

ONE PARAMETER FAMILY OF \mathcal{B} - TANGENT DEVELOPABLE SURFACES OF SPACELIKE BIHARMONIC NEW TYPE \mathcal{B} -SLANT HELICES IN \mathcal{H}^3

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Abstract. In this paper, we study inextensible flows of tangent developable surfaces of biharmonic spacelike new type \mathcal{B} -slant helices according to Bishop frame in the Lorentzian Heisenberg group \mathcal{H}^3 . We give necessary and sufficient conditions for new type \mathcal{B} -slant helices to be biharmonic. We characterize one parameter family of \mathcal{B} -tangent developable surfaces in the Lorentzian Heisenberg group \mathcal{H}^3 . Additionally, we illustrate our results.

1 Introduction

A smooth map $\phi : N \rightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 dv_n,$$

where $\mathcal{T}(\phi) := \text{tr} \nabla^\phi d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \text{tr} R(\mathcal{T}(\phi), d\phi) d\phi,$$

and called the bitension field of ϕ . Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we give necessary and sufficient conditions for new type \mathcal{B} -slant helices to be biharmonic. We characterize this curves in the Lorentzian Heisenberg group \mathcal{H}^3 . Secondly, we study biharmonic \mathcal{B} -tangent developable surfaces of spacelike new type \mathcal{B} -slant helices according to Bishop frame in the Lorentzian Heisenberg group \mathcal{H}^3 . Finally, we illustrate our results.

2 The Lorentzian Heisenberg Group \mathcal{H}^3

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to \mathbb{R}^3 and the group operation is defined as

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x}, y + \bar{y}, z + \bar{z} - \bar{x}y + x\bar{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \quad (1)$$

The characterising properties of this algebra are the following commutation relations, [15]:

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g , defined above the following is true:*

$$\nabla = \frac{1}{2} \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2}$$

where the (i, j) -element in the table above equals $\nabla_{\mathbf{e}_i} \mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\}.$$

3 Spacelike Biharmonic New Type \mathcal{B} –Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group \mathcal{H}^3

Let $\gamma : I \rightarrow \mathcal{H}^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group \mathcal{H}^3 parametrized by arc length. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group \mathcal{H}^3 along γ defined as follows:

\mathbf{t} is the unit vector field γ' tangent to γ , \mathbf{n} is the unit vector field in the direction of $\nabla_{\mathbf{t}} \mathbf{t}$ (normal to γ), and \mathbf{b} is chosen so that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{t}} \mathbf{t} &= \kappa \mathbf{n}, \\ \nabla_{\mathbf{t}} \mathbf{n} &= \kappa \mathbf{t} + \tau \mathbf{b}, \\ \nabla_{\mathbf{T}} \mathbf{B} &= \tau \mathbf{n}, \end{aligned} \tag{3}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g(\mathbf{t}, \mathbf{t}) &= 1, \quad g(\mathbf{n}, \mathbf{n}) = -1, \quad g(\mathbf{b}, \mathbf{b}) = 1, \\ g(\mathbf{t}, \mathbf{n}) &= g(\mathbf{t}, \mathbf{b}) = g(\mathbf{n}, \mathbf{b}) = 0. \end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}} \mathbf{t} &= k_1 \mathbf{m}_1 - k_2 \mathbf{m}_2, \\ \nabla_{\mathbf{t}} \mathbf{m}_1 &= k_1 \mathbf{t}, \\ \nabla_{\mathbf{t}} \mathbf{m}_2 &= k_2 \mathbf{t}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} g(\mathbf{t}, \mathbf{t}) &= 1, \quad g(\mathbf{m}_1, \mathbf{m}_1) = -1, \quad g(\mathbf{m}_2, \mathbf{m}_2) = 1, \\ g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{t}, \mathbf{m}_2) = g(\mathbf{m}_1, \mathbf{m}_2) = 0. \end{aligned}$$

Here, we shall call the set $\{\mathbf{t}, \mathbf{m}_1, \mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures.

Also, $\tau(s) = \psi'(s)$ and $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s) \sinh \psi(s), \\ k_2 &= \kappa(s) \cosh \psi(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1 \mathbf{e}_1 + t^2 \mathbf{e}_2 + t^3 \mathbf{e}_3, \\ \mathbf{m}_1 &= m_1^1 \mathbf{e}_1 + m_1^2 \mathbf{e}_2 + m_1^3 \mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1 \mathbf{e}_1 + m_2^2 \mathbf{e}_2 + m_2^3 \mathbf{e}_3. \end{aligned} \tag{5}$$

Theorem 3.1. $\gamma : I \rightarrow \mathcal{H}^3$ is a spacelike biharmonic curve with Bishop frame if and only if

$$\begin{aligned} k_1^2 - k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' + [k_1^2 - k_2^2] k_1 &= -k_1 [1 + (m_2^1)^2] + k_2 m_1^1 m_2^1, \\ k_2'' + [k_1^2 - k_2^2] k_2 &= -k_1 m_1^1 m_2^1 - k_2 [-1 + (m_1^1)^2]. \end{aligned} \tag{6}$$

4 \mathcal{B} –Tangent Developable Surfaces of Spacelike Biharmonic New Type \mathcal{B} –Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group \mathcal{H}^3

To separate a tangent developable according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as \mathcal{B} –tangent developable.

