c}) \\ & \leq -\frac{\pi}{2} \alpha_1 - \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1 - |c^*|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p-\eta)(1+A)}{(1+B)} + \eta + c \right) (1 + |c^*|) + (\alpha_1 + \alpha_2)(1 - |c^*|) \sin \frac{\pi}{2} t_1} \right\} = -\frac{\pi}{2} \delta_1 \end{aligned}$$

and

$$\begin{aligned} & \arg(d(z_2) + \frac{z_2 d'(z_2)}{(p-\eta)q(z_2) + \eta + c}) \\ & \geq \frac{\pi}{2} \alpha_2 + \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1 - |c^*|) \cos \frac{\pi}{2} t_1}{2 \left(\frac{(p-\eta)(1+A)}{(1+B)} + \eta + c \right) (1 + |c^*|) + (\alpha_1 + \alpha_2)(1 - |c^*|) \sin \frac{\pi}{2} t_1} \right\} = \frac{\pi}{2} \delta_2 \end{aligned}$$

where we have used inequality (2.3), and δ_1, δ_2, t_1 are given by (2.15), (2.16) and (2.17) respectively.

Similarly, for the case $B = -1$, we obtain

$$\arg(d(z_1) + \frac{z_1 d'(z_1)}{(p-\eta)q(z_1) + \eta + c}) \leq -\frac{\pi}{2} \alpha_1$$

and

$$\arg(d(z_2) + \frac{z_2 d'(z_2)}{(p-\eta)q(z_2) + \eta + c}) \geq \frac{\pi}{2} \alpha_2$$

Which contradicts the assumption of the Theorem, and hence the proof is completed.

Corollary 2.2. Let $f(z) \in A(p, n)$, if

$$\left| \arg \left(\frac{z (I_p^{\delta-1} f(z))'}{I_p^{\delta-1} g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

Then

$$\left| \arg \left(\frac{z (I_p^{\delta} f(z))'}{I_p^{\delta} g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha$$

where

$$I_p^{\delta-1} f(z) = J_{1,p,\delta-1} f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{1+p}{1+k} \right)^{\delta-1} \quad (2.22)$$

Proof:

In Theorem (2.1), if we put $m = 1, c = 1, \lambda = 0, \delta_1 = \delta_2 = \delta$ and $\alpha_1 = \alpha_2 = \alpha$, we get

$$I_p^{\delta-1} f(z) = J_{1,p,0}^{1,\delta-1} f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \left(\frac{1+p}{1+k} \right)^{\delta-1}$$

then

$$\left| \arg \left(\frac{z (I_p^{\delta-1} f(z))'}{I_p^{\delta-1} g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta$$

and

$$\left| \arg \left(\frac{z (I_p^{\delta} f(z))'}{I_p^{\delta} g(z)} - \gamma \right) \right| < \frac{\pi}{2} \alpha$$

Theorem 2.2. Let $f(z) \in A(p, n)$ and $0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1$, if

$$-\frac{\pi}{2} \delta_1 < \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma \right) < \frac{\pi}{2} \delta_2$$

for some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$-\frac{\pi}{2} \alpha_1 < \arg \left(\frac{z (J_{c,p,\lambda}^{m,\delta} (L_{\theta} f(z)))'}{J_{c,p,\lambda}^{m,\delta} (L_{\theta} g(z))} - \gamma \right) < \frac{\pi}{2} \alpha_2,$$

where $L_{\theta}(f)$ is defined by (2.8), α_1, α_2 ($0 < \alpha_1, \alpha_2 \leq 1$) is the solutions of the equations

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi} \tan^{-1} \left(\frac{(\alpha_1 + \alpha_2)(1 - |c^*|) \cos \frac{\pi}{2} t_2}{2 \left(\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta \right) (1 + |c^*|) + (\alpha_1 + \alpha_2)(1 - |c^*|) \sin \frac{\pi}{2} t_2} \right) & \text{for } B \neq -1 \\ \alpha_1 & \text{for } B = -1 \end{cases} \quad (2.23)$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi} \tan^{-1} \left(\frac{(\alpha_1 + \alpha_2)(1 - |c^*|) \cos \frac{\pi}{2} t_2}{2 \left(\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta \right) (1 + |c^*|) + (\alpha_1 + \alpha_2)(1 - |c^*|) \sin \frac{\pi}{2} t_2} \right) & \text{for } B \neq -1 \\ \alpha_2 & \text{for } B = -1 \end{cases} \quad (2.24)$$