The purpose of this section is to study \mathcal{B} –tangent developable of biharmonic spacelike new type \mathcal{B} –slant helix in \mathcal{H}^3 . The \mathcal{B} –tangent developable of γ is a ruled surface

$$\mathcal{O}_{new}(s, u) = \gamma(s) + u\gamma'(s). \quad (7)$$

Definition 4.1. A surface evolution $\mathcal{O}_{new}(s, u, t)$ and its flow $\frac{\partial \mathcal{O}_{new}}{\partial t}$ are said to be inextensible if its first fundamental form $\{\mathbf{E}, \mathbf{F}, \mathbf{G}\}$ satisfies

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \mathbf{G}}{\partial t} = 0. \quad (8)$$

Definition 4.2. We can define the following one-parameter family of developable ruled surface

$$\mathcal{O}_{new}(s, u, t) = \gamma(s, t) + u\gamma'(s, t). \quad (9)$$

Hence, we have the following theorem.

Theorem 4.3. Let \mathcal{O}_{new} be one-parameter family of the \mathcal{B} –tangent developable of a unit speed non-geodesic biharmonic new type \mathcal{B} –slant helix. Then $\frac{\partial \mathcal{O}_{new}}{\partial t}$ is inextensible if and only if

$$\begin{aligned} & \frac{\partial}{\partial t} (\sin \varrho(t) - uk_2(t) \cos \varrho(t))^2 + \frac{\partial}{\partial t} (\cos \varrho(t) \cosh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)]) \\ & + uk_1(t) \sinh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] - uk_2(t) \sin \varrho(t) \cosh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)]^2 \\ & + \frac{\partial}{\partial t} (\cos \varrho(t) \sinh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] + uk_1(t) \cosh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)]) \end{aligned} \quad (10)$$

$$-uk_2(t) \sin \varrho(t) \sinh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)]^2 = 0,$$

where $\mathcal{C}_0, \mathcal{C}_1$ are smooth functions of time.

Proof. Assume that $\mathcal{O}_{new}(s, u, t)$ be a one-parameter family of the \mathcal{B} –tangent developable of a unit speed non-geodesic biharmonic new type \mathcal{B} –slant helix.

From our assumption, we get the following equation

$$\mathbf{m}_2 = \cos \varrho(t) \mathbf{e}_1 + \sin \varrho(t) \cosh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] \mathbf{e}_2 + \sin \varrho(t) \sinh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] \mathbf{e}_3. \quad (11)$$

where $\mathcal{C}_0, \mathcal{C}_1$ are smooth functions of time.

On the other hand, using Bishop formulas Eq.(4) and Eq.(1), we have

$$\mathbf{m}_1 = \sinh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] \mathbf{e}_2 + \cosh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] \mathbf{e}_3. \quad (12)$$

Using above equation and Eq.(11), we get

$$\mathbf{t} = \sin \varrho(t) \mathbf{e}_1 + \cos \varrho(t) \cosh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] \mathbf{e}_2 + \cos \varrho(t) \sinh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] \mathbf{e}_3. \quad (13)$$

Furthermore, we have the natural frame $\{(\mathcal{O}_{new})_s, (\mathcal{O}_{new})_u\}$ given by

$$(\mathcal{O}_{new})_s = (\sin \varrho(t) - uk_2(t) \cos \varrho(t)) \mathbf{e}_1 + (\cos \varrho(t) \cosh [\mathcal{C}_0(t)s + \mathcal{C}_1(t)] + \quad (14)$$

$$uk_1(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] - uk_2(t) \sin \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \mathbf{e}_2 + \\ (\cos \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] + uk_1(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] - \\ uk_2(t) \sin \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)]) \mathbf{e}_3,$$

and

$$(\mathcal{O}_{new})_u = \sin \varOmega(t) \mathbf{e}_1 + \cos \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \mathbf{e}_2 + \cos \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \mathbf{e}_3.$$

The components of the first fundamental form are

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= \frac{\partial}{\partial t} g((\mathcal{O}_{new})_s, (\mathcal{O}_{new})_s) = \frac{\partial}{\partial t} (\sin \varOmega(t) - uk_2(t) \cos \varOmega(t))^2 \\ &+ \frac{\partial}{\partial t} (\cos \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] + uk_1(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \\ &- uk_2(t) \sin \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)])^2 \\ &+ \frac{\partial}{\partial t} (\cos \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] + uk_1(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \\ &- uk_2(t) \sin \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)])^2, \\ \frac{\partial \mathbf{F}}{\partial t} &= 0, \end{aligned} \tag{15}$$

$$\frac{\partial \mathbf{G}}{\partial t} = 0.$$

Hence, $\frac{\partial \mathcal{O}_{new}}{\partial t}$ is inextensible if and only if Eq.(10) is satisfied. This concludes the proof of theorem.