Where c^* is given by (2.3) and

$$t_2 = \frac{2}{\pi} \sin^{-1} \left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB) + (\eta+\theta)(1-B^2)} \right) \quad (2.25)$$

Proof:

Let

$$d(z) = \frac{1}{p-\gamma} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta}(L\theta f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L\theta g(z))} - \gamma \right),$$

Since $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then by Corollary (2.1), $L_\theta(g) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$[(p-\gamma)d + \gamma] J_{c,p,\lambda}^{m,\delta}(L_\theta g(z)) = z \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'$$

From Proposition (2.2)

$$\begin{aligned} z \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)' &= (\theta+1) J_{c,p,\lambda}^{m,\delta} f(z) - \theta J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \\ ((p-\gamma)d + \gamma) J_{c,p,\lambda}^{m,\delta}(L_\theta g(z)) &= (\theta+1) J_{c,p,\lambda}^{m,\delta} f(z) - \theta J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \end{aligned}$$

Differentiating both sides with respect to z

$$((p-\gamma)d + \gamma) \left(J_{c,p,\lambda}^{m,\delta}(L_\theta g(z)) \right)' + (p-\gamma) z d'(z) J_{c,p,\lambda}^{m,\delta}(L_\theta g(z)) = (\theta+1) \left(J_{c,p,\lambda}^{m,\delta} f(z) \right)' - \theta \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'$$

Dividing both sides by $J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))$ and multiplying both sides by z , then

$$z \left((p-\gamma)d + \gamma \right) \frac{\left(J_{c,p,\lambda}^{m,\delta}(L_\theta g(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))} + (p-\gamma) z d'(z) = z (\theta+1) \frac{\left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))} - z \theta \frac{\left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))}$$

Let

$$q(z) = \frac{1}{p-\eta} \left(\frac{z \left(J_{c,p,\lambda}^{m,\delta}(L_\theta g(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))} - \eta \right)$$

Then

$$((p-\gamma)d + \gamma)(q(p-\eta) + \eta + \theta) + (p-\gamma) z d'(z) = z (\theta+1) \frac{\left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))}$$

Now dividing both sides by $(p-\gamma)$

$$\left(d + \frac{\gamma}{(p-\gamma)} \right) (q(p-\eta) + \eta + \theta) + z d'(z) = \frac{z (\theta+1) \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{(p-\gamma) J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))}$$

Dividing both sides by $(q(p-\eta) + \eta + \theta)$

$$d(z) + \frac{z d'(z)}{(q(p-\eta) + \eta + \theta)} = \frac{1}{(p-\gamma)} \left[\frac{z (\theta+1) \left(J_{c,p,\lambda}^{m,\delta}(L_\theta f(z)) \right)'}{(q(p-\eta) + \eta + \theta) J_{c,p,\lambda}^{m,\delta}(L_\theta g(z))} - \gamma \right]$$

$$d(z) + \frac{zd'(z)}{(p-\eta)q + \eta + \theta} = \frac{1}{(p-\gamma)} \left[\frac{z(J_{c,p,\delta}^{m,\delta} f(z))'}{J_{c,p,\delta}^{m,\delta} g(z)} - \gamma \right]$$

Then from (2.20) and (2.21), we obtain

$$(p-\eta)q + \eta + \theta = \rho e^{\frac{i\pi\phi}{2}}$$

where

$$\left\{ \begin{array}{l} \frac{(p-\eta)(1-A)}{(1-B)} + \eta + \theta < \rho < \frac{(p-\eta)(1+A)}{(1+B)} + \eta + \theta \\ -t_2 < \phi < t_2 \end{array} \right\} \quad \text{for } B \neq -1$$

where t_2 is given by (2.25), and

$$\left\{ \begin{array}{l} \frac{(p-\eta)(1-A)}{2} + \eta + \theta < \rho < \infty \\ -1 < \phi < 1 \end{array} \right\} \quad \text{for } B = -1$$

Here, we note that $d(z)$ is analytic in U with $d(0) = 1$ in U by applying the assumption and Lemma

$$(2.2) \quad \text{with } w(z) = \frac{1}{(p-\eta)q + \eta + \theta}$$

Hence, $d(z) \neq 0$ in U if the following two points exist $z_1, z_2 \in U$, such that the condition (2.1) is satisfied then (by Lemma 2.3), we obtain (2.2) under the restriction (2.3).

At first, for the case $B \neq -1$

$$\begin{aligned} \arg(d(z_1) + \frac{z_1 d'(z_1)}{(p-\eta)q(z_1) + \eta + \theta}) &\leq -\frac{\pi}{2} \alpha_1 - \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1-|c^*|) \cos \frac{\pi}{2} t_2}{2 \left(\frac{(p-\eta)(1+A)}{(1+B)} + \eta + \theta \right) (1+|c^*|) + (\alpha_1 + \alpha_2)(1-|c^*|) \sin \frac{\pi}{2} t_2} \right\} \\ &= -\frac{\pi}{2} \delta_1 \end{aligned}$$

and

$$\begin{aligned} \arg(d(z_2) + \frac{z_2 d'(z_2)}{(p-\eta)q(z_2) + \eta + \theta}) &\geq \frac{\pi}{2} \alpha_2 + \tan^{-1} \left\{ \frac{(\alpha_1 + \alpha_2)(1-|c^*|) \cos \frac{\pi}{2} t_2}{2 \left(\frac{(p-\eta)(1+A)}{(1+B)} + \eta + \theta \right) (1+|c^*|) + (\alpha_1 + \alpha_2)(1-|c^*|) \sin \frac{\pi}{2} t_2} \right\} \\ &= \frac{\pi}{2} \delta_2 \end{aligned}$$

where we have used the inequality (2.3), and δ_1, δ_2, t_2 are given by (2.15), (2.16) and (2.25) respectively.

Similarly, for the case $B = -1$, we obtain

$$\arg(d(z_1) + \frac{z_1 d'(z_1)}{(p-\eta)q(z_1) + \eta + \theta}) \leq -\frac{\pi}{2} \alpha_1$$

and

$$\arg(d(z_2) + \frac{z_2 d'(z_2)}{(p-\eta)q(z_2) + \eta + \theta}) \geq \frac{\pi}{2} \alpha_2$$

These are contradiction to the assumption of Theorem (2.2).

This completes the proof of the Theorem (2.2).

Corollary 2.3. Let $f(z) \in A(p, n)$ and $0 < \delta \leq 1, 0 \leq \gamma < 1$ if

$$\left| \arg\left(\frac{z (J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma\right) \right| < \frac{\pi}{2} \delta$$

for some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$

, then

$$\left| \arg\left(\frac{z (J_{c,p,\lambda}^{m,\delta} L_\theta f(z))'}{J_{c,p,\lambda}^{m,\delta} L_\theta g(z)} - \gamma\right) \right| < \frac{\pi}{2} \alpha$$

where $L_\theta(f)$ is defined by (2.8), and $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha \cos \frac{\pi}{2} t_2}{\left(\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta \right) + \alpha \sin \frac{\pi}{2} t_2} \right) & \text{for } B \neq -1 \\ \alpha & \text{for } B = -1 \end{cases}$$

where t_2 is given by (2.25)

Proof:

Take $\delta_1 = \delta_2 = \delta$ and $\alpha_1 = \alpha_2 = \alpha$ in Theorem (2.2)

$$\left| \arg\left(\frac{z (J_{c,p,\lambda}^{m,\delta} f(z))'}{J_{c,p,\lambda}^{m,\delta} g(z)} - \gamma\right) \right| < \frac{\pi}{2} \delta$$

, for some $g(z) \in S_{c,p,\lambda}^{m,\delta}(\eta, A, B)$, then

$$\left| \arg\left(\frac{z (J_{c,p,\lambda}^{m,\delta} (L_\theta f(z)))'}{J_{c,p,\lambda}^{m,\delta} (L_\theta g(z))} - \gamma\right) \right| < \frac{\pi}{2} \alpha$$

where

$$\delta = \begin{cases} \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha \cos \frac{\pi}{2} t_2}{\left(\frac{(p-\eta)(1+A)}{1+B} + \eta + \theta \right) + \alpha \sin \frac{\pi}{2} t_2} \right) & \text{for } B \neq -1 \\ \alpha & \text{for } B = -1 \end{cases}$$

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Received: June 4, 2012.

Accepted: September 7, 2012.