Theorem 4.4. *Let \mathcal{O}_{new} be one-parameter family of the \mathcal{B} -tangent developable surface of a unit speed non-geodesic biharmonic new type \mathcal{B} -slant helix. Then, the parametric equations of this family are given by*

$$\begin{aligned} \mathbf{x}_{\mathcal{O}_{new}}(s, u, t) &= \frac{1}{\mathcal{C}_0(t)} \cos \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \\ &+ u \cos \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] + \mathcal{C}_2(t), \\ \mathbf{y}_{\mathcal{O}_{new}}(s, u, t) &= \frac{1}{\mathcal{C}_0(t)} \cos \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \\ &+ u \cos \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] + \mathcal{C}_3(t), \\ \mathbf{z}_{\mathcal{O}_{new}}(s, u, t) &= \sin \varOmega(t) s - \frac{\mathcal{C}_2(t)}{\mathcal{C}_0(t)} \cos \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \end{aligned} \tag{16}$$

$$- \frac{1}{4\mathcal{C}_0} \cos^2 \varOmega(t) (2[\mathcal{C}_0(t) s + \mathcal{C}_1(t)])$$

$$+ \sinh 2[\mathcal{C}_0(t) s + \mathcal{C}_1(t)] + u \sin \varOmega(t) - u \left(\frac{1}{\mathcal{C}_0(t)} \cos \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] \right. \\ \left. + \mathcal{C}_2(t) \right) \cos \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)] + \mathcal{C}_4(t),$$

where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_0$ are smooth functions of time.

Proof. We assume that γ is a unit speed new type \mathcal{B} -slant helix. Substituting Eq.(1) to Eq.(13), we have

$$\mathbf{t} = (\cos \varOmega(t) \sinh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)], \cos \varOmega(t) \cosh [\mathcal{C}_0(t) s + \mathcal{C}_1(t)], \sin \varOmega(t)) \tag{17}$$

$$-\left(\frac{1}{C_0(t)} \cos \Omega(t) \cosh [C_0(t) s + C_1(t)] + C_2(t) \cos \Omega(t) \cosh [C_0(t) s + C_1(t)]\right),$$

Substituting this into the Eq.(15), we have Eq.(16). Thus, the proof is completed.

We can use Mathematica in above theorem, yields

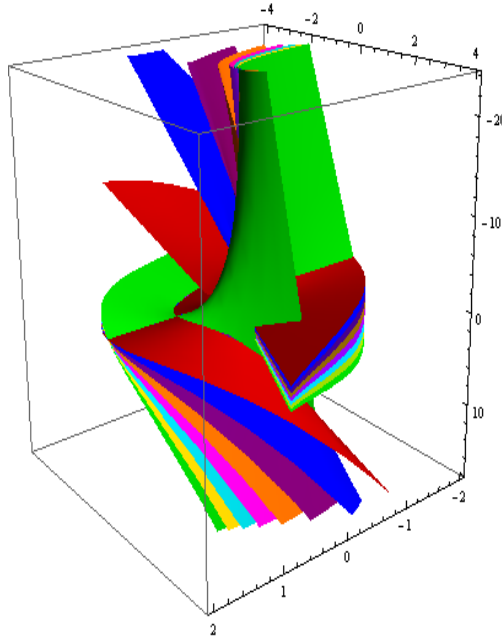


Figure 1.

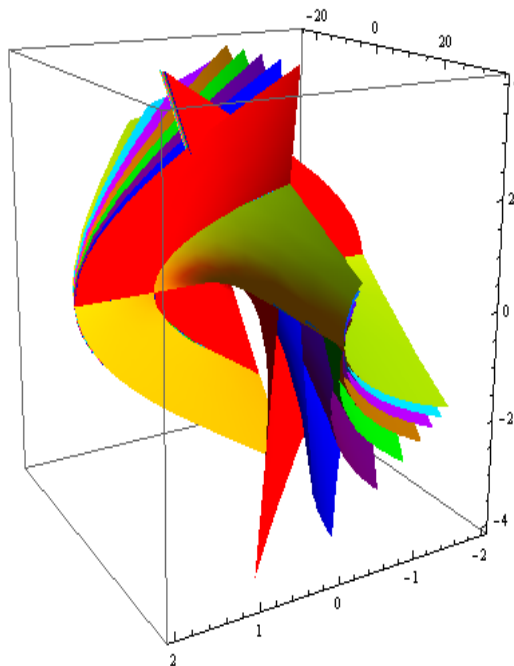


Figure 2.

Fig. 1,2: The equation (16) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time $t = 1, t = 1.2, t = 1.4, t = 1.6, t = 1.8, t = 2, t = 2.2, t = 2.4$, respectively.

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